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On a maximal subgroup of the Thompson simple group*

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Abstract. The present paper deals with a maximal subgroup of the Thompson group, namely the group $2_+^{1+8}{}^{\cdot}A_9 := \overline{G}$. We compute its conjugacy classes using the coset analysis method, its inertia factor groups and Fischer matrices, which are required for the computations of the character table of $\overline{G}$ by means of Clifford-Fischer Theory.
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1. Introduction

Let $\overline{G}$ be the normalizer of the unique class of involutions $2A$ of the sporadic Thompson group Th. Our group $\overline{G}$ is the third largest maximal subgroup of Th (see [7]) and has the form $\overline{G} = 2_+^{1+8}{}^{\cdot}A_9$, the non-split extension of the extraspecial 2-group of order 512 with an outer automorphism group isomorphic to $O_8^+(2)$ by the Alternating group A_9. The group $\overline{G}$ has order 92 897 280 and index 976 841 775 in Th. This is a very good example for the applications of Clifford-Fischer Theory since the group is a non-split extension with an extra-special 2-group as its kernel. Not many examples of this type have been studied via Clifford-Fischer Theory. In this paper, our main aims are to fully study this group, determine its inertia factor groups and compute all Fischer matrices. It will turn out that the character table of $\overline{G}$ is a 52×52 matrix. If one is only interested in the calculation of the character table, then it could be computed by using GAP or Magma and the generators x and y (see below) of $\overline{G}$. But Clifford-Fischer Theory provides much more interesting information on the group and on the character table; in particular, the character table produced by Clifford-Fischer Theory is in a special format that could not be achieved by direct computations using GAP or Magma. Also, providing examples of applications of Clifford-Fischer Theory to both split and non-split extensions is a sensible choice since each group requires an individual approach. The readers (particularly young researchers) will highly benefit from the theoretical background required for these computations. GAP and Magma are computational tools and would not replace good powerful and theoretical arguments.

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Let x and y be in the 248-dimensional matrix group over $\mathbb{F}_2$, that are generators of Th given by the electronic $\mathbb{ATLAS}$ of Wilson [18]. Using [6] and the programm supplied by Wilson, it is possible to construct $\overline{G}$ inside Th. We were also able to locate a normal subgroup $N \lhd \overline{G}$ isomorphic to 2_+^{1+8} by checking the normal subgroups of $\overline{G}$ and the conditions for extraspecial p-groups.

Remark 1. *Due to Havas, Soicher and Wilson [10], a presentation for $\overline{G} = 2_+^{1+8}{}^{\cdot}A_9$ has been restated in Lemmas 12.1.2 and 12.1.3 of [12].*

For the notation used in this paper and the description of the Clifford-Fischer theory technique, we follow [3] and [4].

2. Conjugacy classes of group extensions and of $\overline{G} = 2_+^{1+8}{}^{\cdot}A_9$

In this section, we calculate the conjugacy classes of $\overline{G}$ using the coset analysis technique (for more details see [13] or [14]) as we are interested to organize the classes of $\overline{G}$ corresponding to the classes of A_9. Note that in [12], G. Michler determined the conjugacy classes of $\overline{G} = 2_+^{1+8}{}^{\cdot}A_9$ using the Algorithm of Kratzer (see p. 294 in [12]). One can also use MAGMA or GAP, with the presentation of $\overline{G}$ given by the $\mathbb{ATLAS}$ of Wilson, to compute its conjugacy classes.

Example 1. *Consider the identity coset $N1_{A_9} = N = 2_+^{1+8}$ as this coset gives much information about the structure of the character table of $\overline{G}$. If $g = 1_{A_9}$, then the action of N on $N\overline{1_{A_9}} = N1_{A_9} = N$ produces the conjugacy classes of N, where we know that N has*

- *singleton conjugacy class consisting of 1_N,*
- *singleton conjugacy class consisting of the central involution σ of N,*
- *135 conjugacy classes, each of which consists of two non-central involutions,*
- *120 conjugacy classes, each of which consists only of two elements of order 4.*

Now using MAGMA, the action of $\overline{G}$ on the preceding orbits leaves invariant $\{1_N\} := \Delta_{11}$ and $\{\sigma\} := \Delta_{12}$, while fuses the 135 orbits of non-central involutions into a single orbit Δ_{13} and also fuses the 120 orbits of elements of order 4 altogether into a single orbit Δ_{14}. Thus the identity coset N produces four conjugacy classes $\Delta_{11},\ \Delta_{12},\ \Delta_{13}$ and Δ_{14} in $\overline{G}$, where $|\Delta_{11}| = |\Delta_{12}| = 1,\ |\Delta_{13}| = 270$ and $|\Delta_{14}| = 240$. We let $g_{11} = 1_{\overline{G}},\ g_{12} = \sigma,\ g_{13} \in \Delta_{13}$ and $g_{14} \in \Delta_{14}$ be representatives of the $\overline{G}$-conjugacy classes obtained from N.

In Table 1, we list the conjugacy classes of $\overline{G}$ together with the fusion of its classes into the classes of Thompson group Th. To each conjugacy class of $\overline{G}$, we have attached some weights m_{ij}, which will be used later in computing the Fischer matrices of $\overline{G}$. These weights are computed through the formula

$$m_{ij} = [N_{\overline{G}}(N\overline{g}_i) : C_{\overline{G}}(g_{ij})] = |N|\frac{|C_G(g_i)|}{|C_{\overline{G}}(g_{ij})|}, \text{ where } G = \overline{G}/N \cong A_9. \tag{1}$$

$[g_i]_{A_9}$	m_{ij}	$[g_{ij}]_{2_+^{1+8}\cdot A_9}$	$o(g_{ij})$	$\lvert [g_{ij}]_{2_+^{1+8}\cdot A_9} \rvert$	$\lvert C_{2_+^{1+8}\cdot A_9}(g_{ij}) \rvert$	$\hookrightarrow$ Th
$g_1 = 1A$	$m_{11} = 1$	g_{11}	1	1	92897280	$1A$
	$m_{12} = 1$	g_{12}	2	1	92897280	$2A$
	$m_{13} = 270$	g_{13}	2	270	344064	$2A$
	$m_{14} = 240$	g_{14}	4	240	387072	$4A$
$g_2 = 2A$	$m_{21} = 32$	g_{21}	4	12096	7680	$4B$
	$m_{22} = 480$	g_{22}	4	181440	512	$4B$
$g_3 = 2B$	$m_{31} = 32$	g_{31}	2	30240	3072	$2A$
	$m_{32} = 32$	g_{32}	4	30240	3072	$4A$
	$m_{33} = 192$	g_{33}	4	181440	512	$4B$
	$m_{34} = 256$	g_{34}	8	241920	384	$8A$
$g_4 = 3A$	$m_{41} = 256$	g_{41}	3	43008	2160	$3C$
	$m_{42} = 256$	g_{42}	6	43008	2160	$6A$
$g_5 = 3B$	$m_{51} = 64$	g_{51}	3	143360	648	$3B$
	$m_{52} = 64$	g_{52}	6	143360	648	$6C$
	$m_{53} = 384$	g_{53}	12	860160	108	$12C$
$g_6 = 3C$	$m_{61} = 16$	g_{61}	3	53760	1728	$3A$
	$m_{62} = 16$	g_{62}	6	53760	1728	$6B$
	$m_{63} = 288$	g_{63}	6	967680	96	$6B$
	$m_{64} = 96$	g_{64}	12	322560	288	$12A$
	$m_{65} = 96$	g_{65}	12	322560	288	$12B$
$g_7 = 4A$	$m_{71} = 128$	g_{71}	8	967680	96	$8B$
	$m_{72} = 384$	g_{72}	8	2903040	32	$8B$
$g_8 = 4B$	$m_{81} = 128$	g_{81}	4	1451520	64	$4B$
	$m_{82} = 128$	g_{82}	8	1451520	64	$8A$
	$m_{83} = 256$	g_{83}	8	2903040	32	$8B$
$g_9 = 5A$	$m_{91} = 256$	g_{91}	5	774144	120	$5A$
	$m_{92} = 256$	g_{92}	10	774144	120	$10A$
$g_{10} = 6A$	$m_{10,1} = 512$	$g_{10,1}$	12	3870720	24	$12D$
$g_{11} = 6B$	$m_{11,1} = 128$	$g_{11,1}$	6	3870720	24	$6B$
	$m_{11,2} = 64$	$g_{11,2}$	12	1935360	48	$12A$
	$m_{11,3} = 64$	$g_{11,3}$	12	1935360	48	$12B$
	$m_{11,4} = 128$	$g_{11,4}$	24	3870720	24	$24A$
	$m_{11,5} = 128$	$g_{11,5}$	24	3870720	24	$24B$
$g_{12} = 7A$	$m_{12,1} = 64$	$g_{12,1}$	7	1658880	56	$7A$
	$m_{12,2} = 64$	$g_{12,2}$	14	1658880	56	$14A$
	$m_{12,3} = 128$	$g_{12,3}$	14	3317760	28	$14A$
	$m_{12,4} = 128$	$g_{12,4}$	14	3317760	28	$14A$
	$m_{12,5} = 128$	$g_{12,5}$	28	3317760	28	$28A$
$g_{13} = 9A$	$m_{13,1} = 256$	$g_{13,1}$	9	5160960	18	$9C$
	$m_{13,2} = 256$	$g_{13,2}$	18	5160960	18	$18B$
$g_{14} = 9B$	$m_{14,1} = 64$	$g_{14,1}$	9	1290240	72	$9A$
	$m_{14,2} = 64$	$g_{14,2}$	18	1290240	72	$18A$
	$m_{14,3} = 128$	$g_{14,3}$	36	2580480	36	$36B$
	$m_{14,4} = 128$	$g_{14,4}$	36	2580480	36	$36A$
	$m_{14,5} = 128$	$g_{14,5}$	36	2580480	36	$36C$
$g_{15} = 10A$	$m_{15,1} = 512$	$g_{15,1}$	20	4644864	20	$20A$
$g_{16} = 12A$	$m_{16,1} = 256$	$g_{16,1}$	24	3870720	24	$24C$
	$m_{16,2} = 256$	$g_{16,2}$	24	3870720	24	$24D$
$g_{17} = 15A$	$m_{17,1} = 256$	$g_{17,1}$	15	3096576	30	$15A$
	$m_{17,2} = 256$	$g_{17,2}$	30	3096576	30	$30A$
$g_{18} = 15B$	$m_{18,1} = 256$	$g_{18,1}$	15	3096576	30	$15B$
	$m_{18,2} = 256$	$g_{18,2}$	30	3096576	30	$30B$

Table 1: *The conjugacy classes of* $2_+^{1+8}\cdot A_9$

3. The theory of Clifford-Fischer matrices

We give a brief description on Clifford-Fischer theory for constructing the character table of a group extension $\overline{G}$.

Let $\overline{H} \trianglelefteq \overline{G}$ and let $\phi \in \mathrm{Irr}(\overline{H})$. For $\overline{g} \in \overline{G}$, define $\phi^{\overline{g}}$ by $\phi^{\overline{g}}(h) = \phi(\overline{g}h\overline{g}^{-1})$, $\forall\, h \in \overline{H}$. It follows that $\overline{G}$ acts on $\mathrm{Irr}(\overline{H})$ by conjugation and we define the *inertia group* of ϕ in $\overline{G}$ by $\overline{H}_\phi = \{\overline{g} \in \overline{G} | \ \phi^{\overline{g}} = \phi\}$. Also, for a finite group K, we let $\mathrm{IrrProj}(K, \alpha^{-1})$ denote the set of irreducible projective characters of K with factor set α^{-1}.

Theorem 1 (Clifford Theorem)**.** *Let $\chi \in \mathrm{Irr}(\overline{G})$ and let $\theta_1, \theta_2, \cdots, \theta_t$ be representatives of orbits of $\overline{G}$ on $\mathrm{Irr}(N)$. For $k \in \{1, 2, \cdots, t\}$, let $\theta_k^{\overline{G}} = \{\theta_k = \theta_{k1}, \theta_{k2}, \cdots, \theta_{ks_k}\}$ and let $\overline{H}_k$ be the inertia group in $\overline{G}$ of θ_k. Then*

$$\chi{\downarrow}_N^{\overline{G}} = \sum_{k=1}^{t} e_k \sum_{u=1}^{s_k} \theta_{ku}, \quad \textit{where} \quad e_k = \left\langle \chi{\downarrow}_N^{\overline{G}}, \theta_k \right\rangle.$$

Moreover, for fixed k

$$\begin{aligned} \mathrm{Irr}(\overline{H}_k, \theta_k) \ &:= \ \left\{\psi_k \in \mathrm{Irr}(\overline{H}_k) | \ \left\langle \psi_k{\downarrow}_N^{\overline{H}_k}, \theta_k \right\rangle \neq 0 \right\} \\ &\longleftrightarrow \left\{\chi \in \mathrm{Irr}(\overline{G}) | \ \left\langle \chi{\downarrow}_N^{\overline{G}}, \theta_k \right\rangle \neq 0 \right\} := \mathrm{Irr}(\overline{G}, \theta_k) \end{aligned}$$

under the map $\psi_k \longmapsto \psi_k{\uparrow}_{\overline{H}_k}^{\overline{G}}$.

Proof. See Theorems 4.1.5 and 4.1.7 of Ali [1] with the difference in notations. □

Theorem 2. *Further to the settings of* Theorem 1, *assume that for $k \in \{1, 2, \cdots, t\}$, there exists $\psi_k \in \mathrm{Irr}(\overline{H}_k, \theta_k)$. Then the irreducible characters of $\overline{G}$ are given by*

$$\mathrm{Irr}(\overline{G}) = \bigcup_{k=1}^{t} \left\{ (\psi_k \inf(\zeta)){\uparrow}_{\overline{H}_k}^{\overline{G}} | \ \zeta \in \mathrm{Irr}(\overline{H}_k/N) \right\}. \tag{2}$$

Proof. See Ali [1] or Whitley [17]. □

Remark 2. *It is by no means necessarily the case that there exists an extension ψ_k of θ_k to the inertia group (that is, the case $\mathrm{Irr}(\overline{H}_k, \theta_k) = \emptyset$, the empty set, is feasible). However, there is always a projective extension $\widetilde{\psi}_k \in \mathrm{IrrProj}(\overline{H}_k, \overline{\alpha}_k^{-1})$ for some factor set $\overline{\alpha}_k$ of the Schur multiplier of $\overline{H}_k$. Thus a more appropriate formula for Equation* (2) *is (see Remark 4.2.7 of Ali [1])*

$$\mathrm{Irr}(\overline{G}) = \bigcup_{k=1}^{t} \left\{ (\widetilde{\psi}_k \inf(\zeta)){\uparrow}_{\overline{H}_k}^{\overline{G}} | \widetilde{\psi}_k \in \mathrm{IrrProj}(\overline{H}_k, \overline{\alpha}_k^{-1}), \ \zeta \in \mathrm{IrrProj}(\overline{H}_k/N, \alpha_k^{-1}) \right\}, \tag{3}$$

where the factor set α_k is obtained from $\overline{\alpha}_k$ as described in Corollary 7.3.3 of Whitely [17]. Hence the character table of $\overline{G}$ is partitioned into t blocks $\mathcal{K}_1, \mathcal{K}_2, \cdots, \mathcal{K}_t$, where each block $\mathcal{K}_k$ of characters (ordinary or projective) is produced from the inertia group $\overline{H}_k$.

Note 1. *Let* $[n]$ *denote an equivalence class (containing* n*) of the Schur multiplier of a finite group. It follows that if* $\alpha_k \sim [1]$ *in Equation* (3)*, then we get Equation* (2)*. That is,* $\mathrm{IrrProj}(\overline{H}_k, \overline{1}) = \mathrm{Irr}(\overline{H}_k)$ *and* $\mathrm{IrrProj}(H_k, 1) = \mathrm{Irr}(H_k)$*. By convention, we take* $\theta_1 = \mathbf{1}_N$*, the trivial character of* N*. Thus* $\overline{H}_{\theta_1} = \overline{H}_1 = \overline{G}$ *and thus* $\overline{H}_1/N \cong G$*. Since* $\{\mathbf{1}_{\overline{G}}\} \subseteq \mathrm{Irr}(\overline{G}, \mathbf{1}_N)$ *and such that* $\mathbf{1}_{\overline{G}}{\downarrow}_N^{\overline{G}} = \mathbf{1}_N$*, the block* $\mathcal{K}_1$ *will consist only of the ordinary irreducible characters of* G*.*

We now fix some notations for the conjugacy classes.

- With π being the natural epimorphism from $\overline{G}$ onto G, we use the notation $U = \pi(\overline{U})$ for any subset $\overline{U} \subseteq \overline{G}$. Let us assume that $\pi(g_{ij}) = g_i$ and by convention we may take $g_{11} = 1_{\overline{G}}$. Note that $c(g_1)$ is the number of $\overline{G}$-conjugacy classes obtained from N.

- $$[g_{ij}]_{\overline{G}} \bigcap \overline{H}_k = \bigcup_{n=1}^{c(g_{ijk})} [g_{ijkn}]_{\overline{H}_k},$$

  where $g_{ijkn} \in \overline{H}_k$ and by $c(g_{ijk})$ we mean the number of $\overline{H}_k$-conjugacy classes that form a partition for $[g_{ij}]_{\overline{G}}$. Since $g_{11} = 1_{\overline{G}}$, we have $g_{11k1} = 1_{\overline{G}}$ and thus $c(g_{11k1}) = 1$ for all $1 \leq k \leq t$.

- $$[g_i]_G \bigcap H_k = \bigcup_{m=1}^{c(g_{ik})} [g_{ikm}]_{H_k},$$

  where $g_{ikm} \in H_k$ and by $c(g_{ik})$ we mean the number of H_k-conjugacy classes that form a partition for $[g_i]_G$. Since $g_1 = 1_G$, we have $g_{1k1} = 1_G$ and thus $c(g_{1k1}) = 1$ for all $1 \leq k \leq t$. Also, $\pi(g_{ijkn}) = g_{ikm}$ for some $m = f(j, n)$.

Proposition 1. *With the notations of* Theorem 2 *and the above settings, we have*

$$(\widetilde{\psi}_k \inf(\zeta)){\uparrow}_{\overline{H}_k}^{\overline{G}}(g_{ij}) = \sum_{m=1}^{c(g_{ik})} \zeta(g_{ikm}) \sum_{n=1}^{c(g_{ijk})} \frac{|C_{\overline{G}}(g_{ij})|}{|C_{\overline{H}_k}(g_{ijkn})|} \widetilde{\psi}_k(g_{ijkn}).$$

Proof. See Ali [1] or Barraclough [2]. □

We proceed to define the Fischer matrix $\mathcal{F}_i$ corresponding to the conjugacy class $[g_i]_G$. We label the columns of $\mathcal{F}_i$ by the representatives of $[g_{ij}]_{\overline{G}}$, $1 \leq j \leq c(g_i)$ obtained by the coset analysis, and below each g_{ij} we put $|C_{\overline{G}}(g_{ij})|$. Thus there are $c(g_i)$ columns. To label the rows of $\mathcal{F}_i$ we define the set $\overline{J}_i$ to be (this is an equivalent to the notation $R(g)$ used by Ali [1] (p. 49), where g is a representative of a conjugacy class of G)

$$\overline{J}_i = \{(k, g_{ikm}) |\ 1 \leq k \leq t,\ 1 \leq m \leq c(g_{ik}),\ g_{ikm} \text{ is an } \alpha_k^{-1}\text{-regular class}\},$$

or for more brevity we let

$$J_i = \{(k, m) |\ 1 \leq k \leq t,\ 1 \leq m \leq c(g_{ik}),\ g_{ikm} \text{ is an } \alpha_k^{-1}\text{-regular class}\}. \quad (4)$$

Then each row of $\mathcal{F}_i$ is indexed by a pair $(k, g_{ikm}) \in \overline{J}_i$ or $(k,m) \in J_i$. For fixed $1 \leq k \leq t$, we let $\mathcal{F}_{ik}$ be a sub-matrix of $\mathcal{F}_i$ with rows corresponding to the pairs $(k, g_{ik1}), (k, g_{ik2}), \cdots, (k, g_{ikr_{ik}})$ or for brevity $(k,1), (k,2), \cdots, (k, r_k)$. Now let

$$a_{ij}^{(k,m)} := \sum_{n=1}^{c(g_{ijk})} \frac{|C_{\overline{G}}(g_{ij})|}{|C_{\overline{H}_k}(g_{ijkn})|} \widetilde{\psi}_k(g_{ijkn}), \tag{5}$$

(for which $\pi(g_{ijkn}) = g_{ikm}$). For each i, corresponding to the conjugacy class $[g_i]_G$, we define the Fischer matrix $\mathcal{F}_i = \left(a_{ij}^{(k,m)}\right)$, where $1 \leq k \leq t,\ 1 \leq m \leq c(g_{ik}),\ 1 \leq j \leq c(g_i)$. The Fischer matrix $\mathcal{F}_i$,

$$\mathcal{F}_i = \left(a_{ij}^{(k,m)}\right) = \begin{pmatrix} \mathcal{F}_{i1} \\ \hline \mathcal{F}_{i2} \\ \hline \vdots \\ \hline \mathcal{F}_{it} \end{pmatrix}$$

together with additional information required for their definition, is presented as follows:

$\mathcal{F}_i$					
g_i		g_{i1}	g_{i2}	$\cdots$	$g_{ic(g_i)}$
$\lvert C_{\overline{G}}(g_{ij})\rvert$		$\lvert C_{\overline{G}}(g_{i1})\rvert$	$\lvert C_{\overline{G}}(g_{i2})\rvert$	$\cdots$	$\lvert C_{\overline{G}}(g_{ic(g_i)})\rvert$
(k,m)	$\lvert C_{H_k}(g_{ikm})\rvert$				
$(1,1)$	$\lvert C_G(g_i)\rvert$	$a_{i1}^{(1,1)}$	$a_{i2}^{(1,1)}$	$\cdots$	$a_{ic(g_i)}^{(1,1)}$
$(2,1)$	$\lvert C_{H_2}(g_{i21})\rvert$	$a_{i1}^{(2,1)}$	$a_{i2}^{(2,1)}$	$\cdots$	$a_{ic(g_i)}^{(2,1)}$
$(2,2)$	$\lvert C_{H_2}(g_{i22})\rvert$	$a_{i1}^{(2,2)}$	$a_{i2}^{(2,2)}$	$\cdots$	$a_{ic(g_i)}^{(2,2)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(2,r_2)$	$\lvert C_{H_2}(g_{i2r_{i2}})\rvert$	$a_{i1}^{(2,r_2)}$	$a_{i2}^{(2,r_2)}$	$\cdots$	$a_{ic(g_i)}^{(2,r_2)}$
$(u,1)$	$\lvert C_{H_u}(g_{iu1})\rvert$	$a_{i1}^{(u,1)}$	$a_{i2}^{(u,1)}$	$\cdots$	$a_{ic(g_i)}^{(u,1)}$
$(u,2)$	$\lvert C_{H_u}(g_{iu2})\rvert$	$a_{i1}^{(u,2)}$	$a_{i2}^{(u,2)}$	$\cdots$	$a_{ic(g_i)}^{(u,2)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
(u,r_u)	$\lvert C_{H_u}(g_{iur_{iu}})\rvert$	$a_{i1}^{(u,r_u)}$	$a_{i2}^{(u,r_u)}$	$\cdots$	$a_{ic(g_i)}^{(u,r_u)}$
$(t,1)$	$\lvert C_{H_t}(g_{it1})\rvert$	$a_{i1}^{(t,1)}$	$a_{i2}^{(t,1)}$	$\cdots$	$a_{ic(g_i)}^{(t,1)}$
$(t,2)$	$\lvert C_{H_t}(g_{it2})\rvert$	$a_{i1}^{(t,2)}$	$a_{i2}^{(t,2)}$	$\cdots$	$a_{ic(g_i)}^{(t,2)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
(t,r_t)	$\lvert C_{H_t}(g_{itr_{it}})\rvert$	$a_{i1}^{(t,r_t)}$	$a_{i2}^{(t,r_t)}$	$\cdots$	$a_{ic(g_i)}^{(t,r_t)}$
m_{ij}		m_{i1}	m_{i2}	$\cdots$	$m_{ic(g_i)}$

.

Fischer matrices satisfy some interesting properties, which help in computations of their entries. We gather these properties in the following proposition.

Proposition 2.

(*i*) $\displaystyle\sum_{k=1}^{t} c(g_{ik}) = c(g_i)$.

(*ii*) $\mathcal{F}_i$ *is non-singular for each* i.

(*iii*) $a_{ij}^{(1,1)} = 1,\ \forall\ 1 \leq j \leq c(g_i)$.

(*iv*) *If* $N\overline{g}_i$ *is a split coset, then* $a_{i1}^{(k,m)} = \frac{|C_G(g_i)|}{|C_{H_k}(g_{ikm})|},\ \forall i \in \{1, 2, \cdots, r\}$. *In particular, for the identity coset we have* $a_{11}^{(k,m)} = [G : H_k]\theta_k(1_N),\ \forall\ (k,m) \in J_1$.

(*v*) *If* $N\overline{g}_i$ *is a split coset, then* $|a_{ij}^{(k,m)}| \leq |a_{i1}^{(k,m)}|$ *for all* $1 \leq j \leq c(g_i)$. *Moreover, if* $|N| = p^\alpha$, *for some prime* p, *then* $a_{ij}^{(k,m)} \equiv a_{i1}^{(k,m)} (\text{mod } p)$.

(*vi*) *For each* $1 \leq i \leq r$, *the weights* m_{ij} *satisfy the relation* $\sum_{j=1}^{c(g_i)} m_{ij} = |N|$.

(*vii*) *Column orthogonality relation:*

$$\sum_{(k,m)\in J_i} |C_{H_k}(g_{ikm})| a_{ij}^{(k,m)} \overline{a_{ij'}^{(k,m)}} = \delta_{jj'} |C_{\overline{G}}(g_{ij})|.$$

(*viii*) *Row orthogonality relation:*

$$\sum_{j=1}^{c(g_i)} m_{ij} a_{ij}^{(k,m)} \overline{a_{ij}^{(k',m')}} = \delta_{(k,m)(k',m')} a_{i1}^{(k,m)} |N|.$$

Proof. See Basheer and Moori [4, 5]. □

4. The inertia factors of $\overline{G} = 2_+^{1+8}{\cdot}A_9$

The action of $\overline{G} = 2_+^{1+8}{\cdot}A_9$ on $\mathrm{Irr}(2_+^{1+8})$ produces four orbits of lengths 1, 120, 135 and 1 with representatives $\theta_1,\ \theta_2,\ \theta_3$ and θ_4, respectively, where $\theta_1 = \mathbf{1}_N$ the trivial character of N and θ_4 is the unique faithful irreducible character of N of degree 16. Hence the inertia factor groups $H_1,\ H_2,\ H_3$ and H_4 of $\theta_1,\ \theta_2,\ \theta_3$ and θ_4, respectively, have indices 1, 120, 135 and 1, respectively, in $G \cong A_9$. Clearly, $H_1 = H_4 = A_9$. A subgroup of A_9 of index 120 is a maximal subgroup isomorphic to $PSL(2,8){:}3$ (by considering the maximal subgroups of A_9 given in the $\mathbb{ATLAS}$). A subgroup of A_9 of index 135 can only be contained in a maximal subgroup that is isomorphic to A_8 and must have index 15 in A_8. Again, by considering the maximal subgroups of A_8 given by the $\mathbb{ATLAS}$, it turns out that this subgroup is isomorphic to $2^3{:}GL(3,2)$. Hence we have determined all the inertia factor groups $H_1,\ H_2,\ H_3$ and H_4. We list brief information on these inertia factors as follows:

$H_1 = A_9$,	$[A_9 : H_1] = 1$,	$\lvert\mathrm{Irr}(H_1)\rvert = 18$,
$H_2 = PSL(2,8){:}3$,	$[A_9 : H_2] = 120$,	$\lvert\mathrm{Irr}(H_2)\rvert = 11$,
$H_3 = 2^3{:}GL(3,2)$,	$[A_9 : H_3] = 135$,	$\lvert\mathrm{Irr}(H_3)\rvert = 11$,
$H_4 = A_9$,	$[A_9 : H_4] = 1$,	$\lvert\mathrm{IrrProj}(H_4,2)\rvert = 12$.

Note that above we have listed the number of ordinary irreducible characters of $H_1,\ H_2,\ H_3$ and the number of projective characters of H_4 with the factor set

α^{-1}, $\alpha \sim [2]$ as we shall see later that this projective table is needed for the construction of the character table of $\overline{G} = 2^{1+8}_{+}{}^{\cdot}A_9$. Therefore, the character table of $\overline{G}$ is composed of four blocks of characters which correspond to the ordinary character tables of A_9, $PSL(2,8)$:3, 2^3:$GL(3,2)$ and the projective table of A_9 with factor set α^{-1}, $\alpha \sim [2]$. Hence $\overline{G}$ has altogether 52 irreducible characters, which equals the number of conjugacy classes we obtained in Section 2.

Remark 3. *In [3], the character table of $H_2 = PSL(2,8)$:3 has been constructed using Clifford-Fischer theory. The three Fischer matrices of H_2 have also been listed in [3]. For the sake of convenience, in Table 2 we list the character table of H_2, in the format of Clifford-Fischer theory. For the character table of H_3 we refer to Table 5.4 of [17].*

	$1_{\mathbb{Z}_3}$					ω			ω^{-1}		
	h_{11}	h_{12}	h_{13}	h_{14}	h_{15}	h_{21}	h_{22}	h_{23}	h_{31}	h_{32}	h_{33}
$o(h_{ij})$	1	2	3	7	9	3	6	9	3	6	9
$\lvert C_{H_2}(h_{ij})\rvert$	1512	24	27	7	9	18	6	9	18	6	9
$\lvert [h_{ij}]_{H_2}\rvert$	1	63	56	216	168	84	252	168	84	252	168
χ_1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	A	A	A	$\overline{A}$	$\overline{A}$	$\overline{A}$
χ_3	1	1	1	1	1	$\overline{A}$	$\overline{A}$	$\overline{A}$	A	A	A
χ_4	7	-1	-2	0	1	1	-1	1	1	-1	1
χ_5	7	-1	-2	0	1	A	$-A$	A	$\overline{A}$	$-\overline{A}$	$\overline{A}$
χ_6	7	-1	-2	0	1	$\overline{A}$	$-\overline{A}$	$\overline{A}$	A	$-A$	A
χ_7	8	0	-1	1	-1	2	0	-1	2	0	-1
χ_8	8	0	-1	1	-1	$2A$	0	$-A$	$2\overline{A}$	0	$-\overline{A}$
χ_9	8	0	-1	1	-1	$2\overline{A}$	0	$-\overline{A}$	$2A$	0	$-A$
χ_{10}	21	-3	3	0	0	0	0	0	0	0	0
χ_{11}	27	3	0	-1	0	0	0	0	0	0	0

Table 2: *The character table of $H_2 = PSL(2,8)$:3*

where in Table 2, $A = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$.

5. Fusions of the inertia factors into A_9

In this section, we determine the fusions of the inertia factor groups $H_2 = PSL(2,8)$:3 and $H_3 = 2^3$:$GL(3,2)$ into A_9 using the permutation characters of A_9 on these groups. From the $\mathbb{ATLAS}$, the permutation character of A_9 on H_2 is of the form

$$\chi(A_9|H_2) = \underline{1a} + \underline{35b} + \underline{84a}, \tag{6}$$

where $\underline{1a}$, $\underline{35b}$ and $\underline{84a}$ denote the characters χ_1, χ_8 and χ_{12} of A_9, respectively (in the order listed in $\mathbb{ATLAS}$). Also note that H_3 is a maximal subgroup of A_8 of index 15, which is in turn a maximal subgroup of A_9 of index 9.

Proposition 3. *Let $K_1 \leq K_2 \leq K_3$ and let ψ be a class function on K_1. Then*

$$(\psi\uparrow_{K_1}^{K_2})\uparrow_{K_2}^{K_3} = \psi\uparrow_{K_1}^{K_3}.$$

More generally, if $K_1 \leq K_2 \leq \cdots \leq K_n$ is a nested sequence of subgroups of K_n and ψ is a class function on K_1, then

$$(\psi\uparrow_{K_1}^{K_2})\uparrow_{K_2}^{K_3}\cdots\uparrow_{K_{n-1}}^{K_n} = \psi\uparrow_{K_1}^{K_n}.$$

Proof. See Proposition 3.5.6 of [3]. □

Since we know the permutation characters of A_9 on A_8 and of A_8 on H_3, together with the fusions of classes of H_3 into A_8 and of A_8 into A_9, and by Proposition 3 and the $\mathbb{ATLAS}$, we are able to calculate the permutation character $\chi(A_9|H_3)$ as follows

$$\begin{aligned}\chi(A_9|H_3) &= \mathbf{1}\uparrow_{H_3}^{A_9} = (\mathbf{1}\uparrow_{H_3}^{A_8})\uparrow_{A_8}^{A_9} = (\underline{1a} + \underline{14a})\uparrow_{A_8}^{A_9} \\ &= \underline{1a}\uparrow_{A_8}^{A_9} + \underline{14a}\uparrow_{A_8}^{A_9} = \underline{1a} + \underline{8a} + \underline{14a}\uparrow_{A_8}^{A_9}.\end{aligned}$$

Now it is easy to evaluate the values of $\underline{14a}\uparrow_{A_8}^{A_9}$ using the induction formula for characters. In fact, we have found that $\underline{14a}\uparrow_{A_8}^{A_9} = \underline{42a} + \underline{84a}$ and thus

$$\chi(A_9|H_3) = \underline{1a} + \underline{8a} + \underline{42a} + \underline{84a}. \tag{7}$$

Using Equations (6) and (7), in Table 3 we list the values of $\chi(A_9|H_2)$ and $\chi(A_9|H_3)$ on those classes of A_9, where there are possible fusions from classes of H_2 or H_3. Using the permutation characters of A_9 on H_2 and H_3 together with the size of centralizers, the fusions of H_2 and H_3 into A_9 are completely determined. Let $g_1, g_2, \cdots, g_{18}$ be as in Table 1. To be consistent with the notations of [3], we rename the classes of H_2 and H_3 according to the fusions of these classes into classes of A_9. We list these fusions in Table 4.

$[g]_{A_9}$	$1A$	$2A$	$2B$	$3A$	$3B$	$3C$	$4A$	$4B$	$5A$	$6A$	$6B$	$7A$	$9A$	$9B$
$\chi(A_9\|H_2)$	120	0	8	0	3	6	0	0	0	0	2	1	0	3
$\chi(A_9\|H_3)$	135	15	7	0	0	9	3	3	0	0	1	2	0	0

Table 3: *The values of the permutation characters of A_9 on H_2 and H_3*

Class of H_2	$\hookrightarrow$ Class of G	Class of H_3	$\hookrightarrow$ Class of G
$h_{11} = g_{121}$	$1A$	$1a = g_{131}$	$1A$
$h_{12} = g_{321}$	$2B$	$2a = g_{331}$	$2B$
$h_{13} = g_{521}$	$3B$	$2b = g_{231}$	$2A$
$h_{14} = g_{12,21}$	$7A$	$2c = g_{332}$	$2B$
$h_{15} = g_{14,22}$	$9B$	$3a = g_{631}$	$3C$
$h_{21} = g_{621}$	$3C$	$4a = g_{831}$	$4B$
$h_{22} = g_{11,21}$	$6B$	$4b = g_{731}$	$4A$
$h_{23} = g_{14,22}$	$9B$	$4c = g_{832}$	$4B$
$h_{31} = g_{622}$	$3C$	$6a = g_{11,31}$	$6B$
$h_{32} = g_{11,22}$	$6B$	$7a = g_{12,31}$	$7A$
$h_{33} = g_{14,23}$	$9B$	$7b = g_{12,32}$	$7A$

Table 4: *The fusions of classes of H_2 and H_3 into classes of G*

6. Fischer matrices of $\overline{G} = 2_+^{1+8\cdot}A_9$

In this section, we calculate the Fischer matrices of $\overline{G} = 2_+^{1+8\cdot}A_9$. We have seen in Section 4 that the action of $\overline{G} = 2_+^{1+8\cdot}A_9$ on $\mathrm{Irr}(2_+^{1+8})$ produces four orbits with representatives $\theta_1, \theta_2, \theta_3$ and θ_4, where $|\theta_1^{\overline{G}}| = 1$, $|\theta_2^{\overline{G}}| = 120$, $|\theta_3^{\overline{G}}| = 135$ and $|\theta_4^{\overline{G}}| = 1$. By Proposition 2, it follows that $c(g_1) = 4$. We determine the identity Fischer matrix $\mathcal{F}_1$ in detail as this matrix gives some information about the degrees of the irreducible characters of $\overline{G}$.

Lemma 1. *The identity Fischer matrix $\mathcal{F}_1$ is an integral matrix.*

Proof. The result follows since the irreducible characters of N are integer-valued characters and the rows of $\mathcal{F}_1$ are the orbit sums of characters of N. □

For $\mathcal{F}_1$, which is a 4×4 matrix, let its columns correspond to $g_{11} = 1_N$, $g_{12} = \sigma$, g_{13} and g_{14} in this order, where σ is the central involution of N and g_{13} and g_{14} are elements of orders 2 and 4, respectively. Also, let its rows correspond to $H_1 = A_9$, $H_2 = PSL(2,8){:}3$, $H_3 = 2^3{:}GL(3,2)$ and $H_4 = A_9$ in this order (recall that $\mathcal{F}_1$ corresponds to $g_1 = 1_{A_9}$ and there is a fusion from each inertia factor and hence each of these inertia factors contributes with exactly one row to $\mathcal{F}_1$). Next, we determine some of the entries of $\mathcal{F}_1$. The first row and column of $\mathcal{F}_1$ are determined by properties of Fischer matrices given in Proposition 2. Recall that the first column is

$$[|\theta_1^{\overline{G}}|\deg(\theta_1) \quad |\theta_2^{\overline{G}}|\deg(\theta_2) \quad |\theta_3^{\overline{G}}|\deg(\theta_3) \quad |\theta_4^{\overline{G}}|\deg(\theta_4)]^T = [1 \quad 120 \quad 135 \quad 16]^T.$$

The last row of $\mathcal{F}_1$ is

$$\left[\sum_{\theta_i \in \theta_4^{\overline{G}}} \theta_i(g_{11}) \quad \sum_{\theta_i \in \theta_4^{\overline{G}}} \theta_i(g_{12}) \quad \sum_{\theta_i \in \theta_4^{\overline{G}}} \theta_i(g_{13}) \quad \sum_{\theta_i \in \theta_4^{\overline{G}}} \theta_i(g_{14})\right]$$
$$= [\theta_4(g_{11}) \quad \theta_4(g_{12}) \quad \theta_4(g_{13}) \quad \theta_4(g_{14})] = [16 \quad -16 \quad 0 \quad 0].$$

Recall that the orbits $\theta_1^{\overline{G}} = \{\theta_1\}$, $\theta_2^{\overline{G}}$ and $\theta_3^{\overline{G}}$ consist of irreducible characters of N that contain $Z(N) = \{1_N, \sigma\}$ in their kernel, while $\theta_4^{\overline{G}} = \{\theta_4\}$, where θ_4 is the unique faithful irreducible character of N of degree 16, which does not contain $Z(N)$ in its kernel. Since the second column of $\mathcal{F}_1$ corresponds to σ, entries of this column will coincide with entries of the first column point-wise, except in the last row, where $\theta_4(\sigma) = -16 \neq 16 = \theta_4(1_N)$. Therefore, the second column of $\mathcal{F}_1$ is $[1 \quad 120 \quad 135 \quad -16]^T$. Thus we have found the first two columns together with the first and last row of $\mathcal{F}_1$. So far, the identity Fischer matrix $\mathcal{F}_1$ has the form

$\mathcal{F}_1$

g_1		g_{11}	g_{12}	g_{13}	g_{14}
$o(g_{1j})$		1	2	2	4
$\lvert C_{\overline{G}}(g_{1j})\rvert$		92897280	92897280	344064	387072
(k,m)	$\lvert C_{H_k}(g_{1km})\rvert$				
(1, 1)	181440	1	1	1	1
(2, 1)	1512	120	120	$a_{13}^{(2,1)}$	$a_{14}^{(2,1)}$
(3, 1)	1344	135	135	$a_{13}^{(3,1)}$	$a_{14}^{(3,1)}$
(4, 1)	181440	16	−16	0	0
m_{1j}		1	1	270	240

For simplicity of notation, let $s = a_{13}^{(2,1)}$, $t = a_{14}^{(2,1)}$, $u = a_{13}^{(3,1)}$ and $v = a_{14}^{(3,1)}$. Using the orthogonality relations given in Proposition 2, we get 10 equations in the unknowns s, t, u and v. In fact, the first four of the following five equations formed using the column orthogonality relations suffice to find the values of s, t, u and v.

$$s + u = -1, \quad 9s^2 + 8u^2 = 968,$$
$$t + v = -1, \quad 9t^2 + 8v^2 = 1224 \quad \text{and} \quad 9st + 8uv = -1080.$$

The unique integral solution of these simultaneous equations reveals

$$s\ (=a_{13}^{(2,1)}) = -8, \quad t\ (=a_{14}^{(2,1)}) = 8, \quad u\ (=a_{13}^{(3,1)}) = 7, \quad v\ (=a_{14}^{(3,1)}) = -9.$$

Hence the identity Fischer matrix $\mathcal{F}_1$ will have the form:

$\mathcal{F}_1$

g_1		g_{11}	g_{12}	g_{13}	g_{14}
$o(g_{1j})$		1	2	2	4
$\lvert C_{\overline{G}}(g_{1j})\rvert$		92897280	92897280	344064	387072
$\hookrightarrow$ Th		$1A$	$2A$	$2A$	$4A$
(k,m)	$\lvert C_{H_k}(g_{1km})\rvert$				
(1, 1)	181440	1	1	1	1
(2, 1)	1512	120	120	−8	8
(3, 1)	1344	135	135	7	−9
(4, 1)	181440	16	−16	0	0
m_{1j}		1	1	270	240

.

We proceed to calculate all the Fischer matrices of $\overline{G} = 2_+^{1+8}{\cdot}A_9$. In what follows, by an α^{-1}-*regular Fischer matrix* we mean a Fischer matrix that corresponds to an α^{-1}-regular class of A_9. Also, a class of A_9 is α^{-1}-*irregular* if it is not an α^{-1}-regular class. Since in our case $\alpha \sim [2]$, we only use the terms α-*regular class* and α-*regular Fischer matrix.*

Lemma 2. *For every α-regular Fischer matrix $\mathcal{F}_i$ of size $c(g_i)$, the sum of the first $c(g_i) - 1$ rows equals the (componentwise) square of the last row.*

Proof. The proof is similar to the proof of Lemma 6 of [15]. □

Lemma 3. *For every α-regular Fischer matrix $\mathcal{F}_i$, we can order the g_{ij} for $1 \le j \le c(g_i)$ so that the last row of $\mathcal{F}_i$ is of the form $[q_i \quad -q_i \quad 0 \quad \cdots \quad 0]$ with q_i a power of 2 and we may choose the $g_{i2} = \sigma g_{i1}$, where σ is the central involution in $\overline{G}$. Furthermore,*

$$a_{i1}^{(k,m)} = a_{i2}^{(k,m)} = \frac{|C_{H_k}(g_{i11})|}{|C_{H_k}(g_{ikm})|} \quad \textit{for } 1 \le k \le 3,\ 1 \le m \le c(g_{ik}). \tag{8}$$

Proof. The proof is similar to the proof of Lemma 7 of [15], except that the ordinary character η of degree 2^{11} of $2_+^{1+22}{\cdot}Co_2$ (in [15]) is replaced, in our Group $\overline{G} = 2_+^{1+8}{\cdot}A_9$, by a projective character ξ, with a factor set $\alpha \sim [2]$ of degree 16. □

Note 2. *The proof of Lemma 7 of [15] contained a very important piece of information in that the last row of every Fischer matrix of $2_+^{1+22}{\cdot}Co_2$ is $[\eta(g_{i1}) \quad \eta(g_{i2}) \quad \cdots \quad \eta(g_{is_i})]$, where s_i in the author's notation has the same meaning of $c(g_i)$ in our notation. In our group $\overline{G} = 2_+^{1+8}{\cdot}A_9$, the last row of every α-regular Fischer matrix $\mathcal{F}_i$ is given by $[\xi(g_{i1}) \quad \xi(g_{i2}) \quad \cdots \quad \xi(g_{ic(g_i)})]$. Also, the values of ξ on elements of $[g_{ij}]_{\overline{G}}$, where g_i is a representative of an α-irregular conjugacy class, are zeros. That is, $\xi(g_{ij}) = 0,\ \forall\ 1 \le j \le c(g_i),\ i \in \{2,3,7,8,10,15\}$.*

Note 3. *Observe that with Lemma 2, Equation* (8) *and Note 2 we know the first two columns and the last row of every α-regular Fischer matrix $\mathcal{F}_i$.*

As an example, we compute the Fischer matrix $\mathcal{F}_6$, where $g_6 \in [3C]_{A_9}$. From Table 1 we see that g_6 produces 5 conjugacy classes in $\overline{G}$; namely, $g_{61},\ g_{62},\ g_{63},\ g_{64}$ and g_{65} with respective orders 3, 6, 6, 12 and 12 and respective centralizer sizes 1728, 1728, 96, 288 and 288. Therefore, $\mathcal{F}_6$ is a 5×5 matrix. Note that $|C_{A_9}(g_6)| = 54$. Now from Table 4 we infer that there are two classes; namely, $h_{21} = g_{621}$ and $h_{31} = g_{622}$ of $H_2 = PSL(2,8){:}3$ that fuse to $[g_6]_{A_9}$. Note that $|C_{H_2}(g_{621})| = |C_{H_2}(g_{622})| = 18$. Also, from Table 4 we deduce that there is only one class; namely, $3a = g_{631}$ of $H_3 = 2^3{:}GL(3,2)$ that fuse to $[g_6]_{A_9}$. Note that $|C_{H_3}(g_{631})| = 6$. Furthermore, g_6 is an α-regular class (see the $\mathbb{ATLAS}$) and therefore there is a row of $\mathcal{F}_6$ which corresponds to $H_4 = A_9$. Now, using Equation (1) we find that $m_{61} = m_{62} = 16$, $m_{63} = 288$ and $m_{64} = m_{65} = 96$. It follows from the above assertions that $\mathcal{F}_6$ has the form

$\mathcal{F}_6$

g_6		g_{61}	g_{62}	g_{63}	g_{64}	g_{65}
$o(g_{6j})$		3	6	6	12	12
$\lvert C_{\overline{G}}(g_{6j})\rvert$		1728	1728	96	288	288
(k,m)	$\lvert C_{H_k}(g_{6km})\rvert$					
$(1,1)$	54	$a_{61}^{(1,1)}$	$a_{62}^{(1,1)}$	$a_{63}^{(1,1)}$	$a_{64}^{(1,1)}$	$a_{65}^{(1,1)}$
$(2,1)$	18	$a_{61}^{(2,1)}$	$a_{62}^{(2,1)}$	$a_{63}^{(2,1)}$	$a_{64}^{(2,1)}$	$a_{65}^{(2,1)}$
$(2,2)$	18	$a_{61}^{(2,2)}$	$a_{62}^{(2,2)}$	$a_{63}^{(2,2)}$	$a_{64}^{(2,2)}$	$a_{65}^{(2,2)}$
$(3,1)$	6	$a_{61}^{(3,1)}$	$a_{62}^{(3,1)}$	$a_{63}^{(3,1)}$	$a_{64}^{(3,1)}$	$a_{65}^{(3,1)}$
$(4,1)$	54	$a_{61}^{(4,1)}$	$a_{62}^{(4,1)}$	$a_{63}^{(4,1)}$	$a_{64}^{(4,1)}$	$a_{65}^{(4,1)}$
m_{6j}		16	16	288	96	96

.

Now using Proposition 2(iii), we get that $a_{61}^{(1,1)} = a_{62}^{(1,1)} = a_{63}^{(1,1)} = a_{64}^{(1,1)} = a_{65}^{(1,1)} = 1$. Moreover, using Equation (8) we get

$$a_{61}^{(2,1)} = a_{62}^{(2,1)} = \frac{|C_{H_2}(g_6)|}{|C_{H_2}(g_{621})|} = \frac{54}{18} = 3, \quad a_{61}^{(2,2)} = a_{62}^{(2,2)} = \frac{|C_{H_2}(g_6)|}{|C_{H_2}(g_{622})|} = \frac{54}{18} = 3 \text{ and}$$
$$a_{61}^{(3,1)} = a_{62}^{(3,1)} = \frac{|C_{H_2}(g_6)|}{|C_{H_2}(g_{631})|} = \frac{54}{6} = 9.$$

Using Lemmas 2 and 3 we get

$$(a_{61}^{(4,1)})^2 = q_6^2 = \sum_{k=1}^{3} \sum_{m=1}^{c(g_{6k})} a_{61}^{(k,m)} = 1 + 3 + 3 + 9 = 16.$$

Without loss of generality, we may assume that $a_{61}^{(4,1)} = 4$ and therefore $a_{62}^{(4,1)} = -q_6 = -4$. Also, by Lemma 3 we have $a_{63}^{(4,1)} = a_{64}^{(4,1)} = a_{65}^{(4,1)} = 0$. So far, the Fischer matrix $\mathcal{F}_6$ has the form

$\mathcal{F}_6$

g_6		g_{61}	g_{62}	g_{63}	g_{64}	g_{65}
$o(g_{6j})$		3	6	6	12	12
$\lvert C_{\overline{G}}(g_{6j})\rvert$		1728	1728	96	288	288
(k,m)	$\lvert C_{H_k}(g_{6km})\rvert$					
$(1,1)$	54	1	1	1	1	1
$(2,1)$	18	3	3	$a_{63}^{(2,1)}$	$a_{64}^{(2,1)}$	$a_{65}^{(2,1)}$
$(2,2)$	18	3	3	$a_{63}^{(2,2)}$	$a_{64}^{(2,2)}$	$a_{65}^{(2,2)}$
$(3,1)$	6	9	9	$a_{63}^{(3,1)}$	$a_{64}^{(3,1)}$	$a_{65}^{(3,1)}$
$(4,1)$	54	4	-4	0	0	0
m_{6j}		16	16	288	96	96

.

From the columns and rows orthogonality relations (Proposition 2(vii) and (viii)) we obtain a set of 18 simultaneous equations. With consideration of Lemma 2, these equations give

$$1 + a_{63}^{(2,1)} + a_{63}^{(2,2)} + a_{63}^{(3,1)} = 0, \quad 1 + a_{64}^{(2,1)} + a_{64}^{(2,2)} + a_{64}^{(3,1)} = 0 \text{ and}$$
$$1 + a_{65}^{(2,1)} + a_{65}^{(2,2)} + a_{65}^{(3,1)} = 0.$$

We solve the equations using Maxima [11] and obtain

$$a_{63}^{(2,1)} = a_{63}^{(2,2)} = a_{64}^{(2,2)} = a_{65}^{(2,1)} = -1,\ a_{64}^{(2,1)} = a_{65}^{(2,2)} = 3,$$
$$a_{64}^{(3,1)} = a_{65}^{(3,1)} = -3 \text{ and } a_{63}^{(3,1)} = 1.$$

Thus the Fischer matrix $\mathcal{F}_6$ has the form

$\mathcal{F}_6$						
g_6		g_{61}	g_{62}	g_{63}	g_{64}	g_{65}
$o(g_{6j})$		3	6	6	12	12
$\lvert C_{\overline{G}}(g_{6j})\rvert$		1728	1728	96	288	288
$\hookrightarrow$ Th		$3A$	$6B$	$6B$	$12A$	$12B$
(k,m)	$\lvert C_{H_k}(g_{6km})\rvert$					
$(1,1)$	54	1	1	1	1	1
$(2,1)$	18	3	3	-1	3	-1
$(2,2)$	18	3	3	-1	-1	3
$(3,1)$	6	9	9	1	-3	-3
$(4,1)$	54	4	-4	0	0	0
m_{6j}		16	16	288	96	96

In Table 1, we supplied $|C_{\overline{G}}(g_{ij})|$ and $m_{ij},\ 1 \leq i \leq 18,\ 1 \leq j \leq c(g_i)$. Also, we have obtained the fusions of the inertia factors H_2 and H_3 into $G \cong A_9$ (Table 4). Now, using the properties of the Fischer matrices given in Proposition 2, Lemmas 2 and 3 we plan to compute all Fischer matrices of $\overline{G}$.

Now for any $\chi_n \in \mathrm{Irr}(\mathrm{Th}),\ 1 \leq n \leq 48$, let $\chi_n^{(k)}$ denote a character of $\overline{H}_k$ that $\chi_n^{(k)}\uparrow_{\overline{H}_k}^{\overline{G}}$ is a constituent or a zero of $\chi_n\downarrow_{\overline{G}}^{\mathrm{Th}}$. Since we have the fusions of $N = 2_+^{1+8}$ and $Z(N) = \{1, \sigma\}$ into Th, we have

$$\deg(\chi_n^{(1)}) = \left\langle \chi_n\downarrow_N^{\mathrm{Th}}, \mathbf{1}_N \right\rangle_N,$$
$$\deg(\chi_n^{(4)}) = \left\langle \chi_n\downarrow_{Z(N)}^{\mathrm{Th}}, -\mathbf{1}_{Z(N)} \right\rangle_{Z(N)} = \frac{1}{2}\left(\deg(\chi_n) - \chi_n(\sigma)\right),$$

where $\mathbf{1}_N$ and $-\mathbf{1}_{Z(N)}$ denote the trivial character of N and the non-trivial character of $Z(N)$, respectively. In Table 5, we compute $\deg(\chi_n^{(1)})$ and $\deg(\chi_n^{(4)})$ for $2 \leq n \leq 48$. We use this table to determine whether we are required to use projective characters of inertia factor groups. From Table 5 we can see that $\deg(\chi_n^{(4)}) = k \times 16$,

n	$\deg(\chi_n)$	$\left\langle \chi_n\downarrow_{\overline{G}}^{\mathrm{Th}}, \chi_n\downarrow_{\overline{G}}^{\mathrm{Th}} \right\rangle$	$\deg(\chi_n^{(1)})$	$\deg(\chi_n^{(4)})$
2	248	2	0	8×16
3	4123	7	35	128×16
4	27000	16	120	840×16
5	27000	16	120	840×16
6	30628	23	28	960×16
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
48	190373976	390135998	370008	5949288×16

Table 5: *Degrees of some particular characters of some inertia groups*

for some $k \in \mathbb{N},\ k \geq 2$. That is, $\deg(\chi_n^{(4)}) \neq 16$ for all $n \in \{2, 3, \cdots, 48\}$. We deduce that $\overline{H}_4 = 2_+^{1+8}{\cdot}A_9 = \overline{G}$ has no character of degree 16 that is extended from the faithful irreducible character θ_4 (of degree 16) of N. Thus we have to use a projective character table of $H_4 = A_9$ to construct the character table of $\overline{G}$. Since the Schur multiplier of A_9 is $\mathbb{Z}_2$, we have to consider the projective character table of A_9, which is available in the $\mathbb{ATLAS}$.

Corollary 1. *The Schur multiplier of* $\overline{G}$ *is non-trivial and has a factor set* $\alpha \sim [2]$.

Since we will use the $\mathrm{IrrProj}(A_9, 2^{-1}) = \mathrm{IrrProj}(A_9, 2)$ to construct the character table of $\overline{G}$, there exists (see Remark 2 or Theorem 4.3.2 of [16]) a character $\xi \in \mathrm{IrrProj}(\overline{G}, 2)$ such that $\xi\downarrow_N^{\overline{G}} = \theta_4$. Furthermore, $\xi^2\downarrow_N^{\overline{G}} = \theta_4^2$ is the lift of the regular character of $N/Z(N) \cong 2^8$ and hence the sum of the orbits of θ_1, θ_2 and θ_3 under the action $\overline{G}$. We deduce that there are uniquely determined linear characters $\psi_k \in \mathrm{Irr}(\overline{H}_k)$, $1 \leq k \leq 3$, such that

$$\xi^2 = \sum_{k=1}^{3} \psi_k\uparrow_{\overline{H}_k}^{\overline{G}} \quad \text{and} \quad \psi_k\downarrow_N^{\overline{H}_k} = \theta_k \ \text{ for } k \in \{1,2,3\}. \tag{9}$$

Hence θ_1, θ_2 and θ_3 are extendible to ordinary characters of their respective inertia groups. In summary, to construct the character table of $\overline{G}$, we need the ordinary character tables of $H_1 = A_9$, $H_2 = PSL(2,8){:}3$, $H_3 = 2^3{:}GL(3,2)$ and the projective character table of $H_4 = A_9$ with the factor set α^{-1}, $\alpha \sim [2]$. The character tables of A_9 and $2{\cdot}A_9$ are available in the $\mathbb{ATLAS}$. The character table of H_3 can be constructed easily with GAP, while the character table of H_2 appears as Table 2 of this paper. Note that from Section 4 we get 52 irreducible characters of $\overline{G}$, which is exactly the number of conjugacy classes of $\overline{G}$ given in Table 1.

Note 4. *Observe that the character table of* $2{\cdot}A_9$ *indicates that classes of* A_9 *represented by* $g_2 \sim 2A$, $g_3 \sim 2B$, $g_7 \sim 4A$, $g_8 \sim 4B$, $g_{10} \sim 6A$ *and* $g_{15} \sim 10A$ *are* α^{-1}*-irregular classes, where* $\alpha \sim [2]$. *This shows that the number of projective characters is exactly the number of* α^{-1}*-regular classes of* A_9.

Now note that with the help of Note 3 the computations of Fischer matrices $\mathcal{F}_i$, correspond to g_i, for $i \in \{1,4,5,6,9,11,12,13,14,16,17,18\}$ are reduced to computations of 1×1, 2×2 and 3×3 matrices. Also, note that all α-irregular Fischer matrices are of small size (the size of every Fischer matrix (regular or irregular) is $c(g_i)$ and we have $c(g_i) \in \{1,2,3,4\}$, $\forall\, i \in \{2,3,7,8,10,15\}$). The columns and rows orthogonality relations are sufficient to compute both regular and irregular Fischer matrices of $\overline{G}$. Hence we have all the Fischer matrices of $\overline{G}$, which we list below.

$\mathcal{F}_1$

g_1		g_{11}	g_{12}	g_{13}	g_{14}
$o(g_{1j})$		1	2	2	4
$\lvert C_{\overline{G}}(g_{1j})\rvert$		92897280	92897280	344064	387072
$\hookrightarrow$ Th		$1A$	$2A$	$2A$	$4A$
(k,m)	$\lvert C_{H_k}(g_{1km})\rvert$				
$(1,1)$	181440	1	1	1	1
$(2,1)$	1512	120	120	-8	8
$(3,1)$	1344	135	135	7	-9
$(4,1)$	181440	16	-16	0	0
m_{1j}		1	1	270	240

$\mathcal{F}_2$

g_2		g_{21}	g_{22}
$o(g_{2j})$		4	4
$\lvert C_{\overline{G}}(g_{2j})\rvert$		7680	5120
$\hookrightarrow$ Th		$4B$	$4B$
(k,m)	$\lvert C_{H_k}(g_{2km})\rvert$		
$(1,1)$	480	1	1
$(2,1)$	32	15	-1
m_{2j}		32	480

$\mathcal{F}_3$

g_3		g_{31}	g_{32}	g_{33}	g_{34}
$o(g_{3j})$		4	2	4	8
$\lvert C_{\overline{G}}(g_{3j})\rvert$		3072	3072	512	384
$\hookrightarrow$ Th		$4A$	$2A$	$4B$	$8A$
(k,m)	$\lvert C_{H_k}(g_{3km})\rvert$				
$(1,1)$	192	1	1	1	1
$(2,1)$	24	8	-8	0	0
$(3,1)$	192	1	1	1	-1
$(3,2)$	32	6	6	-2	0
m_{3j}		32	32	192	256

$\mathcal{F}_4$

g_4		g_{41}	g_{42}
$o(g_{4j})$		3	6
$\lvert C_{\overline{G}}(g_{4j})\rvert$		2160	2160
$\hookrightarrow$ Th		$3C$	$6A$
(k,m)	$\lvert C_{H_k}(g_{4km})\rvert$		
(1, 1)	1080	1	1
(4, 1)	1080	1	−1
m_{4j}		256	256

$\mathcal{F}_5$

g_5		g_{51}	g_{52}	g_{53}
$o(g_{5j})$		6	3	12
$\lvert C_{\overline{G}}(g_{5j})\rvert$		648	648	108
$\hookrightarrow$ Th		$6C$	$3B$	$12C$
(k,m)	$\lvert C_{H_k}(g_{5km})\rvert$			
(1, 1)	81	1	1	1
(2, 1)	27	3	3	−1
(4, 1)	81	2	−2	0
m_{5j}		64	64	384

$\mathcal{F}_6$

g_6		g_{61}	g_{62}	g_{63}	g_{64}	g_{65}
$o(g_{6j})$		3	6	6	12	12
$\lvert C_{\overline{G}}(g_{6j})\rvert$		1728	1728	96	288	288
$\hookrightarrow$ Th		$3A$	$6B$	$6B$	$12A$	$12B$
(k,m)	$\lvert C_{H_k}(g_{6km})\rvert$					
(1, 1)	54	1	1	1	1	1
(2, 1)	18	3	3	−1	3	−1
(2, 2)	18	3	3	−1	−1	3
(3, 1)	6	9	9	1	−3	−3
(4, 1)	54	4	−4	0	0	0
m_{6j}		16	16	288	96	96

$\mathcal{F}_7$

g_{71}	g_{72}		
$o(g_{7j})$		8	8
$\lvert C_{\overline{G}}(g_{7j})\rvert$		96	32
$\hookrightarrow$ Th		$8B$	$8B$
(k,m)	$\lvert C_{H_k}(g_{7km})\rvert$		
(1, 1)	24	1	1
(3, 1)	8	3	−1
m_{7j}		128	384

$\mathcal{F}_8$

g_8		g_{81}	g_{82}	g_{83}
$o(g_{8j})$		8	4	8
$\lvert C_{\overline{G}}(g_{8j})\rvert$		64	64	32
$\hookrightarrow$ Th		$8A$	$4B$	$8B$
(k,m)	$\lvert C_{H_8}(g_{8km})\rvert$			
(1, 1)	16	1	1	1
(3, 1)	16	1	1	−1
(3, 2)	8	2	−2	0
m_{8j}		128	128	256

$\mathcal{F}_9$

g_9		g_{91}	g_{92}
$o(g_{9j})$		5	10
$\lvert C_{\overline{G}}(g_{9j})\rvert$		120	120
$\hookrightarrow$ Th		$5A$	$10A$
(k,m)	$\lvert C_{H_k}(g_{9km})\rvert$		
(1, 1)	60	1	1
(4, 1)	60	1	−1
m_{9j}		256	256

$\mathcal{F}_{10}$

g_{10}		$g_{10,1}$
$o(g_{10j})$		12
$\lvert C_{\overline{G}}(g_{10j})\rvert$		24
$\hookrightarrow$ Th		$12D$
(k,m)	$\lvert C_{H_k}(g_{10km})\rvert$	
(1, 1)	24	1
m_{10j}		512

$\mathcal{F}_{11}$

g_{11}		$g_{11,1}$	$g_{11,2}$	$g_{11,3}$	$g_{11,4}$	$g_{11,5}$
$o(g_{11j})$		12	12	6	24	24
$\lvert C_{\overline{G}}(g_{11j})\rvert$		48	48	24	24	24
$\hookrightarrow$ Th		$12B$	$12A$	$6B$	$24A$	$24B$
(k,m)	$\lvert C_{H_k}(g_{11km})\rvert$					
(1, 1)	6	1	1	1	1	1
(2, 1)	6	1	1	−1	1	−1
(2, 2)	6	1	1	−1	−1	1
(3, 1)	6	1	1	1	−1	−1
(4, 1)	6	2	−2	0	0	0
m_{11j}		64	64	128	128	128

$\mathcal{F}_{12}$

g_{12}		$g_{12,1}$	$g_{12,2}$	$g_{12,3}$	$g_{12,4}$	$g_{12,5}$
$o(g_{12j})$		7	14	14	14	28
$\lvert C_{\overline{G}}(g_{12j})\rvert$		56	56	28	28	28
$\hookrightarrow$ Th		$7A$	$14A$	$14A$	$14A$	$28A$
(k,m)	$\lvert C_{H_k}(g_{12km})\rvert$					
(1, 1)	7	1	1	1	1	1
(2, 1)	7	1	1	−1	−1	1
(3, 1)	7	1	1	−1	1	−1
(3, 2)	7	1	1	1	−1	−1
(4, 1)	7	2	−2	0	0	0
m_{12j}		64	64	128	128	128

$\mathcal{F}_{13}$

g_{13}		$g_{13,1}$	$g_{13,2}$
$o(g_{13j})$		9	18
$\lvert C_{\overline{G}}(g_{13j})\rvert$		18	18
$\hookrightarrow$ Th		$9C$	$18B$
(k,m)	$\lvert C_{H_k}(g_{13km})\rvert$		
(1, 1)	9	1	1
(4, 1)	9	1	−1
m_{13j}		256	256

$\mathcal{F}_{14}$

g_{14}		$g_{14,1}$	$g_{14,2}$	$g_{14,3}$	$g_{14,4}$	$g_{14,5}$
$o(g_{14j})$		18	9	36	36	36
$\lvert C_{\overline{G}}(g_{14j})\rvert$		72	72	36	36	36
$\hookrightarrow$ Th		$18A$	$9A$	$36B$	$36A$	$36C$
(k, m)	$\lvert C_{H_k}(g_{14km})\rvert$					
$(1, 1)$	9	1	1	1	1	1
$(2, 1)$	9	1	1	-1	1	-1
$(2, 2)$	9	1	1	1	-1	-1
$(2, 3)$	9	1	1	-1	-1	1
$(4, 1)$	9	2	-2	0	0	0
m_{14j}		64	64	128	128	128

$\mathcal{F}_{15}$

g_{15}		$g_{15,1}$
$o(g_{15j})$		20
$\lvert C_{\overline{G}}(g_{15j})\rvert$		20
$\hookrightarrow$ Th		$20A$
(k, m)	$\lvert C_{H_k}(g_{15km})\rvert$	
$(1, 1)$	20	1
m_{15j}		512

$\mathcal{F}_{16}$

g_{16}		$g_{16,1}$	$g_{16,2}$
$o(g_{16j})$		24	24
$\lvert C_{\overline{G}}(g_{16j})\rvert$		24	24
$\hookrightarrow$ Th		$24D$	$24C$
(k, m)	$\lvert C_{H_k}(g_{16km})\rvert$		
$(1, 1)$	12	1	1
$(4, 1)$	12	1	-1
m_{16j}		256	256

$\mathcal{F}_{17}$

g_{17}		$g_{17,1}$	$g_{17,2}$
$o(g_{17j})$		15	30
$\lvert C_{\overline{G}}(g_{17j})\rvert$		30	30
$\hookrightarrow$ Th		$15A$	$30A$
(k, m)	$\lvert C_{H_k}(g_{17km})\rvert$		
$(1, 1)$	15	1	1
$(4, 1)$	15	1	-1
m_{17j}		256	256

$\mathcal{F}_{18}$

g_{18}		$g_{18,1}$	$g_{18,2}$
$o(g_{18j})$		15	30
$\lvert C_{\overline{G}}(g_{18j})\rvert$		30	30
$\hookrightarrow$ Th		$15B$	$30B$
(k, m)	$\lvert C_{H_k}(g_{18km})\rvert$		
$(1, 1)$	15	1	1
$(4, 1)$	15	1	-1
m_{18j}		256	256

Using the Fischer matrices we can now compute the character table of $\overline{G}$. For example, the part $\mathcal{K}_{62}\mathcal{F}_{62}$ of the character table of $\overline{G}$ can be derived as follows: From Table 4, two conjugacy classes; namely, $h_{21} = g_{621}$ and $h_{31} = g_{622}$ of $H_2 = PSL(2,8){:}3$, fuse to class $[g_6]_{A_9} = 3C$. Let $\mathcal{K}_{62}$ be the fragment of the character table of H_2 on these two classes, which can be extracted from Table 2. The Fischer matrix $\mathcal{F}_6$ has two rows corresponding to the classes h_{21} and h_{31} of H_2. Let $\mathcal{F}_{62}$ be the partial Fischer matrix of $\mathcal{F}_6$ consisting of these two rows. Therefore the values of the irreducible characters of $\overline{G}$ (corresponding to H_2) on the classes $g_{61}, g_{62}, g_{63}, g_{64}$ and g_{65} are given by

$$\mathcal{K}_{62}\mathcal{F}_{62} = \begin{pmatrix} 1 & 1 \\ A & \overline{A} \\ \overline{A} & A \\ 1 & 1 \\ A & \overline{A} \\ \overline{A} & A \\ 2 & 2 \\ 2A & 2\overline{A} \\ 2\overline{A} & 2A \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 3 & 3 & -1 & 3 & -1 \\ 3 & 3 & -1 & -1 & 3 \end{pmatrix} = \begin{pmatrix} g_{61} & g_{62} & g_{63} & g_{64} & g_{65} \\ 6 & 6 & -2 & 2 & 2 \\ -3 & -3 & 1 & B & \overline{B} \\ -3 & -3 & 1 & \overline{B} & B \\ 6 & 6 & -2 & 2 & 2 \\ -3 & -3 & 1 & B & \overline{B} \\ -3 & -3 & 1 & \overline{B} & B \\ 12 & 12 & -4 & 4 & 4 \\ -6 & -6 & 2 & 2B & 2\overline{B} \\ -6 & -6 & 2 & 2\overline{B} & 2B \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where $A = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $B = -1 + 2i\sqrt{3}$. Similarly one can obtain all the 72 blocks $\mathcal{K}_{ik}\mathcal{F}_{ik}$, $1 \leq i \leq 18$, $1 \leq k \leq 4$ and hence the full character table of $\overline{G}$ is computed, which is supplied in the format of Clifford-Fischer Theory as Table 11.10 of [3].

Corollary 2. *The group $\overline{G}$ has 12 faithful irreducible characters.*

Proof. Note that $a_{11}^{(k,m)} = a_{12}^{(k,m)}$ for $1 \leq k \leq 3$, while $a_{11}^{(4,m)} = 16 \neq -16 = a_{12}^{(4,m)}$. Thus $\ker(\chi) \supset [g_{12}]_{\overline{G}}$, $\forall \chi \in \mathcal{K}_1,\ \mathcal{K}_2,\ \mathcal{K}_3$, where $\mathcal{K}_1,\ \mathcal{K}_2$ and $\mathcal{K}_3$ are the blocks of irreducible characters of $\overline{G}$ obtained through characters of the inertia factor groups $H_1,\ H_2$ and H_3, respectively. Hence if $\overline{G}$ has a faithful irreducible character χ, then $\chi \in \mathcal{K}_4 = \{\xi\beta_d |\ \beta_d \in \mathrm{IrrProj}(A_9, 2)\}$. Since the values of ξ are completely known (see Note 2) and since the character table of $2{\cdot}A_9$ is also known, we can see that for all $d \in \{1, 2, \cdots, 12\}$

$$\xi\beta_d(g_{ij}) \neq \deg(\xi\beta_d),\ \forall\ 1 \leq i \leq 18,\ 1 \leq j \leq c(g_i),\ (i,j) \neq (1,1).$$

Thus $\overline{G}$ has 12 faithful irreducible characters as claimed. □

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