Brodogradnja/Shipbilding Volume 66 Number 3, 2015

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ISSN 0007-215X
eISSN 1845-5859

ANALYTICAL SOLUTION OF BASIC SHIP HYDROSTATICS INTEGRALS USING POLYNOMIAL RADIAL BASIS FUNCTIONS

UDC 629.5.015.1
Original scientific paper

Summary

One of the main tasks of ship's computational geometry is calculation of basic integrals of ship's hydrostatics. In order to enable direct computation of those integrals it is necessary to describe geometry using analytical methods, like description using radial basis functions (RBF) with L_1 norm. Moreover, using the composition of cubic and linear Polynomial radial basis functions, it is possible to give analytical solution of general global 2D description of ship geometry with discontinuities in the form of polynomials, thus enabling direct calculation of basic integrals of ship hydrostatics.

Key words: *analytical, solution, integrals, intersection, ship, hydrostatics, polynomial, RBF, L_1 norm*

1 Introduction

From the beginning of naval architecture as science, in the works of its founders Chapman, [1] and Euler, [2] and others, there have been efforts for analytical description of ship geometry using polynomials in order to enable direct, analytical solution of ship's computational geometry problems, like direct solving of intersection problem or direct solving of basic hydrostatic integrals. At the beginning of 20th century, Taylor, [3], described ship's hydrostatic and geometric particulars approximating them with polynomials, thus showing the possibility of their usage in ship computational geometry calculations. Nevertheless, his method did not solve all computational geometry problems like solving belonging geometric and hydrostatic integrals or intersection problem, with possible multiple roots over single ordered pair of function domain values or not finding solutions for nearly parallel lines. That has not been achieved until the advances in the computer technology in recent decades that results in the development of meshless methods based on piecewise radial basis functions.

In dissertation, [4], and paper, [5], it is shown that there exist global solution of description problem of 2D ship geometry with discontinuities using composition of cubic and linear Polynomial radial basis functions with dense description of discontinuities. Except very high accuracy, this solution gives possibility of direct solution of basic integrals of ship hydrostatics expanded for the integral for the determination of wetted area of immersed part of the ship's hull, which is much faster and more accurate than numerical integration methods.

In order to show the efficiency of the global 2D radial basis functions (RBFs) description method using Polynomial RBFs as the direct solution of basic ship hydrostatic integrals, two test frames are chosen: test frame of car-truck carrier with flat side and camber, and test frame in the shape of semicircle. Former is frame section with discontinuities which is complex from the ship's computational geometry point of view, and latter is theoretical frame section that enables analytical checking of integration results. Those test frames will enable the testing of novel calculation method using Polynomial RBFs, in the calculation of hydrostatic particulars for actual ship frame with discontinuities, and for the calculation of hydrostatic particulars of theoretical frame section in the shape of semicircle.

2 Calculation of ship hydrostatics particulars

There are five basic integrals defined for the calculation of ship hydrostatic particulars, with the assumption of using 2D description methods. Those integrals are all univariant definite integrals whose upper limit depends on the actual waterline position, defined for one of the coordinate axis of the ship coordinate system, Figure 1.

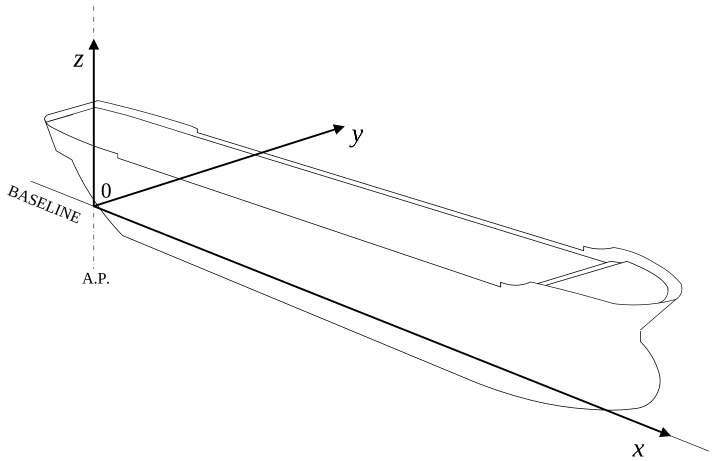

Figure 1 Ship coordinate system with origin set to aft perpendicular and baseline

According to the hydrostatic values to be calculated, they can be divided to integrals for the calculation of actual waterline particulars and actual displacement properties.

The ship waterline particulars and their belonging integrals to be determined are:

– Waterplane area A_{WL} ,

– The centroid of waterplane area $X_{WL} \equiv \{x_{WL}, y_{WL}\}$, and

– The moment of inertia of waterplane $I_{WL} \equiv \{I_L, I_B\}$.

The displacement particulars and their belonging integrals are:

– The volume of displacement ∇, and

– The centre of buoyancy $X_B \equiv \{x_B, y_B, z_B\}$.

After obtaining the results of frame section areas and belonging static moments around *y* axis, it is necessary to integrate along remaining *x* axis in order to obtain ship displacement particulars values.

Respective integrals for the calculation of Bonjean curves correspond to the waterplane area integrals with:

- Section area *A*, and
- Section area moment around *y* axis, M_y.

In order to calculate the centre of waterline area, it is necessary to determine belonging area moments as:

$$M_{WL} = A_{WL} \cdot \mathrm{X}_{WL}, \; M_{WL} \equiv \{M_{WL,x}, M_{WL,y}\}$$

Respective displacement volume moments for the calculation of centre of buoyancy are:

$$M_B = \nabla \cdot \mathrm{X}_B, M_B \equiv \{M_{B,x}, M_{B,y}, M_{B,z}\}$$

In the case of 3D description of ship geometry, the formulae for the calculation of volume displacement ∇ for some predefined waterline, and known geometry description $y = f(x,z)$ using double integral definition, is:

$$\nabla = \int_{x_1}^{x_2} \int_{z_1(x)}^{z_2(x)} f(x,z) \cdot dx \cdot dz$$

Where $x_1, x_2, z_1(x), z_2(x)$ are integration limits defined by the intersection of actual waterline with ship geometry and ship bottom line.

Accordingly, belonging moment M_B can be also determined using double integral as:

$$M_B = \int_{x_1}^{x_2} \int_{z_1(x)}^{z_2(x)} f_M(x,z) \cdot dx \cdot dz$$

Nevertheless, the 3D RBF description of ship geometry is not available yet, in the form suitable for direct integration of ship hydrostatic particulars. Therefore, only 2D Polynomial RBF direct integration methods will be shown in this paper.

Except above mentioned, the surface area of the immersed ship's hull can be also included in basic hydrostatic particulars, with the calculation of:

- Wetted area S_W.

When 2D calculation methods are used, it is first necessary to calculate:

- Frame section curve lengths L_W.

After that, it is necessary to integrate along remaining *x* axis.

In order to solve above stated integrals it is necessary to determine their limits first, by calculating the intersection of some arbitrary waterline with ship geometry, both described by computationally compatible RBF methods that will be shown in the next paragraph.

2.1 Five basic integrals of ship hydrostatics

There are five basic integrals to be solved for the determination of ship's hydrostatics particulars. When defined for general coordinate x and function description $f(x)$, those definite integrals are:

1. $$\int_{x_1}^{x_2} f(x)dx \tag{1}$$

2. $$\int_{x_1}^{x_2} xf(x)dx \tag{2}$$

3. $$\int_{x_1}^{x_2} [f(x)]^2 dx \tag{3}$$

4. $$\int_{x_1}^{x_2} [f(x)]^3 dx \tag{4}$$

5. $$\int_{x_1}^{x_2} x^2 f(x)dx \tag{5}$$

The limits of above integrals are defined for the keel of the ship, x_1, and the intersection of frame section curve with actual ship waterline, x_2.

The descriptions of the ship geometry in general case, contain translated points by elastic shift method and can be rotated for some angle φ, [1]. Therefore, general definitions of above integrals contain translation corrections and rotation terms, as will be shown in further papers.

2.2 Curve length integral

The curve length integral for 2D calculation of wetted area of the ship is:

$$\int_C \sqrt{d^2x + d^2y} = \int_x \sqrt{1 + y'^2}\,dx$$

Using function description $f(x)$:

$$\int_{x_1}^{x_2} \sqrt{1 + [f'(x)]^2}\,dx$$

More generally, basic integral to be solve is:

6. $$\int_{x_1}^{x_2} \sqrt{1 + g(x)}\,dx \tag{6}$$

3 Polynomial Radial Basis Functions

3.1 RBF Definition

The main reason for using analytical curve description using functions is to enable direct 2D and 3D integration, as well as solving intersection problem. The possibility of solving integrals for belonging functional curve description depends on computational characteristics of function chosen. In the case of radial basis functions, except their basis type, there is additional problem of their definition with norm as function argument.

Radial basis function networks are generally defined as the linear combination of basis functions, which depend on L_2 norm, i.e. the distance $\|x - t_i\|$ between input data set points, **x**, and the points of centres, **t**, around which the function is developed. Therefore, RBFs as direct feed-forward neural networks, Figure 2, can be represented in the form:

$$\mathbf{y} = f(\mathbf{x}) = \sum_{i=1}^{O} w_i \Phi_i(\|x, t_i\|), \mathbf{x} \in \Omega \subseteq IR^d, \mathbf{y} \in IR^l$$

where Φ_i is radial basis function, **x** is input variables data points set, **y** is output variables data points set, **t** is the set of O centres radial basis functions are developed for, **w** is the matrix of the weight coefficients, d is the dimension of input data set, and l is the dimension of output data set.

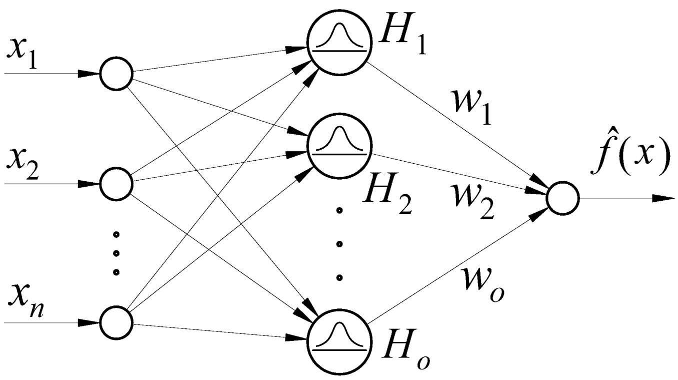

Figure 2 Single-layered, Feed-forward RBF Neural Network

The main advantage of RBFs, and the reason why they are widely used, is that they are the solution of scattered data interpolation problem, where the number of centres equal the number of input points with $O = N$, i.e. **t = x**. The solution of above interpolation problem can be obtained by determination of weight coefficient vector/matrix **w**, using inversion of interpolating matrix **H** as:

$$\mathbf{w} = \mathbf{H}^{-1} \cdot \mathbf{y} \tag{7}$$

The problem of the curve or some surface reconstruction is the problem of the determination of the function f for curve description, based on the input data set $X \equiv \{x_j, j = 1,\ldots, N\}$ and output data set points $Y \equiv \{y_j, j = 1,\ldots, N\}$, on the range $[a, b]$. Overall data set is then divided in two sets, one data set X for RBF calculation and one set X_G for generalization of the description, with $X_G \equiv \{x_j, j = 1,\ldots, N_G\}$.

3.2 Polynomial RBF definition

Although widely used, none of standard RBFs defined with L_2 norm used for 3D ship's geometry description are not twice integrable for their argument $||x - x_i||$. Besides, it is not possible to solve intersection problem directly for 3D case using L_2 norm, [4]. That is one of the reasons for choosing 2D ship geometry description using Polynomial RBFs with L_1 norm and argument $|x - x_i|$.

Polynomial radial basis functions for 2D problems can be generally defined as:

$$f(x) = \sum_{j=1}^{N} w_i |x - t_i|^{\beta} + c, \ \beta \in IR \setminus 2 \cdot IN \tag{8}$$

With shape parameter c set outside weighted sum, $c \in IR$, and function exponent β defined in the whole space real numbers IR restricted for even integers.

Moreover, it is shown in [4] and [5] that it is possible to solve general 2D description problem of ship geometry with discontinuities using composition of cubic and linear Polynomial RBFs with L_1 norm and dense points around discontinuities.

Thus Polynomial RBFs with L_1 norm have simple form with odd integer exponents that enable direct integral solutions.

3.3 Polynomial RBFs in general polynomial form

Above mentioned Polynomial RBFs (PRBF) can be defined for general polynomial basis with polynomial coefficients as:

$$f(x) = \sum_{i=1}^{n} C_i x^i \tag{9}$$

where $C_i, i = 1, \ldots, n$ are polynomial coefficients.

Regarding polynomial degree n of developed Polynomial RBF, it is equal to the function exponent β, i.e.:

$$n = \beta \tag{10}$$

Due to limitations of contemporary mathematics, the polynomial degree n is limited to direct univariate polynomial solutions with $n \leq 6$, as will be described in chapter 5 of this paper.

The integration of polynomials is the simplest possible, and the integration of cubic Polynomial radial basis functions will be shown in this paper. Moreover, the solutions of polynomial roots exist also, for some polynomial degrees as will be described later in the text.

4 Precision of the description methods

Except integrability and solvability of the basis function, it is necessary to ensure high precision of the ship geometry description, in order to enable integration procedure.

Global and local measures of the accuracy of the description are defined as RMSE and Err_{max}, for chosen number of input points N. Global error is defined as:

$$\text{RMSE} = \sqrt{\frac{\sum_{i=1}^{N}(f(x_i)-y_i)^2}{N}}$$

Where N is the number of input data set y_i, i = 1, …, N is the output data set of points, and $f(x)$ is radial basis function.

Corresponding local error is defined as:

$$\text{Err}_{\max} = \max(\text{Err}_i), \text{Err}_i = |y_i - f(x_i)|, i = 1, \ldots, N_G$$

Where N_G is the number of the points used for generalization of the description.

The problem with precision usually occurs for the integration of the interpolating functions with small global precision of the description, where the effect of error grouping in a point occur, shown on Figure 3 for calculation of vertical centre of buoyancy z_B, below, with low Root Mean Square Error (RMSE) value.

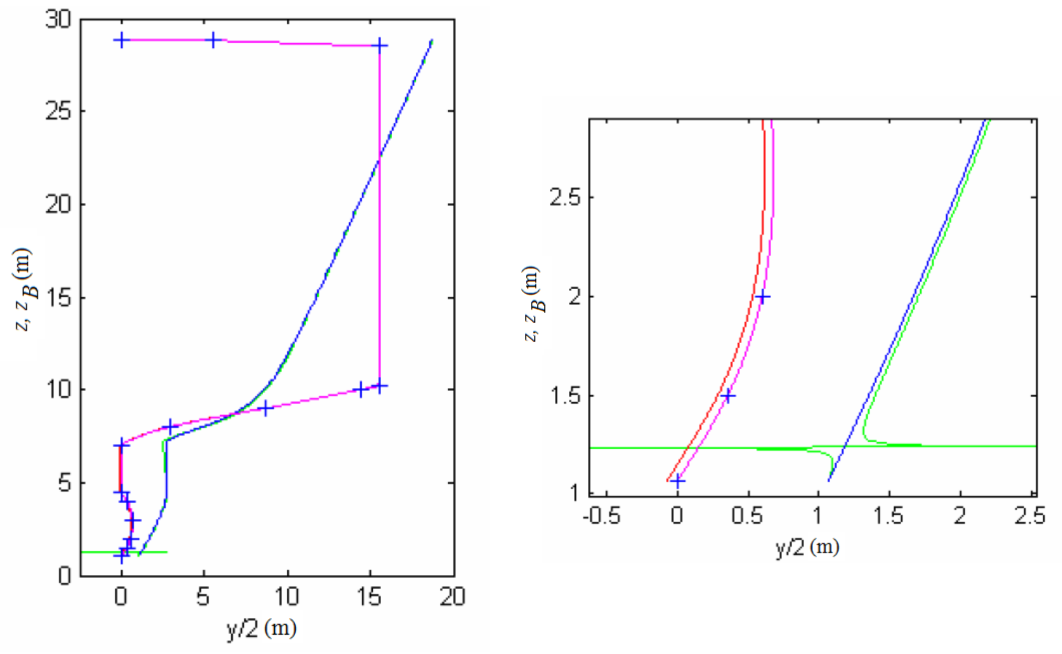

Figure 3 Error grouping in a point with low precision integration of RMSE = $4.16 \cdot 10^{-2}$

This problem is one of the main reasons why approximation methods, like decomposition methods, fast computing methods and the development into series methods, or approximation methods in general, are not suitable in computational geometry for the calculation of ship hydrostatics particulars using 2D methods.

Opposite to above, Polynomial radial basis functions ensure high precision of the description, even for frame sections with discontinuities. For example, global accuracy of the description for Test frame No. 1 using composition of Polynomial RBFs with β = {3, 1} is RMSE = $3.5 \cdot 10^{-10}$, while local accuracy is $\text{Err}_{\max}$ = $1.168 \cdot 10^{-6}$ (m), as shown on Figure 4.

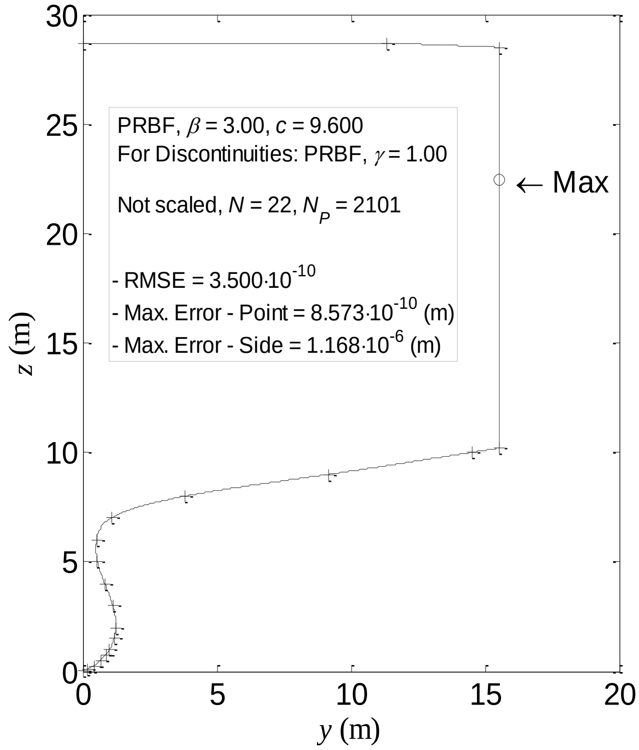

Figure 4 The description of Test Frame No. 1 with camber and knuckles, using composition of PRBF functions with $\beta = \{3, 1\}$

It is obvious that the accuracy of the description of Test frame No. 1 using composition of Polynomial RBFs with $\beta = \{3, 1\}$ is very high, much higher than required 10^{-4} (m).

5 Solution of the intersection problem

One of basic computational geometry tasks, together with solving basic hydrostatic integrals, is solving the problem of ship geometry intersection with actual waterline. In order to enable direct solution of this problem, both ship geometry and surrounding waterplane should be described in the same manner, analytically. The method that ensures such solution is polynomial description of ship geometry with corresponding waterplane, using Polynomial RBFs with L_1 norm.

It is known form the theory of polynomials that direct solution of roots of general polynomials are available for degrees lower than seven, only, [6]. Also, their direct solutions always exist for polynomials with exponents lower and including four, and those solutions can be obtained by basic arithmetic operations, adding, subtracting, multiplication and division, according to Abel's theorem, [7]. The roots of the polynomials for general quintic and sextic cannot always be obtained, and their available solutions are given using hypergeometric functions. The solution of septic equation is not always available too, and when it is, it can be solved using Galois groups, [6], that require hyperelliptic functions for their solution, or by superimposing continuous functions of two variables.

Above degree restrictions of directly solvable polynomials is the reason for the limitation of the polynomial degree for ship geometry description, and therefore it is set to:

$$n \leq 4 \tag{11}$$

Another restriction for polynomial degree comes from the definition of polynomial RBF where only odd integer values are allowed:

$$\beta = 1, 3, 5, \dots \tag{12}$$

In order to produce smooth curve, Polynomial RBFs must be at least twice integrable, thus giving yet another restriction with:

$$\beta \geq 3 \tag{13}$$

As mentioned before, belonging polynomial degree n of developed Polynomial RBF is equal to the function exponent β. It can be concluded from above restrictions that the only Polynomial RBF exponent β that satisfies all three requirements is:

$$\beta = 3 \tag{14}$$

As stated before, the solution of the global description of ship geometry with discontinuities is given using cubic-linear Polynomial RBFs, thus fulfilling above exponent conditions, and enabling direct solution of intersection problem. After writing it in the form with polynomial coefficients (9), cubic polynomial can be obtained with direct root solutions are always available.

$$C_3 \cdot x^3 + C_2 \cdot x^2 + C_1 \cdot x + C_0 = 0,\ C_3 \neq 0 \tag{15}$$

The roots of above cubic polynomial (15) are found in 16^{th} century by Nicolo Tartaglia, as published by Gerolamo Cardano, [8].

General cubic with real coefficients (15) always has at least one real solution thus fulfilling the requirement for the existence of the solution. The solution of cubic polynomials equation is well known, and more detailed description of the cubic roots determination can be found in [4].

6 Area integral

6.1 General

General area integral for some ship section area calculation can be written as a plane curve integral along some coordinate axis, as given for basic definite integral (1). The section area of a frame using integration along z axis is than:

$$A = \int_{z_1}^{z_2} y \cdot dz \tag{16}$$

Where z_1 and z_2 are the limits of definite integral.

In the case of the functional description of the ship's frame section curve with $y = f(z)$ we have:

$$A = \int_{z_1}^{z_2} f(z) \cdot dz \tag{17}$$

If we choose the point on the ship keel z_0 as the lower integral limit, and the point of intersection with waterline z_{WL} as upper limit of the integral limit, we obtain:

$$A = \int_{z_0}^{z_{WL}} y \cdot dz \tag{18}$$

Regarding Polynomial RBFs chosen, there are two integration procedures that can be done: by direct integration, or by transformation of radial basis into polynomial basis, and those two ways will be chosen in the text below.

6.2 Direct Polynomial RBF integration

6.2.1 Integration of compatible cubic-linear PRBF

General form of cubic-linear compatible PRBF for z axis with $\beta = \{3, 1\}$ is:

$$y = \sum_{\substack{i \\ \beta=3}} w_i \left(\left| z - z_i \right|^3 + c_1^3 \right) + \sum_{\substack{i \\ \beta=1}} w_i \left(\left| z - z_i \right| + c_2 \right)$$

Belonging integral can be written as:

$$A = \int_{z_1}^{z_2} \left[\sum_{\substack{i \\ \beta=3}} w_i \left(\left| z - z_i \right|^3 + c_1^3 \right) + \sum_{\substack{i \\ \beta=1}} w_i \left(\left| z - z_i \right| + c_2 \right) \right] dz \tag{19}$$

The direct solution of above integral is:

$$A = \left[\sum_{\substack{i \\ \beta=3}} w_i \left(\frac{\left| z - z_i \right|^4}{4} + c_1^3 z \right) + \sum_{\substack{i \\ \beta=1}} w_i \left(\frac{\left| z - z_i \right|^2}{2} + c_2 \cdot z \right) \right] \Bigg|_{z_1}^{z_2} \tag{20}$$

6.2.2 Integration of Polynomial RBFs with general polynomial basis

Since Polynomial RBFs can be written with polynomial basis, the area integral can be generally written as:

$$A = \int_{z_1}^{z_2} \left(\sum_{k=0}^{n} C_k \cdot z^k \right) \cdot dz \tag{21}$$

Belonging area integral can be than written as:

$$A = \left(\sum_{k=0}^{n} \frac{1}{k+1} \cdot C_k \cdot z^{k+1} \right) \Bigg|_{z_1}^{z_2} \tag{22}$$

Regarding solution is than:

$$A = \sum_{k=0}^{n} \frac{1}{k+1} \cdot C_k \cdot \left(z_2^{k+1} - z_1^{k+1} \right) \tag{23}$$

7 Calculation of the static section area moment for y axis

7.1 General integral form

General integral for the calculation of ship's static section area moment for *y* axis in (2) can be rewritten as:

$$M_y = \int_{z_1}^{z_2} z \cdot y \cdot dz \tag{24}$$

For $y = f(z)$ we obtain integral:

$$M_y = \int_{z_1}^{z_2} z \cdot f(z) \cdot dz \tag{25}$$

The solutions of above integral will be shown below.

7.2 Integral of compatible cubic-linear Polynomial RBF for y axis

The integral of static section area moment for *y* axis in (2), using cubic-linear PRBFs, can be written as:

$$M_y = \int_{z_1}^{z_2} z \left[\sum_{\substack{i \\ \beta=3}} w_i \left(\left| z - z_i \right|^3 + c_1^3 \right) + \sum_{\substack{i \\ \beta=1}} w_i \left(\left| z - z_i \right| + c_2 \right) \right] dz \tag{26}$$

Above integral can be further separated in two integrals as:

$$M_y = \int_{z_1}^{z_2} \sum_{\substack{i \\ \beta=3}} w_i \left(\left| z - z_i \right|^3 + c_1^3 \right) \cdot z \cdot dz + \int_{z_1}^{z_2} \sum_{\substack{i \\ \beta=1}} w_i \left(\left| z - z_i \right| + c_2 \right) \cdot z \cdot dz$$

In order to solve the first integral in above, it is necessary to develop it as:

$$z \cdot \left| z - z_i \right|^3 = z \cdot |z|^3 - 3 \cdot z \cdot |z|^2 |z_i| + 3 \cdot z \cdot |z| |z_i|^2 - z \cdot |z_i|^3$$

$$\int_{z_1}^{z_2} z \cdot \left[(z - z_i)^3 + c_1^3 \right] dz = \int_{z_1}^{z_2} \left(z^4 - 3 \cdot z^3 |z_i| + 3 \cdot z^2 \cdot |z_i|^2 + z \cdot \left(c_1^3 - |z_i|^3 \right) \right) dz =$$

$$= \left[\frac{z^5}{5} - 3 \cdot \frac{z^4}{4} \cdot |z_i| + 3 \cdot \frac{z^3}{3} |z_i|^2 + \frac{z^2}{2} \cdot \left(c_1^3 - |z_i|^3 \right) \right] \Bigg|_{z_1}^{z_2}$$

The same has to be done with the second integral as:

$$z \cdot \left| z - z_i \right| = z \cdot |z| - z |z_i|$$

$$\int_{z_1}^{z_2} z \cdot (z - z_i + c_2) \cdot dz = \int_{z_1}^{z_2} \left[z^2 + z \cdot \left(c_2 - |z_i| \right) \right] \cdot dz = \left[\frac{z^3}{3} + \frac{z^2}{2} \left(c_2 - |z_i| \right) \right] \Bigg|_{z_1}^{z_2}$$

Final statement for calculation of static section area moment for *y* axis is:

$$M_y = \sum_{\substack{i \\ \beta=3}} w_i \left[\frac{z^5}{5} - 3\frac{z^4}{4}|z_i| + 3\frac{z^3}{3}|z_i|^2 + \frac{z^2}{2}\left(c_1^3 - |z_i|^3\right) \right]\Bigg|_{z_1}^{z_2} + \sum_{\substack{i \\ \beta=1}} w_i \left[\frac{z^3}{3} + \frac{z^2}{2}\left(c_2 - |z_i|\right) \right]\Bigg|_{z_1}^{z_2} \tag{27}$$

Or we can write differently as

$$z \cdot |z - z_i|^3 = (z - z_i) \cdot |z - z_i|^3 + z_i \cdot |z - z_i|^3$$

And

$$z \cdot |z - z_i| = (z - z_i) \cdot |z - z_i| + z_i \cdot |z - z_i|$$

Final statement for direct calculation of static section area moment for y axis is than:

$$M_y = \sum_{\substack{i \\ \beta=3}} w_i \left[\frac{(z - z_i)^5}{5} + \frac{(z - z_i)^4}{4} z_i + c_1^3 \frac{z^2}{2} \right]\Bigg|_{z_1}^{z_2} + \sum_{\substack{i \\ \beta=1}} w_i \left[\frac{(z - z_i)^3}{3} + \frac{(z - z_i)^2}{2} z_i + c_2 \frac{z^2}{2} \right]\Bigg|_{z_1}^{z_2} \tag{28}$$

7.3 Integral of compatible cubic-linear PRBF for y axis written for polynomial basis

The equation for static section area moment calculation using PRBFs with polynomial basis can be generally written as:

$$M_y = \int_{z_1}^{z_2} z \left(\sum_{k=0}^{n} C_k \cdot z^k \right) \cdot dz \tag{29}$$

Belonging solution of moment integral can be than written as:

$$M_y = \left(\sum_{k=0}^{n} \frac{1}{k+2} \cdot C_k \cdot z^{k+2} \right)\Bigg|_{z_1}^{z_2}$$

After including limit values we obtain:

$$M_y = \sum_{k=0}^{n} \frac{1}{k+2} \cdot C_k \cdot \left(z_2^{k+2} - z_1^{k+2} \right) \tag{30}$$

8 Calculation of the static section area moment for z axis

8.1 General integral

The integral for the calculation of static section area moment for z axis in (3) can be rewritten as:

$$M_z = \frac{1}{2} \int_{z_1}^{z_2} y^2 dz \tag{31}$$

If we insert explicit curve definition $y = f(z)$ we obtain integral:

$$M_z = \frac{1}{2}\int_{z_1}^{z_2} f^2(z)\cdot dz \tag{32}$$

This integral has form of basic integral number 3, as defined in chapter 2.

Similarly to the calculation of static area moment for y axis, direct integration of above equation with radial basis functions is very complex even for Polynomial RBFs, and therefore will not be done. In order to solve integral (32) above, Polynomial RBFs will be written in the form of algebraic polynomials, instead. Additionally, it will be shown that another efficient solution of above integral can be obtained by using the solution of interpolation problem for y^2 .

8.2 Integral of static area moment for z axis using compatible cubic-linear PRBF

The integral of static section area moment for z axis using cubic-linear PRBF can be written as:

$$M_z = \frac{1}{2}\int_{z_1}^{z_2}\left[\sum_{\substack{i\\ \beta=3}} w_i\left(|z-z_i|^3 + c_1^3\right) + \sum_{\substack{i\\ \beta=1}} w_i\left(|z-z_i| + c_2\right)\right]^2 dz \tag{33}$$

This integral can be divided in three integrals as:

$$M_y = \frac{1}{2}\int_{z_1}^{z_2}\left[\sum_{\substack{i\\ \beta=3}} w_i\left(|z-z_i|^3 + c_1^3\right)\right]^2 \cdot dz + \int_{z_1}^{z_2}\left[\sum_{\substack{i\\ \beta=3}} w_i\left(|z-z_i|^3 + c_1^3\right)\right]\cdot\left[\sum_{\substack{i\\ \beta=1}} w_i\left(|z-z_i| + c_2\right)\right]\cdot dz +$$
$$+\frac{1}{2}\int_{z_1}^{z_2}\left[\sum_{\substack{i\\ \beta=1}} w_i\left(|z-z_i| + c_2\right)\right]^2 \cdot dz$$

This equation with RBF description contains translated weighted sums with basis functions and therefore has multiple integral terms non suitable for direct calculation. More suitable polynomial form will be used instead, as will be shown below.

8.3 Static area moment integral for *z* axis using PRBFs in polynomial form

When radial basis function can be developed in polynomial form, static area moment integral for z axis can be generally written as:

$$M_z = \frac{1}{2}\int_{z_1}^{z_2}\left(\sum_{k=0}^{n} C_k \cdot z^k\right)^2 \cdot dz \tag{34}$$

By squaring we get:

$$M_z = \frac{1}{2}\int_{z_1}^{z_2}\sum_{k=0}^{2n} D_k \cdot z^k \cdot dz \tag{35}$$

where $D_k = f(C_k)$ are polynomial coefficients.

In the case of cubic-linear Polynomial RBFs, belonging solution of the moment integral can be written as:

$$M_z = \frac{1}{2}\int_{z_1}^{z_2}\left(C_3 \cdot z^3 + C_2 \cdot z^2 + C_1 \cdot z + C_0\right)^2 \cdot dz \quad (36)$$

By squaring we get:

$$M_z = \frac{1}{2}\int_{z_1}^{z_2}\left(D_6 \cdot z^6 + D_5 \cdot z^5 + D_4 \cdot z^4 + D_3 \cdot z^3 + D_2 \cdot z^2 + D_1 \cdot z + D_0\right) \cdot dz$$

The solution can be then written as

$$M_z = \frac{1}{2}\left(\frac{D_6}{7} \cdot z^7 + \frac{D_5}{6} \cdot z^6 + \frac{D_4}{5} \cdot z^5 + \frac{D_3}{4} \cdot z^4 + \frac{D_2}{3} \cdot z^3 + \frac{D_1}{2} \cdot z + D_0 z\right)\Bigg|_{z_1}^{z_2}$$

Or

$$M_z = \frac{1}{2}\sum_{k=0}^{2n}\frac{1}{k+1}D_k z^{k+1}\Bigg|_{z_1}^{z_2} = \frac{1}{2}\sum_{k=0}^{2n}\frac{1}{k+1}D_k\left(z_2^{k+1} - z_1^{k+1}\right) \quad (37)$$

8.4 Direct integration using y^2

Instead of direct solving of basic integral for M_z by developing complex RBF form with norms, that integral can be solved by determining weight coefficients w_2 for curve equation $y = f(z)$ squared, i.e. for y^2. Therefore, we are introducing the variable $q = y^2$ and obtaining the integral with the same form as the area integral 1, i.e.:

$$M_z = \frac{1}{2}\int_{z_1}^{z_2} q dz \quad (38)$$

By inserting $q = f_2(z)$ instead of y^2 following integral is obtained then

$$M_z = \frac{1}{2}\int_{z_1}^{z_2} f_2(z) \cdot dz \quad (39)$$

with the same form as the area integral (1).

Nevertheless, the function f_2 has different weight coefficients w_2 than interpolation function f, with coefficients w. It is therefore necessary to define separate interpolation problem for y^2 that will be shown below.

8.4.1 Interpolation problem for y^2

Instead of basic interpolation problem $y = H \cdot w$, we are defining the problem:

$$y^2 = f_2(z) = H_2 \cdot w_2 \quad (40)$$

Where H_2 is interpolating matrix for input data set, and w_2 is weight coefficients vector.

Similar to basic scattered data interpolation problem, the solution of this problem can be obtained by the inversion of interpolating matrix H_2, and according to (7) we have:

$$w_2 = H_2^{-1} \cdot y^2$$

Since interpolating matrix does not depend on output data, but input data set, the interpolating matrix for y and y^2 are equal:

$$H_2 = H$$

Since interpolating matrices H and H_2 equal, it is not necessary to do another matrix inversion that can be advantageous, i.e. we have:

$$w_2 = H^{-1} \cdot y^2 \tag{41}$$

Therefore, after solving the inversion of basic interpolating matrix, it is possible to determine weight coefficients of any output data set, and therefore it is possible to scale output data set. Following definition can be written:

Definition 1: The RBF description has basic property of affine transformation by scaling output data set *Y*.

8.5 Solution of interpolating problem for y^2 using cubic-linear PRBF

The solution of the static area moment integral for z axis is identical to the solution of the area integral (20) for cubic-linear Polynomial RBFs but with weight coefficients w_2 used:

$$M_z = \left[\sum_{\substack{i \\ \beta=3}} w_{2i} \left(\frac{|z - z_i|^4}{4} + c_1^3 z \right) + \sum_{\substack{i \\ \beta=1}} w_{2i} \left(\frac{|z - z_i|^2}{2} + c_2 \cdot z \right) \right] \Bigg|_{z_1}^{z_2} \tag{42}$$

Therefore, if direct solution of the basic RBF integral is known, it is easy to determine static moment for z axis then. This is the easiest way for solving above integral, and it will be applied in further calculations.

9 Calculation of waterplane inertia moment for x axis

9.1 Basic integral

The basic integral for calculation of waterplane inertia moment around x axis, in (4), can be rewritten as:

$$I_x = \frac{1}{3} \int_{x_1}^{x_2} y^3 dx \tag{43}$$

By inserting curve functional definition in explicit form $y = f(x)$ in it, following integral is obtained:

$$I_x = \frac{1}{3} \int_{x_1}^{x_2} f^3(x) \cdot dx \tag{44}$$

This integral represents fourth basic integral for ship's hydrostatic particulars calculation to be solved and its solution follows below.

9.2 Integral of inertia moment for x axis using cubic-linear Polynomial RBFs

Belonging inertia moment integral for cubic-linear Polynomial RBFs can be written as:

$$I_x = \frac{1}{3}\int_{x_1}^{x_2}\left[\sum_{\substack{i \\ \beta=3}} w_i\left(|x-x_i|^3 + c_1^3\right) + \sum_{\substack{i \\ \beta=1}} w_i\left(|x-x_i| + c_2\right)\right]^3 dx \tag{45}$$

Above integral can be separated into four integrals as:

$$I_x = \frac{1}{3}\int_{x_1}^{x_2}\left[\sum_{\substack{i \\ \beta=3}} w_i\left(|x-x_i|^3 + c_1^3\right)\right]^3 \cdot dx + \int_{x_1}^{x_2}\left[\sum_{\substack{i \\ \beta=3}} w_i\left(|x-x_i|^3 + c_1^3\right)\right]^2 \cdot \left[\sum_{\substack{i \\ \beta=1}} w_i\left(|x-x_i| + c_2\right)\right] \cdot dx +$$

$$+ \int_{x_1}^{x_2}\left[\sum_{\substack{i \\ \beta=3}} w_i\left(|x-x_i|^3 + c_1^3\right)\right] \cdot \left[\sum_{\substack{i \\ \beta=1}} w_i\left(|x-x_i| + c_2\right)\right]^2 \cdot dx + \frac{1}{3}\int_{x_1}^{x_2}\left[\sum_{\substack{i \\ \beta=1}} w_i\left(|x-x_i| + c_2\right)\right]^3 \cdot dx$$

This integral contains multiple terms because of basic RBF's definition with translated sums, and therefore it is not suitable for calculation in this form. Therefore, it will be solved for Polynomial RBFs written in basic polynomial form, (9).

9.3 Moment of Inertia Integral for PRBFs written in general polynomial form

The integral of inertia moment around x axis written for PRBF in general polynomial form can be written as:

$$I_x = \frac{1}{3}\int_{x_1}^{x_2}\left(\sum_{j=0}^{n} C_j \cdot x^j\right)^3 \cdot dx \tag{46}$$

By cubing it we obtain:

$$I_x = \frac{1}{3}\int_{x_1}^{x_2}\sum_{j=0}^{3n} D_j \cdot x^j \cdot dx$$

Where: $D_j = f\left(C_j\right)$

Belonging integral solution, in the case of cubic-linear PRBF, can be written as:

$$I_x = \frac{1}{3}\int_{x_1}^{x_2}\left(C_3 \cdot x^3 + C_2 \cdot x^2 + C_1 \cdot x + C_0\right)^3 \cdot dx \tag{47}$$

By cubing it, complex solution is obtained, that can be written as:

$$I_x = \frac{1}{3}\sum_{j=0}^{2n}\frac{1}{j+1} D_j x^{j+1}\Bigg|_{x_1}^{x_2} = \frac{1}{3}\sum_{j=0}^{3n}\frac{1}{j+1} D_j\left(x_2^{j+1} - x_1^{j+1}\right) \tag{48}$$

9.4 Direct integration using y^3

Instead of solving basic moment of inertia integral I_x by developing complex RBF form with norms, corresponding integral can be solved using weight coefficients w_3 for cubed curve equation, i.e. for y^3, as shown before for y^2 in (41). Therefore, by substituting $q = y^3$ the same integral form as basic area integral is obtained, i.e. we obtain:

$$I_x = \frac{1}{3}\int_{x_1}^{x_2} q dx \tag{49}$$

And by inserting $f_3(x)$, following integral is obtained:

$$I_x = \frac{1}{3}\int_{x_1}^{x_2} f_3(x)\cdot dx$$

The function f_3 has different weight coefficients w_3 than function f with coefficients w, and it is necessary to define separate interpolation problem for y^3, as will be shown below.

9.5 Interpolation problem for y^3

Instead of interpolation problem for $y = H \cdot w$ we define problem:

$$y^3 = f_3(x) = H \cdot w_3 \tag{50}$$

Where H is interpolating matrix for input data set, and w_3 is weight coefficients vector.

But interpolating matrix does not depend on output, but only input data set, as shown before, so we have:

$$H_3 = H \tag{51}$$

Similar to basic scattered data interpolation problem, the solution of this problem is obtained by inverting interpolating matrix H which is according to (7) known as:

$$w_3 = H^{-1} \cdot y^3 \tag{52}$$

9.6 The solution of interpolation problem for y^3 using cubic-linear PRBF

The solution of the inertia moment integral for x axis (4) using cubic-linear Polynomial RBF is identical to the solution of the area integral (20) with weight coefficients w_3 used:

$$I_x = \left[\sum_{\substack{i \\ \beta=3}} w_{3i} \left(\frac{|x - x_i|^4}{4} + c_1^3 x \right) + \sum_{\substack{i \\ \beta=1}} w_{3i} \left(\frac{|x - x_i|^2}{2} + c_2 \cdot x \right) \right]_{x_1}^{x_2} \tag{53}$$

Therefore, if direct solution of the integral for general radial function is known, corresponding solution of the waterplane moment of inertia for x axis is known, also.

10 Calculation of waterplane inertia moment for y axis

10.1 Basic integral

Basic integral form for the calculation of waterplane moment of inertia for *y* axis shown in (5) can be rewritten as:

$$I_y = \int_{x_1}^{x_2} x^2 \cdot y \cdot dx \tag{54}$$

By inserting waterline curve description in explicit form $y = f(x)$ following integral can be obtained:

$$I_y = \int_{x_1}^{x_2} x^2 \cdot f(x) \cdot dx \tag{55}$$

This integral represents basic fifth integral of ship's hydrostatics.

Direct integration of above integral can be very complex when general Polynomial RBFs are used. Therefore, the integration of cubic-linear Polynomial RBFs will be shown instead, i.e. the integration of RBFs written in polynomial form.

10.2 Moment of inertia integral for y axis using compatible cubic-linear PRBF

Belonging integral of waterplane moment of inertia for *y* axis using cubic-linear Polynomial RBF can be written as:

$$I_y = \int_{x_1}^{x_2} x^2 \cdot \left[\sum_{\substack{i \\ \beta=3}} w_i \left(|x - x_i|^3 + c_1^3 \right) + \sum_{\substack{i \\ \beta=1}} w_i \left(|x - x_i| + c_2 \right) \right] dx \tag{56}$$

I.e., above integral can be separated in two integrals as:

$$I_y = \int_{x_1}^{x_2} x^2 \cdot \left[\sum_{\substack{i \\ \beta=3}} w_i \left(|x - x_i|^3 + c_1^3 \right) \right] dx + \int_{x_1}^{x_2} x^2 \cdot \left[\sum_{\substack{i \\ \beta=1}} w_i \left(|x - x_i| + c_2 \right) \right] dx$$

In order to solve first integral in above equation (56), it is necessary to develop the argument of L_1 norm, i.e. develop $|x - x_i|^3$ and multiply it with x^2 as:

$$x^2 \cdot |x - x_i|^3 = x^2 \cdot |x|^3 - 3 \cdot x^2 \cdot |x|^2 |x_i| + 3 \cdot x^2 \cdot |x| |x_i|^2 - x^2 \cdot |x_i|^3$$

After inserting we have

$$\int_{x_1}^{x_2} x^2 \cdot \left[|x - x_i|^3 + c_1^3 \right] dx = \int_{z_1}^{z_2} \left(x^5 - 3 \cdot x^4 |x_i| + 3 \cdot x^3 \cdot |x_i|^2 + x^2 \cdot \left(c_1^3 - |x_i|^3 \right) \right) dx =$$

$$= \left[\frac{x^6}{6} - 3 \cdot \frac{x^5}{5} \cdot |x_i| + 3 \cdot \frac{x^4}{4} |x_i|^2 + \frac{x^3}{3} \cdot \left(c_1^3 - |x_i|^3 \right) \right] \Bigg|_{x_1}^{x_2}$$

Similar has to be done with second integral with:

$$x^2 \cdot |x - x_i| = x^2 \cdot |x| - x|x_i|$$

We get

$$\int_{z_1}^{z_2} x^2 \cdot (x - x_i + c_2) \cdot dx = \int_{z_1}^{z_2} \left[x^3 + x^2 \cdot (c_2 - |x_i|)\right] \cdot dx = \left[\frac{x^4}{4} + \frac{x^3}{3}(c_2 - |x_i|)\right]\Bigg|_{x_1}^{x_2}$$

Final solution for ship's waterplane moment of inertia for *y* axis is then:

$$I_y = \sum_{\substack{i \\ \beta=3}} w_i \left[\frac{x^6}{6} - 3\frac{x^5}{5}|x_i| + 3\frac{x^4}{4}|x_i|^2 + \frac{x^3}{3}\left(c_1^3 - |x_i|^3\right)\right]\Bigg|_{x_1}^{x_2} + \sum_{\substack{i \\ \beta=1}} w_i \left[\frac{x^4}{4} + \frac{x^3}{3}(c_2 - |x_i|)\right]\Bigg|_{x_1}^{x_2} \quad (57)$$

10.3 Moment of inertia Integral for y axis using Polynomial RBF in general polynomial form

For general RBF written in polynomial form, the integral for moment of inertia for *y* axis can be generally written as:

$$I_y = \int_{x_1}^{x_2} x^2 \cdot \left(\sum_{j=0}^{n} C_j \cdot x^j\right) \cdot dx \quad (58)$$

I.e., we have:

$$I_y = \int_{x_1}^{x_2} \left(\sum_{j=0}^{n} C_j \cdot x^{j+2}\right) \cdot dx \quad (59)$$

In the case of cubic-linear Polynomial RBF, belonging moment of inertia integral solution can be written as:

$$I_y = \int_{x_1}^{x_2} \left(C_3 \cdot x^5 + C_2 \cdot x^4 + C_1 \cdot x^3 + C_0 x^2\right) \cdot dx \quad (60)$$

The solution of this integral is simply obtained by integrating every polynomial term as:

$$I_y = \left(\frac{1}{6}C_3 \cdot x^6 + \frac{1}{5}C_2 \cdot x^5 + \frac{1}{4}C_1 \cdot x^4 + \frac{1}{3}C_0 x^3\right)\Bigg|_{x_1}^{x_2} \quad (61)$$

Or:

$$I_y = \left(\sum_{j=0}^{n} \frac{1}{j+3} C_j \cdot x^{j+3}\right)\Bigg|_{x_1}^{x_2}$$

Finally, when limits are inserted, the solution can be written as:

$$I_y = \sum_{j=0}^{n} \frac{1}{j+3} C_j \cdot \left(x_2^{j+3} - x_1^{j+3}\right) \quad (62)$$

This solution of fifth basic integral (5) used for calculation of waterplane inertia moment for *y* axis is the easiest possible, and can be used in practice.

11 Curve length calculation using Polynomial RBF

Except five basic integrals of ship' hydrostatics, there is a need for determination of wetted area of immersed ship hull on actual waterline, in ship resistance calculation. The integral of curve length for some frame section therefore can be included into basic integrals to be determined, too. Similarly to first five integrals, this integral should be always solvable.

In order to define the length integral for curve description defined with $f(x)$, it is necessary to differentiate it once, obtaining $g(x) = f'(x)$. If Polynomial RBF is written in general polynomial form, with polynomial coefficients introduced for $f(x)$, the function $g(x)$ can be obtained after differentiation with:

$$g(x) = \left(\sum_{k=1}^{n} k \cdot C_k \cdot z^{k-1} \right)^2 \tag{63}$$

where C_k are polynomial coefficients.

After inserting $g(x)$ in integral (6) following integral is obtained:

$$L = \int_{x_1}^{x_2} \sqrt{1 + \left(\sum_{k=1}^{n} k \cdot C_k \cdot z^{k-1} \right)^2} \, dx \tag{64}$$

Written by coefficients it becomes:

$$L = \int_{x_1}^{x_2} \sqrt{1 + c_0 + c_1 x + \ldots + c_{2k-2} x^{2k-2}} \, dx \tag{65}$$

where c_k are polynomial coefficients after derivation and squaring of polynomial.

However, as described before, only Polynomial RBFs with integer exponent β equal three is acceptable as smooth description solution, and therefore that case will be investigated here. After squaring Polynomial RBF with $\beta = 3$, belonging polynomial function $g(x)$ with degree four is obtained.

The indefinite integral for Polynomial RBF with exponent $\beta = 3$ is:

$$L = \int_{x} \sqrt{1 + c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4} \, dx \tag{66}$$

The solution contains elliptic integrals of the first kind $F(x|m)$, elliptic integrals of the second kind $E(x|m)$ and elliptic integrals of the third kind $\Pi(x|m)$.

For special case where $g(x) = (x - a)^2$ the solution is:

$$L = \frac{1}{3} \left\{ \sqrt{(x-a)^4 + 1} \cdot (x-a) + 2 \cdot \sqrt[4]{-1} \cdot F\left[i \cdot \sinh^{-1}\left(\sqrt[4]{-1} \cdot (x-a) \middle| -1 \right) \right] \right\}$$

Where $F(x|m)$ is elliptic integral of the first kind.

Above length integrals are not always solvable and their existing general solutions are complex. Therefore, it is more efficient to calculate curve length by segments using curve description with Polynomial RBFs, for total input data set X_G used for generalization, with:

$$L = \sum_{j=1}^{N_G} \sqrt{\left(x_j - x_{j-1}\right)^2 + \left(y_j - y_{j-1}\right)^2} \tag{67}$$

where curve segments represent l_2 norm of input and output point pairs.

It can be concluded, that there is no efficient polynomial method available for direct solution of basic curve length integral using Polynomial RBFs with exponent $\beta \geq 3$. Therefore, the combination of analytical and numerical calculation method should be used for curve length determination, instead of solely analytical method.

12 The example of the frame section area particulars calculation

The solutions of above integrals are tested for actual ship's Test frame No. 1 shown on Figure 3, aft frame of car-truck carrier with flat of the side, camber and discontinuities, [9]; and theoretical semicircle test frame section, Test frame No. 2, Figure 5.

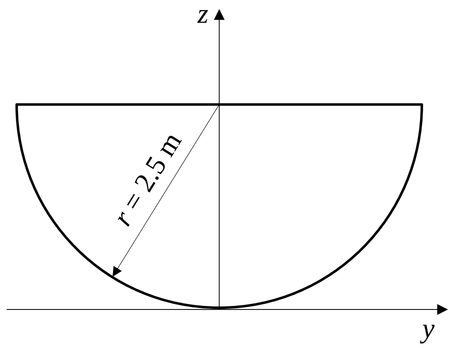

Figure 5: Theoretical Test frame No. 2 in the form of semicircle

The integration results will be tested for test frame sections Bonjean curves as:

$$\{A, y_B, z_B\} = f(z)$$

The comparison of calculated results using composition of cubic-linear Polynomial RBFs and corresponding test frames are shown in Table 1, below.

Table 1: The comparison of direct Polynomial RBF integration results and actual values for test frames

	Test frame No. 1 until camber		Test frame No. 2	
	PRBF	Trim & Stability Booklet	PRBF	Analytical values
A (m^2)	311.3901	311.3901	9.8178	9.8175
z_B (m)		-	1.4389	1.4389

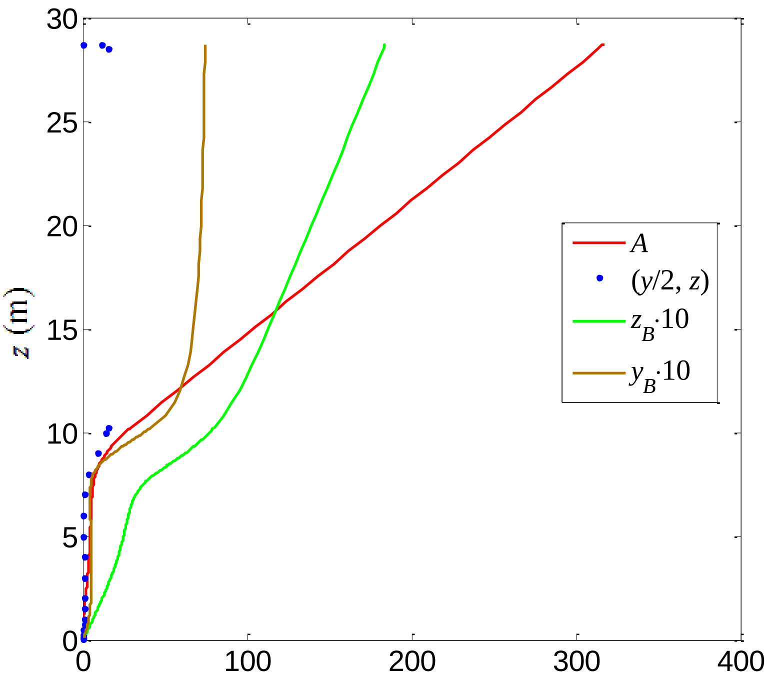

Figure 6: The distribution of Test frame No. 1 area particulars for frame draft

Theoretically, vertical centroid of the semicircle area z_B can be obtained by:

$$z_B = r - \frac{4r}{3\pi}$$

Vertical position of the centre of semicircle area for r = 2.5 (m) is than z_B = 1.4389 (m). It can be seen from Table 1 that the same vertical centre of semicircle area is obtained by the calculations with Polynomial RBFs, while area value differs in forth decimal, thus satisfying required calculating precision of 1 (mm) in shipbuilding.

Also, it is visible from Polynomial RBF integration results of area particulars for Test frame No. 1 shown on Figure 6, that their distribution is smooth for whole draft, as required by ship computational geometry. Table 1, above, shows that result for actual ship test frame are very accurate and correspond to the values from Trim and Stability Booklet for Test frame No. 1, [9].

13 Conclusion

Global curve description using composition of cubic-linear polynomial radial basis functions enables direct solution of five basic integrals of ship's hydrostatics for whole drafts range. Besides integrability, high precision of the geometric and hydrostatic particulars is shown, with the elimination of the effect of grouping of error in a point, connected with global geometry description. Belonging integral for curve length calculation is not always solvable, but functional geometry description using cubic-linear Polynomial radial basis functions enables its calculation using summation of segments used for its generalization.

Except above mentioned, the direct solution of RBF integrals eliminates the need for numerical integration procedures that dominated ship's computational geometry calculations for a long time, and dictated the distribution of ship frame sections and waterlines in wire-frame modelling of ship geometry. Direct RBF integration enables arbitrary distribution of ship frame sections using meshless RBF principle, as well as high precision of calculations, even for the existence of the discontinuities in ship geometry.

It can be concluded that solution of arbitrary ship geometry using composition of cubic-linear Polynomial RBFs represents required analytical solution of all ship's computational geometry problems in the case of global, non-two manifold 2D description, with remark of using the combination of analytical and numerical solution for curve length calculation.

Corresponding analytical solution for 3D problem of ship geometry description is not found yet, and that will be the subject of further research of authors of this paper.

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Submitted:16.05.2015.
Accepted: 10.09.2015.

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