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ELEMENTARY EXAMPLES OF ESSENTIAL PHANTOM MAPPINGS

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Abstract. It is known that essential phantom mappings (of the second kind) between connected CW-complexes do exist. However, it appears that in the literature there are few explicit examples of such mappings. One usually finds descriptions of the domain and the codomain and an existence proof that the set of homotopy classes of mappings from the domain to the codomain is infinite. The purpose of the present paper is to describe some elementary examples of essential phantom mappings. The codomain is the n-sphere S^n, $n \geq 2$, and the domain is the telescope T^n, associated with the sequence of copies of the canonical mapping $f\colon S^{n-1} \to S^{n-1}$ of odd degree $p > 1$. There are no essential phantom mappings whose codomain is the 1-sphere S^1.

1. Introduction

1.1. A mapping $h\colon Z \to W$ between connected CW-complexes with basepoints $*$ is said to be a phantom mapping (of the second kind, see [9]) provided the restriction $h|C$ to any compact subset (equivalently, to any finite subcomplex) $C \subseteq Z$ is homotopic to a constant mapping. Since connected CW-complexes are pathwise connected, there is no loss of generality in requiring that $h|C \simeq *$, for all C. Clearly, if a mapping h is homotopic to $*$, it is a phantom mapping. Therefore, of primary interest are phantom mappings, which are not homotopic to $*$, i.e., are essential mappings. It is known that essential phantom mappings do exist [9].

It appears that in the literature there are few explicit examples of essential phantom mappings. One usually finds explicit descriptions of the domain and the codomain and an existence proof that the set of homotopy classes of mappings from the domain to the codomain is infinite.

1.2. Professor Jaka Smrekar of the University of Ljubljana kindly communicated to the author the following proof (here slightly modified) of the existence of essential phantom mappings $h\colon T^n \to S^n$, $n \geq 2$. Here S^n is

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the n-sphere and T^n is the telescope, associated with the canonical mapping $f\colon S^{n-1} \to S^{n-1}$ of prime degree $p > 1$. A detailed description of T^n is given in Section 2. It is easy to show that every mapping $h\colon T^n \to S^n$ is a phantom mapping (see Lemma 1 in Section 4). Therefore, in order to construct essential phantom mappings $h\colon T^n \to S^n$, it suffices to construct essential mappings $h\colon T^n \to S^n$.

By a generalization of the Hopf classification theorem, proved by C. H. Dowker (see [2], Theorem 7.5), there is a bijection between the set $[P^n, S^n]$ of homotopy classes of mappings from an n-dimensional polyhedron P^n to S^n and the n-th integral cohomology group $H^n(P^n;\mathbb{Z})$ of that polyhedron. Since $\dim T^n = n$, we see that there is a bijection between the sets $[T^n, S^n]$ and $H^n(T^n;\mathbb{Z})$. Therefore, it suffices to show that $H^n(T^n;\mathbb{Z}) \neq 0$. Actually, the latter group is uncountable. Indeed, it is known that $H^n(T^n;\mathbb{Z}) \approx \widehat{\mathbb{Z}}_p/\mathbb{Z}$, where $\widehat{\mathbb{Z}}_p$ denotes the group of p-adic integers and $\mathbb{Z}$ is naturally embedded in $\widehat{\mathbb{Z}}_p$ ([4], Example 3F.9). It is also known that the group $\widehat{\mathbb{Z}}_p$ is uncountable ([4], Example 3F.6). Since $\mathbb{Z}$ is countable, $\widehat{\mathbb{Z}}_p/\mathbb{Z}$ is also uncountable.

1.3. The following explicit example of an essential phantom mapping $h\colon Z \to S^n$, $n \geq 3$, is described in [5] (pp. 83-84) and [9] (pp. 1211-1212). Let A and B be nonempty complementary sets of primes and let $\phi_A\colon S^{n-1} \to S^{n-1}_A$ and $\phi_B\colon S^{n-1} \to S^{n-1}_B$ be the respective localizations of the $(n-1)$-sphere S^{n-1}. Let $\lambda\colon S^{n-1} \to S^{n-1}_A \vee S^{n-1}_B$ be the composition of two mappings. The first one $S^{n-1} \to S^{n-1} \vee S^{n-1}$ is the quotient mapping, obtained by collapsing to a point the equator of S^{n-1}. The second one is $\phi_A \vee \phi_B\colon S^{n-1} \vee S^{n-1} \to S^{n-1}_A \vee S^{n-1}_B$. Let Z be the CW-complex obtained by attaching the n-cell D^n to $S^{n-1}_A \vee S^{n-1}_B$ via the mapping $\lambda\colon \partial D^n \to S^{n-1}_A \vee S^{n-1}_B$. Then the quotient mapping $h\colon Z \to S^n$, obtained from Z by collapsing $S^{n-1}_A \vee S^{n-1}_B$ to a point, is an essential phantom mapping $h\colon Z \to S^n$.

The present paper describes explicitly elementary examples of some essential phantom mappings $h\colon T^n \to S^n$, $n \geq 2$. There are no essential phantom mappings, whose codomain is S^1 [7].

2. The telescope T^n

2.1. To fix notations we recall the definition of the telescope T^n. First consider the 1-sphere $S^1 = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$ with the basepoint $* = 1$. The canonical mapping $f\colon S^1 \to S^1$ of degree $p > 1$ is defined by $f(\zeta) = \zeta^p$, $\zeta \in S^1$. Note that $f(*) = *$. In order to define the canonical mapping $f\colon S^{n-1} \to S^{n-1}$ of degree $p > 1$, for $n > 2$, one views the $(n-1)$-sphere S^{n-1} as the (unreduced) $(n-2)$-fold suspension $\Sigma^{n-2}(S^1)$ with basepoint

$* \in S^1 \subseteq S^{n-1}$. Then $f\colon S^{n-1} \to S^{n-1}$ is the $(n-2)$-fold suspension of $f\colon S^1 \to S^1$. Note that also in this case $f(*) = *$.

The mapping cylinder M associated with $f\colon S^{n-1} \to S^{n-1}$ is obtained from $S^{n-1} \times [0,1]$ by identifying the points $(\zeta_1, 1)$ and $(\zeta_2, 1)$, $\zeta \in S^{n-1}$, whenever $f(\zeta_1) = f(\zeta_2)$. The corresponding quotient mapping will be denoted by $\phi\colon S^{n-1} \times [0,1] \to M$. The natural CW-structure of M consists of two 0-cells, $v^- = \phi(*, 0)$ and $v^+ = \phi(*, 1)$, called the initial and the terminal basepoints of M, of the arc $A = \phi(* \times [0,1])$ as the only 1-cell, called the spine of M, of two $(n-1)$-spheres $B^- = \phi(S^{n-1} \times 0)$ and $B^+ = \phi(S^{n-1} \times 1)$, called the initial and the terminal bases of M and of the n-cell $M = \phi(S^{(n-1)} \times [0,1])$. Note that M is a connected 2-dimensional polyhedron.

2.2. The telescope T^n, associated with the sequence $S^{n-1} \xrightarrow{f} S^{n-1} \xrightarrow{f} S^{n-1} \to \dots$ is obtained from the direct sum $M_1 \sqcup M_2 \sqcup \dots$ of the sequence $M_1, M_2, \dots$ of copies of the mapping cylinder M of $f\colon S^{n-1} \to S^{n-1}$, by identifying the terminal base of M_i with the initial base of M_{i+1}, $i \in \mathbb{N}$. The $(n-1)$-spheres obtained in this way will be denoted by B_i, $i = 0, 1, \dots$. The mapping cylinders M_i and their spines A_i are naturally embedded in T^n. The two endpoints of A_i are denoted by v_{i-1} and v_i. Note that $T^n = M_1 \cup M_2 \cup \dots$ is a connected 2-dimensional polyhedron. It has a natural CW-structure, which consists of the points v_i as 0-cells, of the arcs A_i as 1-cells, of the $(n-1)$-spheres B_i as the $(n-1)$-cells and of mapping cylinders M_i as the n-cells.

3. The mapping $h\colon T^n \to S^n$

3.1. If in the cylinder $S^{n-1} \times [0,1]$ we collapse $S^{n-1} \times 0$ to a point v^- and we collapse $S^{n-1} \times 1$ to a point v^+, we obtain the (unreduced) suspension ΣS^{n-1} of S^{n-1} with vertices v^-, v^+ and we obtain the corresponding quotient mapping $\psi\colon S^{n-1} \times [0,1] \to \Sigma S^{n-1}$. Note that, whenever $y \in M$ belongs to the terminal base B^+ of M, then $\phi^{-1}(y) \subseteq S^{n-1} \times 1$ and thus, $\psi(\phi^{-1}(y)) = v^+$. If $y \in M$ does not belong to B^+, then $\phi^{-1}(y)$ is a single point of $S^{n-1} \times [0,1)$ and so is $\psi(\phi^{-1}(y))$. Therefore, there is a unique mapping $\chi\colon M \to \Sigma S^{n-1} = S^n$ such that $\chi\phi = \psi$. Clearly, χ is the quotient mapping, which collapses the initial base of M to v^- and collapses the terminal base of M to v^+.

We now repeat the described procedure for every mapping cylinder M_i, $i \in \mathbb{N}$, of the telescope T^n, i.e., we collapse to a point v_i each $(n-1)$-sphere B_i (including B_0) and we consider the corresponding quotient mappings χ_i. We obtain a CW-complex T^{n*}, called the pinched telescope. We also obtain a quotient mapping $\chi\colon T^n \to T^{n*}$ such that $\chi | M_i = \chi_i$. Note that χ maps the initial and the terminal bases B_{i-1} and B_i of M_i to the vertices v_{i-1} and v_i of the n-sphere $S^n_i = \chi_i(M_i)$, respectively. The CW-structure of T^{*n}

consists of the points $v_0, v_1, \dots$ as 0-cells, of the arcs $A_1, A_2, \dots$ as 1-cells and of the n-spheres S_i^n as n-cells, $i \in \mathbb{N}$. Notice that T^{2*} looks like an infinite string-of-beads.

3.2. Our next goal is to define a mapping χ' of T^{n*} to the wedge of a sequence of pointed n-spheres, $S = S_1^n \vee S_2^n \vee \dots$. In the proof we need the following fact. Let v^- and v^+ be the vertices of the (unreduced) suspension ΣS^{n-1}, let $* \in S^{n-1}$ be the basepoint of S^{n-1} and let A be the arc $\Sigma(*)$. Then the quotient space $(\Sigma S^{n-1})/A$ is an n-sphere S^n. If $\chi' \colon \Sigma S^{n-1} \to S^n$ is the corresponding quotient mapping and we denote the point A of $(\Sigma S^{n-1})/A$ by $*$, then $\chi'(*) = *$.

The assertion is a very special case of well-known facts concerning decompositions of manifolds. If X is a cellular subset of a closed n-manifold N, i.e., X is the intersection of a sequence of n-disks $D_i \subseteq N$, where $D_{i+1} \subseteq \mathrm{Int} D_i$, $i \in \mathbb{N}$, then the decomposition G_X of N, whose only nondegenerate element is X, is a shrinkable upper semicontinuous decomposition ([1], 1, Proposition 4 and 6, Proposition 2). Moreover, if G is a shrinkable upper semicontinuous decomposition of a compact metric space N, then the corresponding quotient mapping $\pi \colon N \to N/G$ is a near-homeomorphism, i.e., it can be approximated by homeomorphisms ([1], 5, Theorem 2) and thus, N/G is homeomorphic to N.

In our case, A is a cellular subset of the n-sphere ΣS^{n-1} and thus, $\Sigma S^{n-1}/A$ is also an n-sphere. It is now clear that, by collapsing the union of arcs $A = A_1 \cup A_2 \cup \dots \subseteq T^{n*}$ to a point $*$, one obtains the wedge $S = S_1^n \vee S_2^n \vee \dots$ and a quotient mapping $\chi' \colon T^{n*} \to S$, $\chi'(*) = *$. Note that the natural CW-structure of S consists of a single 0-cell $*$, and of a sequence of n-cells $S_1^n, S_2^n, \dots$.

The next mapping considered is the folding map $\chi'' \colon S \to S^n$. It maps each summand S_i^n of S to S^n by the identity mapping. Note that $\chi''(*) = *$. Finally, we define the desired mapping $h \colon T^n \to S^n$, by putting $h = \chi''\chi'\chi$. Note that $h(*) = *$.

Theorem 1. *If $p > 1$ is an odd integer, the mapping $h \colon T^n \to S^n$, $n \geq 2$, is an essential phantom mapping.*

4. Proof of Theorem 1

4.1. We first prove the following lemma.

Lemma 1. *Every mapping $k \colon T^n \to S^n$ from the telescope T^n to the n-sphere S^n, $n \geq 2$, is a phantom mapping.*

Proof. It is well known that there is a strong deformation retraction r of M to its terminal base B^+. Denote by $r_i \colon M_i \to B_i$, $i \in \mathbb{N}$, the mappings which correspond to r. For $i \leq n$, put $T_{in} = M_i \cup \dots \cup M_n$ and note that

$T_{in} = M_i \cup T_{i+1n}$, $i < n$. Also consider the deformation retractions $r_{in}\colon T_{in} \to T_{i+1n}$, where $r_{in}|M_i = r_i$ and $r_{in}|T_{i+1n}$ is the identity mapping. Note that $T_{nn} = M_n$ and put $r_{nn} = r_n\colon M_n \to B_n$. Clearly, the composition $r_1^n = r_{nn} \dots r_{2n} r_{1n}\colon T_{1n} \to B_n$ is a strong deformation retraction. In particular, the inclusion $i_n\colon B_n \to T_{1n}$ has the property that $i_n r_1^n$ is homotopic to the identity mapping on T_{1n}. Therefore, $k|T_{1n} \simeq k i_n r_1^n = (k|B_n) r_1^n$. Now note that every mapping of S^{n-1} to S^n is homotopic to the constant mapping $*$. Since B_n is an $(n-1)$-sphere, $k|B_n \simeq *$. However, this implies that also $k|T_{1n} \simeq *$. Now consider a compact subset C of T^n. Since $T_{11} \subseteq T_{12} \dots$ and $T_{11} \cup T_{12} \dots = T^n$, one concludes that there is an $n \in \mathbb{N}$ such that $C \subseteq T_{1n}$. Consequently, $k|T_{1n} \simeq *$ implies $k|C \simeq *$ and k is indeed a phantom mapping. ∎

4.2. In view of Lemma 1, the proof of Theorem 1 will be completed if we prove the following lemma.

Lemma 2. *If $p > 1$ is an odd integer, then the mapping $h\colon T^n \to S^n$, $n \geq 2$, is essential, i.e., it is not homotopic to the constant mapping $*$.*

Proof. In proving the assertion, we will use cellular cohomology groups of CW-complexes with coefficients in $\mathbb{Z}$ (see e.g., [6], [10], [4], [3]). The mapping $h\colon T^n \to S^n$ induces a homomorphism $h^{n*}\colon H^n(S^n;\mathbb{Z}) \to H^n(T^n;\mathbb{Z})$. Since homotopic cellular mappings induce equal homomorphisms of cellular cohomology groups (see e.g., [3], Theorem 12.1.9 or [4], p. 201) and the homomorphism $H^2(Y_\mu;\mathbb{Z}) \to H^2(T_{i_\mu};\mathbb{Z})$, induced by the constant mapping $*$, equals 0, Lemma 2 will be proved if we show that $h^{n*} \neq 0$.

Recall that S^n has a CW-structure, which consists of a single 0-cell $*$ and a single n-cell S^n. Let a be the cellular n-cochain of S^n, which assumes an odd value α at the n-cell S^n. Since in S^n there are no $(n+1)$-cells, a is a cocycle. The folding mapping χ'' associates with a the n-cocycle $\chi''^n(a)$, which on each n-cell S_i^n of the cellular chain complex of S assumes the same value α. Furthermore, the mapping χ' associates with the n-cocycle $\chi''^n(a)$ the n-cocycle $\chi'^n\chi''^n(a)$, which on each n-cell ΣS_i^1 of the cellular chain complex of T^{n*} assumes the same value α. Finally, the mapping χ associates with the n-cocycle $\chi'^n\chi''^n(a)$ the n-cocycle $h^*(a) = \chi^*\chi'^n\chi'^n(a)$, which on each n-cell M_i of the cellular chain complex of T^n assumes the same value α. Therefore, to complete the proof of Lemma 2, it suffices to show that $h^n(a)$ is not the coboundary of some $(n-1)$-cochain of the cellular chain complex of T^n.

4.3. The latter assertion is an immediate consequence of the next lemma.

Lemma 3. *Let $p > 1$ be an odd integer and let a be the n-cochain of the cellular chain complex of T^n, which on each n-cell M_i, $i \in \mathbb{N}$, of T^n assumes the same odd value $\alpha \in \mathbb{Z}$, i.e., $a(M_i) = \alpha$, for all $i \in \mathbb{N}$. Then a is an n-cocycle, which is not a coboundary.*

Proof. Consider the telescope T^n and assume that there is an $(n-1)$-cochain b such that $\delta b = a$. Since the n-cells of T^n are the mapping cylinders M_i, we must have $(\delta b)(M_i) = a(M_i) = \alpha$. Note that, for $n > 2$, the $(n-1)$-cochain b is a function on the set $\{B_i | i \in \mathbb{N}\}$. Put $b(B_i) = \beta_i$. If $n = 2$, the set of 1-cells also includes the arcs A_i, $i \in \mathbb{N}$. By definition, $(\delta b)(M_i) = b(\partial M_i)$ and thus, $b(\partial M_i) = \alpha$. Since $\partial M_i = pB_i - B_{i-1}$, one concludes that $b(\partial M_i) = b(pB_i - B_{i-1}) = p\beta_i - \beta_{i-1}$. Consequently,

$$p\beta_i - \beta_{i-1} = \alpha,\ i \in \mathbb{N}. \tag{4.1}$$

Subtracting pairs of consecutive equalities (4.1), one sees that

$$\beta_1 - \beta_0 = p(\beta_2 - \beta_1),\ \beta_2 - \beta_1 = p(\beta_3 - \beta_2),\ \ldots,\ \beta_i - \beta_{i-1} = p(\beta_{i+1} - \beta_i) = \ldots. \tag{4.2}$$

The equalities (4.2) show that

$$\beta_1 - \beta_0 = p(\beta_2 - \beta_1) = p^2(\beta_3 - \beta_2) = \ldots = p^i(\beta_{i+1} - \beta_i) = \ldots. \tag{4.3}$$

Since $p > 1$, formula (4.3) shows that the integer $\beta_1 - \beta_0$ has arbitrarily large divisors p^i. This is possible only when $\beta_1 - \beta_0 = 0$, i.e., $\beta_0 = \beta_1$. In that case, for $i = 1$, (4.1) becomes $\alpha = p\beta_1 - \beta_0 = (p-1)\beta_0$. Since it was assumed that p is odd, $p - 1$ is even and so is $\alpha = (p-1)\beta_0$. However, this is in contradiction with the assumption that α is an odd number. □

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Elementarni primjeri bitnih fantomskih preslikavanja

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Sažetak. Poznato je da bitna fantomska preslikavanja (druge vrste) između povezanih CW-kompleksa postoje. U literaturi su rijetki eksplicitni primjeri takvih preslikavanja. Obično je dana domena i kodomena preslikavanja te egzistencijski dokaz da je skup klasa homotopije iz domene u kodomenu beskonačan. U ovom radu opisuju se neki elementarni primjeri bitnih fantomskih preslikavanja. Kodomena je n-sfera S^n, $n \geq 2$, a domena je teleskop T^n pridružen nizu kopija kanonskoga preslikavanja $f\colon S^{n-1} \to S^{n-1}$, neparnoga stupnja $p > 1$. Nema bitnih fantomskih preslikavanja čija je kodomena 1-sfera S^1.

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