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A note on Mid rules optimization of distance on the sphere

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ABSTRACT

In this article Mid-latitude and mid-longitude rules are used to find the free turning point along the Great Circle ensuing optimized distance. Namely, to reach Great Circle vertex with two rhumb line legs ensuing optimized distance, an initial rhumb line course equal to the orthodromic course at Mid-latitude may be used. The initial course is thereupon optimized by the incremental value steps. Alternatively, mid-longitude point at the same Great Circle can be beneficial in optimizing process. Optimized distance is achieved if the rhumb line course is altered towards the vertex at the orthodrome-loxodrome intersection point.

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1 Introduction

The orthodrome (Great Circle) is the intersection of the sphere with a plane containing the centre of the sphere. Finding the infinitesimal distance dD_0 that the vessel must cover in Great Circle sailing for orthodromic course to change by a small value dC_0 i.e. for the unit course alteration the unit distance is defined:

$$dD_0 = -dC_0 \sec\varphi_V \cot\varphi \cos\varphi,$$

where φ denotes geographic (geodetic) latitude whilst φ_V stands for vertex latitude.

Since the orthodrome is curved line whose true direction changes continually (except for a meridian or the equator), a number of points along the Great Circle are selected, connected by loxodromes and followed by rhumb line courses. By using the latitude equation of the mid-longitude an orthodrome (Great Circle) is split into the even number of legs with equal difference of longitude.

For the shortest overall distance of two piecewise rhumb-lines the essay [Han-Fei et al., 1991] inferred that there must exist a turn point along the initial rhumb line course prior to reaching vertex latitude (φ_V). The paper [Petrović, 2014] proved the existence of a turn point at the orthodrome-loxodrome intersection point. The initial rhumb line course equals to the orthodromic course at the Mid-latitude. The rule is mnemonic and straightforward for practical navigation. More about the intersection problem of two loxodromes or two geodesics can be found in the paper [Sjöberg, 2002].

In this article the Mid-latitude and mid-longitude rules are used to find the free turning point along the Great Circle ensuing optimized distance. The results prove that optimized distances have been reached i.e. required accuracy for marine navigation has been achieved.

2 Analysis

2.1 Finding orthodrome-loxodrome intersection on the sphere

In an oblique spherical triangle PTV (Figure 1) difference of longitude ($\Delta\lambda$) can be divided in an even number of equal parts i.e. $\Delta\lambda/2$ (named mid-longitude), $\Delta\lambda/4$ etc.

By applying the four parts equation of the spherical trigonometry the latitude equation of the mid-longitude for the Geodesic (Great Circle) on spherical surface is derived as follows [Petrović, 1990]:

$$\mathrm{ctg}\left(180^\circ - \alpha\right)\sin\frac{\Delta\lambda}{2} = \mathrm{tg}\varphi_T \cos\varphi_m - \sin\varphi_m \cos\frac{\Delta\lambda}{2}, \qquad (1)$$

$$\mathrm{ctg}\alpha \sin\frac{\Delta\lambda}{2} = \mathrm{tg}\varphi_V \cos\varphi_m - \sin\varphi_m \cos\frac{\Delta\lambda}{2}. \qquad (2)$$

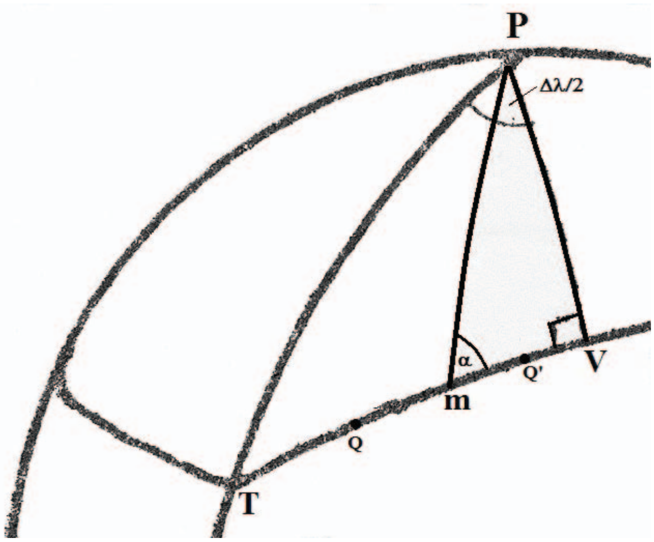

Figure 1 mid-longitude rule on a sphere

Adding the above two equations and rearranging yields:

$$\mathrm{tg}\varphi_m = \frac{\mathrm{tg}\varphi_T + \mathrm{tg}\varphi_V}{2\cos\frac{\Delta\lambda}{2}}, \quad (3)$$

The final formula (3) shows that the sum of the tangents of two latitudes divided by the double cosine of mid-longitude is equal to the tangent of latitude at mid-longitude. By connecting departure (*T*) and vertex (*V*) with loxodromes intersecting at mid-longitude (*m*) point, two step distance (*TmV*) is easily deduced.

If difference of longitude $\Delta\lambda/4$ is used, symmetric equations are derived for positions *Q* and *Q'*:

$$\mathrm{tg}\varphi_Q = \frac{\mathrm{tg}\varphi_T + \mathrm{tg}\varphi_m}{2\cos\frac{\Delta\lambda}{4}},$$

and

$$\mathrm{tg}\varphi_{Q'} = \frac{\mathrm{tg}\varphi_V + \mathrm{tg}\varphi_m}{2\cos\frac{\Delta\lambda}{4}}.$$

Introducing Mid-latitude (φ_{Mid}) at which the arc length of the parallel is equal to the *departure* in proceeding between two points, shown with the relation:

$$\varphi_{Mid} = \arccos\frac{DLat}{DMP}, \quad (4)$$

orthodromic course at Mid-latitude ($C_{O_{Mid}}$) equals to:

$$C_{O_{Mid}} = \arcsin\frac{\cos\varphi_V}{\cos\varphi_{Mid}}, \quad (5)$$

where *DLat* is the *Difference of Latitude*, and *DMP* stands for the *Difference of Meridional Parts for the Sphere.*

Taking $C_{O_{Mid}}$ as an initial rhumb line course (Θ_T) from the point of departure (*T*) leads to the intersection (*I*) of a loxodrome with an orthodrome (Great Circle) as shown on Figure 2.

The transcendental differentiable function $f(\lambda_I)$ defined on the interval $[\lambda_T, \lambda_V]$ expressed with the equation [Petrović, 2014]:

$$f(\lambda_I) = \sinh\left[\cot\Theta_T(\lambda_I \sim \lambda_0)\right] - \tan\left|\varphi_V\right| \cos(\lambda_V \sim \lambda_I), \quad (6)$$

(λ_0 – equatorial intersection longitude of the loxodrome) form a base for an iterative solution of intersection using *Newton-Raphson* method. If $\varphi_T = 0 \Longrightarrow \lambda_T = \lambda_0$. In order to use the above formulas for all quadrants, the difference of longitude sign (~) represents the shorter arc of the equator between the two meridians. Vertex latitude (φ_V) is taken as an absolute value whilst the rhumb line course ($0 < \Theta_T < \pi/2$). Thus, determining the zero (root) of $f(\lambda_I)$ crossing is obtained. As an initial estimate (guess) for the longitude of intersection (λ_I), geographic vertex longitude (λ_V) which satisfies the condition $f(\lambda_V) \cdot f''(\lambda_V) > 0$ may be used.

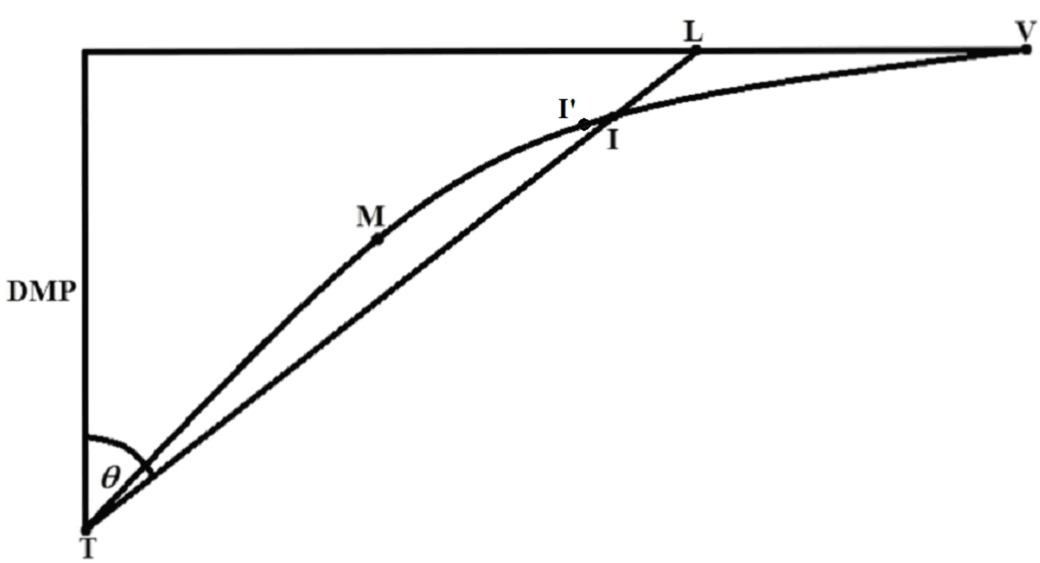

Figure 2 Mid-latitude rule on a Mercator chart

Table 1 Mid-latitude vs. mid-longitude (Sphere)

φ_T/φ_V	λ_0/λ_V	φ_I/λ_I	θ_T/θ_I	D_{TLV}/D_{TIV}	D_{TmV}
5°/25°	-14°17.6'/79°11.2'	-22°25.3'/51°25.1'	70.7°/84.2°	4702.7'/4695.1'	4695.0'
15°/35°	-33°24.8'/67°30.0'	-32°11.2'/41°30.7'	65.6°/82.6°	3813.8'/3803.4'	3802.7'
25°/45°	-46°02.0'/62°12.3'	-42°05.5'/36°48.0'	60.7°/81.0°	3226.4'/3213.7'	3212.6'
35°/55°	-54°22.5'/60°38.4'	-52°03.9'/34°34.8'	55.5°/79.3°	2767.0'/2752.1'	2750.8'
45°/65°	-58°28.0'/62°12.3'	-62°05.6'/33°54.3'	49.2°/77.0°	2361.7'/2344.6'	2343.5'

Table 2 Distance optimized with the mid-latitude rule (Sphere)

φ_T/φ_V	λ_0/λ_V	$\varphi_{I'}/\lambda_{I'}$	$\theta'_T/\theta_{I'}$	$D_{TI'V}$	D_{GC}
5°/25°	-13°36.4'/79°11.2'	-21°08.2'/45°11.4'	69.8°/83.0°	4694.39'	4685.9'
15°/35°	-31°40.3'/67°30.0'	-30°52.4'/36°08.0'	64.4°/81.1°	3802.48'	3790.6'
25°/45°	-43°30.5'/62°12.3'	-40°48.0'/31°52.7'	59.3°/79.3°	3212.54'	3197.8'
35°/55°	-51°17.7'/60°38.4'	-50°50.1'/29°54.9'	53.9°/77.3°	2750.82'	2733.4'
45°/65°	-55°06.6'/62°12.3'	-60°59.0'/29°24.8'	47.5°/74.9°	2343.34'	2323.2'

2.2 Computation algorithm for optimized distance

In the examples presented in Table 1, Greenwich meridian is taken as a longitude of departure ($\lambda_T = 0$), whilst θ_I represents a rhumb line course connecting intersection point (I) with vertex (V). Distance (D_{TLV}) is based on rhumb line (θ_T)sailing to vertex latitude then due east (or west) along parallel to vertex [Han-Fei et al., 1991]. Shorter distance (D_{TIV}) is obtained if course is altered to rhumb line (θ_I) at I, then proceeding towards vertex [Petrović, 2014].

The free turning point (I') along the Great Circle ensuing optimized distance can be reached using longitude of I or m as an initial value in formula (6). For optimization purpose the initial rhumb line course (θ_T) is then altered by an incremental value of ±0.1° (converted in radians) and inserted into formula (6) as θ'_T. After few iterative steps (*Newton-Raphson* method) θ'_T reaches free turning point I' on the same great circle (Fig. 2). Optimized distance ($D_{TI'V}$) is obtained if course is altered to rhumb line ($\theta_{I'}$) at I', then proceeding towards vertex as shown in Table 2. Great Circle Distance D_{GC} serves as a reference value.

Whilst comparing results it can be noted that the two-step rhumb line distance (D_{TmV}) (Table 1) based on mid-longitude method is very close to the optimized distance ($D_{TI'V}$) (Table 2).

The above concept may be used on the rotational ellipsoid (spheroid) but the flow of geodesics on the ellipsoid of revolution (spheroid) differs from the geodesic on the sphere and it is not a great ellipse. As the Geodesic (line) on the spheroid is defined by the differential equations, finding its vertex longitude and intersection with a loxodrome requires a different mathematical model.

3 Conclusion

To reach Great Circle vertex with two rhumb line legs ensuing optimized distance, the initial rhumbline course (θ_T) equal to the orthodromic course at the Mid-latitude ($C_{0_{Mid}}$) may be used. Alternatively, mid-longitude (m) point at the same Great Circle can give initial rhumbline course (θ_T) as well. The initial course is thereupon optimized by the incremental value steps. The method finds the free turn point (I') from where a rhumbline course ($\theta_{I'}$) is followed towards vertex which gives an optimized overall distance ($D_{TI'V}$). Moreover, results show that the distance saving obtained by optimization is not substantial i.e. the Mid-latitude (D_{TIV}) and mid-longitude (D_{TmV}) rules hold good for practical marine navigation.

References

[1] Petrović, M. (2014). Orthodrome-Loxodrome Correlation by the Middle Latitude Rule. *The Journal of Navigation*, 67, 539–543.

[2] Han-Fei Lu, Hsin-Hsiung Fang and Chung-Hsiung Chiang. (1991). Trans-oceanic Passages by Rhumbline Sailing. *The Journal of Navigation*, 44, 423–428.

[3] L. E. Sjöberg. (2002). Intersections on the sphere and ellipsoid. *Journal of Geodesy*, 76, 115–120.

[4] Petrović, M. (1990). Orthodrome. Graduation thesis at the College of Maritime Studies, Dubrovnik, Croatia.