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WEIGHTED POPOVICIU TYPE INEQUALITIES VIA GENERALIZED MONTGOMERY IDENTITIES

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Abstract. We obtained useful identities via generalized Montgomery identities, by which the inequality of Popoviciu for convex functions is generalized for higher order convex functions. We investigate the bounds for the identities related to the generalization of the Popoviciu inequality using inequalities for the Čebyšev functional. Some results relating to the Grüss and Ostrowski type inequalities are constructed. Further, we also construct new families of exponentially convex functions and Cauchy-type means by looking at linear functionals associated with the obtained inequalities.

1. Introduction

The theory developed under the theme of convex functions, arising from intuitive geometrical observations, may be readily applied to topics in real analysis and economics. Convexity is a simple and natural notion which can be traced back to Archimedes (circa 250 B.C.), in connection with his famous estimate of the value of π (using inscribed and circumscribed regular polygons). He noticed the important fact that the perimeter of a convex figure is smaller than the perimeter of any other convex figure, surrounding it. In modern Era, their occurs a rapid development in the theory of convex functions. Their are serval reasons behind it: firstly, so many areas in modern analysis directly or indirectly involve the application of convex functions; secondly, convex functions are closely related to the theory of inequalities and many important inequalities are consequences of the applications of convex functions (see [16]).

A characterization of convex function established by T. Popoviciu [17] is studied by many people (see [16, 18] and references with in). For recent work, we refer [7, 10, 11, 13, 14]. The following form of Popoviciu's inequality is by Vasić and Stanković in [18] (see also page 173 [16]):

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Theorem 1.1. *Let* $m, k \in \mathbb{N}$, $m \geq 3$, $2 \leq k \leq m-1$, $[\alpha, \beta] \subset \mathbb{R}$, $\mathbf{x} = (x_1, ..., x_m) \in [\alpha, \beta]^m$, $\mathbf{p} = (p_1, ..., p_m)$ *be a positive* m*-tuple such that* $\sum_{i=1}^m p_i = 1$. *Also let* $f : [\alpha, \beta] \to \mathbb{R}$ *be a convex function. Then*

$$p_{k,m}(\mathbf{x}, \mathbf{p}; f) \leq \frac{m-k}{m-1} p_{1,m}(\mathbf{x}, \mathbf{p}; f) + \frac{k-1}{m-1} p_{m,m}(\mathbf{x}, \mathbf{p}; f), \tag{1.1}$$

where

$$p_{k,m}(\mathbf{x}, \mathbf{p}; f) = p_{k,m}(\mathbf{x}, \mathbf{p}; f(x)) :=$$

$$\frac{1}{C_{k-1}^{m-1}} \sum_{1 \leq i_1 < \ldots < i_k \leq m} \left(\sum_{j=1}^k p_{i_j} \right) f \left(\frac{\sum_{j=1}^k p_{i_j} x_{i_j}}{\sum_{j=1}^k p_{i_j}} \right)$$

is the linear functional with respect to f.

By inequality (1.1), we write

$$\mathbf{P}(\mathbf{x}, \mathbf{p}; f) := \frac{m-k}{m-1} p_{1,m}(\mathbf{x}, \mathbf{p}; f) + \frac{k-1}{m-1} p_{m,m}(\mathbf{x}, \mathbf{p}; f) - p_{k,m}(\mathbf{x}, \mathbf{p}; f). \tag{1.2}$$

Remark 1.2. It is important to note that under the assumptions of Theorem 1.1, if the function f is convex then $\mathbf{P}(\mathbf{x}, \mathbf{p}; f) \geq 0$ and $\mathbf{P}(\mathbf{x}, \mathbf{p}; f) = 0$ for $f(x) = x$ or f is a constant function.

The mean value theorems and exponential convexity of the linear functional $\mathbf{P}(\mathbf{x}, \mathbf{p}; f)$ are given in [10] for a positive m-tuple $\mathbf{p}$. Some special classes of convex functions are considered to construct the exponential convexity of $\mathbf{P}(\mathbf{x}, \mathbf{p}; f)$ in [10]. In [11] (see also [7]), the results related to $\mathbf{P}(\mathbf{x}, \mathbf{p}; f)$ are generalized with help of Green function and $n-$exponential convexity is proved instead of exponential convexity.

In order to obtain our main results, we use the generalized Montgomery identities via Taylor's formula given in papers [3] and [4].

Theorem 1.3. *Let* $n \in \mathbb{N}$, $\lambda : I \to \mathbb{R}$ *be such that* $\lambda^{(n-1)}$ *is absolutely continuous,* $I \subset \mathbb{R}$ *an open interval,* $\alpha, \beta \in I$, $\alpha < \beta$. *Then the following identity holds*

$$\begin{aligned} \lambda(x) = {} & \frac{1}{\beta - \alpha} \int_\alpha^\beta \lambda(t)\, dt + \sum_{l=0}^{n-2} \frac{\lambda^{(l+1)}(\alpha)}{l!\,(l+2)} \frac{(x-\alpha)^{l+2}}{\beta - \alpha} \\ & - \sum_{l=0}^{n-2} \frac{\lambda^{(l+1)}(\beta)}{l!\,(l+2)} \frac{(x-\beta)^{l+2}}{\beta - \alpha} + \frac{1}{(n-1)!} \int_\alpha^\beta R_n(x,s)\, \lambda^{(n)}(s)\, ds \end{aligned} \tag{1.3}$$

where

$$R_n(x,s) = \begin{cases} -\frac{(x-s)^n}{n(\beta-\alpha)} + \frac{x-\alpha}{\beta-\alpha}(x-s)^{n-1}, & \alpha \leq s \leq x, \\ -\frac{(x-s)^n}{n(\beta-\alpha)} + \frac{x-\beta}{\beta-\alpha}(x-s)^{n-1}, & x < s \leq \beta. \end{cases} \tag{1.4}$$

Theorem 1.4. *Let $n \in \mathbb{N}$, $\lambda : I \to \mathbb{R}$ be such that $\lambda^{(n-1)}$ is absolutely continuous, $I \subset \mathbb{R}$ an open interval, $\alpha, \beta \in I$, $\alpha < \beta$. Then the following identity holds*

$$\lambda(x) = \frac{1}{\beta-\alpha}\int_\alpha^\beta \lambda(t)\,dt + \sum_{l=0}^{n-2}\lambda^{(l+1)}(x)\frac{(\alpha-x)^{l+2}-(\beta-x)^{l+2}}{(l+2)!(\beta-\alpha)}$$

$$+\frac{1}{(n-1)!}\int_\alpha^\beta \widehat{R}_n(x,s)\,\lambda^{(n)}(s)\,ds \tag{1.5}$$

where

$$\widehat{R}_n(x,s) = \begin{cases} \frac{-1}{n(\beta-\alpha)}(\alpha-s)^n, & \alpha \le s \le x, \\ \frac{-1}{n(\beta-\alpha)}(\beta-s)^n, & x < s \le \beta. \end{cases} \tag{1.6}$$

In case $n = 1$ the sum $\sum_{l=0}^{n-2}\cdots$ is empty, so identity (1.3) and (1.5) reduces to well-known Montgomery identity (see for instance [12])

$$\lambda(x) = \frac{1}{\beta-\alpha}\int_\alpha^\beta \lambda(t)\,dt + \int_\alpha^\beta P(x,s)\,\lambda'(s)\,ds$$

where $P(x,s)$ is the Peano kernel, defined by

$$P(x,s) = \begin{cases} \frac{s-\alpha}{\beta-\alpha}, & \alpha \le s \le x, \\ \frac{s-b}{\beta-\alpha}, & x < s \le \beta. \end{cases}$$

The organization of the paper follows the following pattern: In Section 2 and 3, we present generalization of the Popoviciu's inequality by using generalized Montgomery identities for higher order convex functions. In Section 4, we use classical Čebyšev functional and obtained results related to Grüss-type inequalities and Ostrowski-type inequalities. In Section 5, we study the functional defined as the difference between the R.H.S. and the L.H.S. of the generalized inequality and our goal is to investigate the n-exponential and logarithmic convexity of the obtained functional. Furthermore, we prove monotonicity property of the generalized Cauchy means obtained via this functional. Finally, we conclude our paper by giving several examples of the families of functions for which the obtained results can be applied.

2. Popoviciu's Inequality by extension of Montgomery identity via Taylor's Formula I

Motivated by identity (1.2), we construct the following identity with help of the generalized Montgomery identity (1.3).

Theorem 2.1. *Let all the assumptions of Theorem 1.3 be valid and let $m, k \in \mathbb{N}$, $m \ge 3$, $2 \le k \le m-1$, $[\alpha,\beta] \subset \mathbb{R}$, $\mathbf{x} = (x_1, ..., x_m) \in [\alpha,\beta]^m$, $\mathbf{p} = (p_1, ..., p_m)$ be a real m-tuple such that $\sum_{j=1}^k p_{i_j} \neq 0$ for any*

$1 \le i_1 < \ldots < i_k \le m$ *and* $\sum_{i=1}^{m} p_i = 1$. *Also let* $\frac{\sum_{j=1}^{k} p_{i_j} x_{i_j}}{\sum_{j=1}^{k} p_{i_j}} \in [\alpha, \beta]$ *for any* $1 \le i_1 < \ldots < i_k \le m$ *with* $R_n(x, v)$ *be the same as defined in* (1.4). *Then we have the following identity:*

$$\mathbf{P}(\mathbf{x}, \mathbf{p}; \lambda(x)) = \frac{1}{\beta - \alpha} \sum_{l=0}^{n-2} \left(\frac{1}{l!(l+2)} \right) \times \left(\lambda^{(l+1)}(\alpha) \, \mathbf{P}(\mathbf{x}, \mathbf{p}; (x-\alpha)^{l+2}) - \lambda^{(l+1)}(\beta) \, \mathbf{P}(\mathbf{x}, \mathbf{p}; (x-\beta)^{l+2}) \right) + \frac{1}{(n-1)!} \int_{\alpha}^{\beta} \mathbf{P}(\mathbf{x}, \mathbf{p}; R_n(x, v)) \lambda^{(n)}(v) dv. \tag{2.1}$$

Proof. Using (1.3) in (1.2) and using linearity of the functional $\mathbf{P}(\cdot)$, we have

$$\mathbf{P}(\mathbf{x}, \mathbf{p}; \lambda(x)) = \mathbf{P}\left(\mathbf{x}, \mathbf{p}; \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} \lambda(t) dt \right) + \frac{1}{\beta - \alpha} \sum_{l=0}^{n-2} \left(\frac{1}{l!(l+2)} \right) \lambda^{(l+1)}(\alpha) \, \mathbf{P}(\mathbf{x}, \mathbf{p}; (x-\alpha)^{l+2}) - \frac{1}{\beta - \alpha} \sum_{l=0}^{n-2} \left(\frac{1}{l!(l+2)} \right) \lambda^{(l+1)}(\beta) \, \mathbf{P}(\mathbf{x}, \mathbf{p}; (x-\beta)^{l+2}) + \frac{1}{(n-1)!} \int_{\alpha}^{\beta} \mathbf{P}(\mathbf{x}, \mathbf{p}; R_n(x, v)) \lambda^{(n)}(v) dv. \tag{2.2}$$

After simplification and following constant property of our functional, we get (2.1). ◻

In the following theorem we obtain generalizations of Popoviciu's inequality for $n-$convex functions.

Theorem 2.2. *Let all the assumptions of Theorem 2.1 be satisfied and let for* $n \ge 2$

$$\mathbf{P}(\mathbf{x}, \mathbf{p}; R_n(x, v)) \ge 0, \quad v \in [\alpha, \beta]. \tag{2.3}$$

If λ *is* $n-$*convex function such that* $\lambda^{(n-1)}$ *is absolutely continuous, then we have*

$$\mathbf{P}(\mathbf{x},\mathbf{p};\lambda(x)) \geq \frac{1}{\beta-\alpha}\sum_{l=0}^{n-2}\left(\frac{1}{l!(l+2)}\right) \times \left(\lambda^{(l+1)}(\alpha)\,\mathbf{P}(\mathbf{x},\mathbf{p};(x-\alpha)^{l+2}) - \lambda^{(l+1)}(\beta)\,\mathbf{P}(\mathbf{x},\mathbf{p};(x-\beta)^{l+2})\right). \tag{2.4}$$

PROOF. Since $\lambda^{(n-1)}$ is absolutely continuous on $[\alpha,\beta]$, $\lambda^{(n)}$ exists almost everywhere. As λ is $n-$convex, so $\lambda^{(n)}(x) \geq 0$ for all $x \in [\alpha,\beta]$ (see [16], p. 16). Hence we can apply Theorem 2.1 to obtain (2.4). ◻

REMARK 2.3. *The inequality* (2.4) *holds in reverse direction if reverse of* (2.3) *holds.*

Now we will give generalization of Popoviciu's inequality for m-tuples.

THEOREM 2.4. *Let all the assumptions of Theorem 2.1 be satisfied in addition with the condition that* $\mathbf{p} = (p_1,...,p_m)$ *be a positive m-tuple such that* $\sum_{i=1}^{m} p_i = 1$ *and consider* $\lambda : [\alpha,\beta] \to \mathbb{R}$ *is $n-$convex function.*

(i) If n is even and $n \geq 2$, then (2.4) holds.
(ii) Let the inequality (2.4) be satisfied. If the function

$$F(x) = \frac{1}{\beta-\alpha}\sum_{l=0}^{n-2}\left(\frac{\lambda^{(l+1)}(\alpha)(x-\alpha)^{l+2} - \lambda^{(l+1)}(\beta)(x-\beta)^{l+2}}{l!(l+2)}\right) \tag{2.5}$$

is convex, the R.H.S. of (2.4) is non negative and we have inequality

$$\mathbf{P}(\mathbf{x},\mathbf{p};\lambda(x)) \geq 0. \tag{2.6}$$

PROOF. (i) Since

$$R_n(x,v) = \begin{cases} -\frac{(x-v)^n}{n(\beta-\alpha)} + \frac{x-\alpha}{\beta-\alpha}(x-v)^{n-1}, & \alpha \leq v \leq x \leq \beta, \\ -\frac{(x-v)^n}{n(\beta-\alpha)} + \frac{x-\beta}{\beta-\alpha}(x-v)^{n-1}, & \alpha \leq x < v \leq \beta. \end{cases}$$

So

$$\frac{d}{dx}R_n(x,v) =$$

$$\begin{cases} \frac{1}{\beta-\alpha}\left[-(x-v)^{n-1} + (x-v)^{n-1} + (n-1)(x-\alpha)(x-v)^{n-2}\right], \ \alpha \leq v \leq x \leq \beta, \\ \frac{1}{\beta-\alpha}\left[-(x-v)^{n-1} + (x-v)^{n-1} + (n-1)(x-\beta)(x-v)^{n-2}\right], \ \alpha \leq x < v \leq \beta. \end{cases}$$

and

$$\frac{d^2}{dx^2}R_n(x,v) = \begin{cases} \frac{n-1}{\beta-\alpha}\left[(x-v)^{n-2} + (n-2)(x-\alpha)(x-v)^{n-3}\right], \ \alpha \leq v \leq x \leq \beta, \\ \frac{n-1}{\beta-\alpha}\left[(x-v)^{n-2} + (n-2)(x-\beta)(x-v)^{n-3}\right], \ \alpha \leq x < v \leq \beta. \end{cases}$$

showing that $R_n(\cdot, v)$ is convex for even n, where $n \geq 2$. Hence by virtue of Remark 1.2, (2.3) holds for even n, where $n \geq 2$. Therefore following Theorem 2.2, we can obtain (2.4).

(ii) $\mathbf{P}$ is linear functional, so we can rewrite the R.H.S. of (2.4) in the form $\mathbf{P}(x, p; F(x))$ where F is defined in (2.5) and will be obtained after reorganization of this side. Since F is assumed to be convex, therefore using the given conditions and by following Remark 1.2, the non negativity of the R.H.S. of (2.4) is immediate and we have (2.6) for m-tuples .

□

3. Popoviciu's Inequality by extension of Montgomery identity via Taylor's Formula II

Motivated by identity (1.2), we construct the following identity with help of generalized Montgomery identity (1.5).

Theorem 3.1. *Let all the assumptions of Theorem 1.4 valid and let $m, k \in \mathbb{N}$, $m \geq 3$, $2 \leq k \leq m-1$, $[\alpha, \beta] \subset \mathbb{R}$, $\mathbf{x} = (x_1, ..., x_m) \in [\alpha, \beta]^m$, $\mathbf{p} = (p_1, ..., p_m)$ be a real m-tuple such that $\sum_{j=1}^{k} p_{i_j} \neq 0$ for any $1 \leq i_1 < ... < i_k \leq m$ and $\sum_{i=1}^{m} p_i = 1$. Also let $\frac{\sum_{j=1}^{k} p_{i_j} x_{i_j}}{\sum_{j=1}^{k} p_{i_j}} \in [\alpha, \beta]$ for any $1 \leq i_1 < ... < i_k \leq m$ with $\widehat{R}_n(x, v)$ be the same as defined in* (1.6). *Then we have the following identity:*

$$\mathbf{P}(\mathbf{x}, \mathbf{p}; \lambda(x)) = \frac{1}{\beta - \alpha} \sum_{l=0}^{n-2} \left(\frac{1}{(l+2)!} \right) \times \left(\mathbf{P}(\mathbf{x}, \mathbf{p}; \lambda^{(l+1)}(x) (\alpha - x)^{l+2}) - \mathbf{P}(\mathbf{x}, \mathbf{p}; \lambda^{(l+1)}(x) (\beta - x)^{l+2}) \right) + \frac{1}{(n-1)!} \int_{\alpha}^{\beta} \mathbf{P}(\mathbf{x}, \mathbf{p}; \widehat{R}_n(x, v)) \lambda^{(n)}(v) dv. \tag{3.1}$$

Proof. Using (1.5) in (1.2) and using linearity of the functional $\mathbf{P}(\cdot)$, we have

$$\mathbf{P}(\mathbf{x},\mathbf{p};\lambda(x)) = \mathbf{P}\left(\mathbf{x},\mathbf{p};\frac{1}{\beta-\alpha}\int_\alpha^\beta \lambda(t)dt\right) \tag{3.2}$$
$$+\frac{1}{\beta-\alpha}\sum_{l=0}^{n-2}\left(\frac{1}{(l+2)!}\right)\mathbf{P}(\mathbf{x},\mathbf{p};\lambda^{(l+1)}(x)\,(\alpha-x)^{l+2})$$
$$-\frac{1}{\beta-\alpha}\sum_{l=0}^{n-2}\left(\frac{1}{(l+2)!}\right)\mathbf{P}(\mathbf{x},\mathbf{p};\lambda^{(l+1)}(x)\,(\beta-x)^{l+2})$$
$$+\frac{1}{(n-1)!}\int_\alpha^\beta \mathbf{P}(\mathbf{x},\mathbf{p};\widehat{R}_n\,(x,v))\lambda^{(n)}(v)dv.$$

After simplification and following constant property of our functional, we get (3.1). ∎

In the following theorem we obtain generalizations of Popoviciu's inequality for $n-$convex functions.

THEOREM 3.2. *Let all the assumptions of Theorem 3.1 be satisfied and let for* $n \geq 2$

$$\mathbf{P}(\mathbf{x},\mathbf{p};\widehat{R}_n\,(x,v)) \geq 0, \quad v \in [\alpha,\beta]. \tag{3.3}$$

If λ *is* $n-$*convex function such that* $\lambda^{(n-1)}$ *is absolutely continuous, then we have*

$$\mathbf{P}(\mathbf{x},\mathbf{p};\lambda(x)) \geq \frac{1}{\beta-\alpha}\sum_{l=0}^{n-2}\left(\frac{1}{(l+2)!}\right) \tag{3.4}$$
$$\times\left(\mathbf{P}(\mathbf{x},\mathbf{p};\lambda^{(l+1)}(x)\,(\alpha-x)^{l+2}) - \mathbf{P}(\mathbf{x},\mathbf{p};\lambda^{(l+1)}(x)\,(\beta-x)^{l+2})\right).$$

PROOF. Since $\lambda^{(n-1)}$ is absolutely continuous on $[\alpha,\beta]$, $\lambda^{(n)}$ exists almost everywhere. As λ is $n-$convex, so $\lambda^{(n)}(x) \geq 0$ for all $x \in [\alpha,\beta]$ (see [16], p. 16). Hence we can apply Theorem 3.1 to obtain (3.4). ∎

REMARK 3.3. *The inequality* (3.4) *holds in reverse direction if reverse of* (3.3) *holds.*

REMARK 3.4. *We can also get results similar to other previous results given in Theorem 2.4.*

4. BOUNDS FOR IDENTITIES RELATED TO GENERALIZATION OF POPOVICIU'S INEQUALITY

In this section we present some interesting results by using Čebyšev functional and Grüss type inequalities. For two Lebesgue integrable functions

$f, h : [\alpha, \beta] \to \mathbb{R}$, we consider the Čebyšev functional

$$\Delta(f,h) = \frac{1}{\beta-\alpha}\int_\alpha^\beta f(t)h(t)dt - \frac{1}{\beta-\alpha}\int_\alpha^\beta f(t)dt.\frac{1}{\beta-\alpha}\int_\alpha^\beta h(t)dt.$$

The following Grüss type inequalities are given in [6].

Theorem 4.1. *Let $f : [\alpha, \beta] \to \mathbb{R}$ be a Lebesgue integrable function and $h : [\alpha, \beta] \to \mathbb{R}$ be an absolutely continuous function with $(. - \alpha)(\beta - .)[h']^2 \in L[\alpha, \beta]$. Then we have the inequality*

$$(4.1)\quad |\Delta(f,h)| \leq \frac{1}{\sqrt{2}}[\Delta(f,f)]^{\frac{1}{2}}\frac{1}{\sqrt{\beta-\alpha}}\left(\int_\alpha^\beta (x-\alpha)(\beta-x)[h'(x)]^2dx\right)^{\frac{1}{2}}.$$

The constant $\frac{1}{\sqrt{2}}$ in (4.1) is the best possible.

Theorem 4.2. *Assume that $h : [\alpha, \beta] \to \mathbb{R}$ is monotonic nondecreasing on $[\alpha, \beta]$ and $f : [\alpha, \beta] \to \mathbb{R}$ be an absolutely continuous with $f' \in L_\infty[\alpha, \beta]$. Then we have the inequality*

$$(4.2)\qquad |\Delta(f,h)| \leq \frac{1}{2(\beta-\alpha)}||f'||_\infty \int_\alpha^\beta (x-\alpha)(\beta-x)[h'(x)]^2dh(x).$$

The constant $\frac{1}{2}$ in (4.2) is the best possible.

In the sequel, we consider above theorems to derive generalizations of the results proved in the previous section. In order to avoid many notions let us denote

$$(4.3)\qquad \mathfrak{O}(v) = \mathbf{P}(\mathbf{x}, \mathbf{p}; R_n(x,v)) \geq 0, \;\; v \in [\alpha, \beta]$$

and

$$(4.4)\qquad \widehat{\mathfrak{O}}(v) = \mathbf{P}(\mathbf{x}, \mathbf{p}; \widehat{R}_n(x,v)) \geq 0, \;\; v \in [\alpha, \beta].$$

Consider the Čebyšev functional $\Delta(\mathfrak{O}, \mathfrak{O})$ and $\Delta(\widehat{\mathfrak{O}}, \widehat{\mathfrak{O}})$ given as:

$$(4.5)\qquad \Delta(\mathfrak{O},\mathfrak{O}) = \frac{1}{\beta-\alpha}\int_\alpha^\beta \mathfrak{O}^2(v)dv - \left(\frac{1}{\beta-\alpha}\int_\alpha^\beta \mathfrak{O}(v)dv\right)^2$$

and

$$(4.6)\qquad \Delta(\widehat{\mathfrak{O}},\widehat{\mathfrak{O}}) = \frac{1}{\beta-\alpha}\int_\alpha^\beta \widehat{\mathfrak{O}}^2(v)dv - \left(\frac{1}{\beta-\alpha}\int_\alpha^\beta \widehat{\mathfrak{O}}(v)dv\right)^2,$$

respectively.

Theorem 4.3. *Let $\lambda : [\alpha, \beta] \to \mathbb{R}$ be such that for $n \geq 2$, $\lambda^{(n)}$ is absolutely continuous with $(. - \alpha)(\beta - .)[\lambda^{(n+1)}]^2 \in L[\alpha, \beta]$. Let $m, k \in \mathbb{N}$, $m \geq 3$, $2 \leq k \leq m-1$, $[\alpha, \beta] \subset \mathbb{R}$, $\mathbf{x} = (x_1, ..., x_m) \in [\alpha, \beta]^m$, $\mathbf{p} = (p_1, ..., p_m)$ be a real m-tuple such that $\sum_{j=1}^k p_{i_j} \neq 0$ for any $1 \leq i_1 < ... < i_k \leq m$ and*

$\sum_{i=1}^{m} p_i = 1$. *Also let* $\frac{\sum_{j=1}^{k} p_{i_j} x_{i_j}}{\sum_{j=1}^{k} p_{i_j}} \in [\alpha, \beta]$ *for any* $1 \le i_1 < \ldots < i_k \le m$ *with* $\mathfrak{O}$, $\widehat{\mathfrak{O}}$ *defined in* (4.3) *and* (4.4) *respectively. Then*

$$\mathbf{P}(\mathbf{x},\mathbf{p};\lambda(x)) = \frac{1}{\beta-\alpha}\sum_{l=0}^{n-2}\left(\frac{1}{l!(l+2)}\right) \times \left(\lambda^{(l+1)}(\alpha)\,\mathbf{P}(\mathbf{x},\mathbf{p};(x-\alpha)^{l+2}) - \lambda^{(l+1)}(\beta)\,\mathbf{P}(\mathbf{x},\mathbf{p};(x-\beta)^{l+2})\right) + \frac{\lambda^{(n-1)}(\beta)-\lambda^{(n-1)}(\alpha)}{(\beta-\alpha)\,(n-1)!}\int_\alpha^\beta \mathfrak{O}(v)dv + \mathfrak{K}_n(\alpha,\beta;\lambda) \tag{4.7}$$

and

$$\mathbf{P}(\mathbf{x},\mathbf{p};\lambda(x)) = \frac{1}{\beta-\alpha}\sum_{l=0}^{n-2}\left(\frac{1}{(l+2)!}\right) \times \left(\mathbf{P}(\mathbf{x},\mathbf{p};\lambda^{(l+1)}(x)\,(\alpha-x)^{l+2}) - \mathbf{P}(\mathbf{x},\mathbf{p};\lambda^{(l+1)}(x)\,(\beta-x)^{l+2})\right) + \frac{\lambda^{(n-1)}(\beta)-\lambda^{(n-1)}(\alpha)}{(\beta-\alpha)\,(n-1)!}\int_\alpha^\beta \widehat{\mathfrak{O}}(v)dv + \widehat{\mathfrak{K}}_n(\alpha,\beta;\lambda) \tag{4.8}$$

where the remainder $\mathfrak{K}_n(\alpha,\beta;\lambda)$ *and* $\widehat{\mathfrak{K}}_n(\alpha,\beta;\lambda)$ *satisfies the bound*

$$|\mathfrak{K}_n(\alpha,\beta;\lambda)| \le \frac{1}{(n-1)!}[\Delta(\mathfrak{O},\mathfrak{O})]^{\frac{1}{2}}\sqrt{\frac{\beta-\alpha}{2}}\left|\int_\alpha^\beta (v-\alpha)(\beta-v)[\lambda^{(n+1)}(v)]^2dv\right|^{\frac{1}{2}}, \tag{4.9}$$

$$|\widehat{\mathfrak{K}}_n(\alpha,\beta;\lambda)| \le \frac{1}{(n-1)!}[\Delta(\widehat{\mathfrak{O}},\widehat{\mathfrak{O}})]^{\frac{1}{2}}\sqrt{\frac{\beta-\alpha}{2}}\left|\int_\alpha^\beta (v-\alpha)(\beta-v)[\lambda^{(n+1)}(v)]^2dv\right|^{\frac{1}{2}} \tag{4.10}$$

respectively.

Proof. If we apply Theorem 4.1 for $f \mapsto \mathfrak{O}$ and $h \mapsto \lambda^{(n)}$, we get

$$\left|\frac{1}{\beta-\alpha}\int_\alpha^\beta \mathfrak{O}(v)\lambda^{(n)}(v)dv - \frac{1}{\beta-\alpha}\int_\alpha^\beta \mathfrak{O}(v)dv.\frac{1}{\beta-\alpha}\int_\alpha^\beta \lambda^{(n)}(v)dv\right| \le \frac{1}{\sqrt{2}}[\Delta(\mathfrak{O},\mathfrak{O})]^{\frac{1}{2}}\frac{1}{\sqrt{\beta-\alpha}}\left|\int_\alpha^\beta (v-\alpha)(\beta-v)[\lambda^{(n+1)}(v)]^2dv\right|^{\frac{1}{2}}. \tag{4.11}$$

Divide both sides of (4.11) by $(n-1)!$ and multiplying by $(\beta-\alpha)$, we have

(4.12)
$$\left|\frac{1}{(n-1)!}\int_{\alpha}^{\beta}\mathfrak{O}(v)\lambda^{(n)}(v)dv-\frac{1}{(n-1)!}\int_{\alpha}^{\beta}\mathfrak{O}(v)dv.\frac{\lambda^{(n-1)}(\beta)-\lambda^{(n-1)}(\alpha)}{(\beta-\alpha)}\right|$$
$$\leq\frac{1}{(n-1)!}[\Delta(\mathfrak{O},\mathfrak{O})]^{\frac{1}{2}}\frac{\sqrt{\beta-\alpha}}{\sqrt{2}}\left|\int_{\alpha}^{\beta}(v-\alpha)(\beta-v)[\lambda^{(n+1)}(v)]^{2}dv\right|^{\frac{1}{2}}.$$

By denoting

$$\mathfrak{K}_n(\alpha,\beta;\lambda)=\frac{1}{(n-1)!}\int_{\alpha}^{\beta}\mathfrak{O}(v)\lambda^{(n)}(v)dv-\frac{1}{(n-1)!}\int_{\alpha}^{\beta}\mathfrak{O}(v)dv.\frac{\lambda^{(n-1)}(\beta)-\lambda^{(n-1)}(\alpha)}{(\beta-\alpha)} \tag{4.13}$$

in (4.12), we have (4.9). Hence, we have

$$\frac{1}{(n-1)!}\int_{\alpha}^{\beta}\mathfrak{O}(v)\lambda^{(n)}(v)dv=\frac{\lambda^{(n-1)}(\beta)-\lambda^{(n-1)}(\alpha)}{(\beta-\alpha)\,(n-1)!}\int_{\alpha}^{\beta}\mathfrak{O}(v)dv+\mathfrak{K}_n(\alpha,\beta;\lambda),$$

where the remainder $\mathfrak{K}_n(\alpha,\beta;\lambda)$ satisfies the estimation (4.9). Now from identity (2.1), we obtain (4.7).

Similarly from identity (3.1), we get (4.10). □

The following Grüss type inequalities can be obtained by using Theorem 4.2.

THEOREM 4.4. *Let* $\lambda:[\alpha,\beta]\to\mathbb{R}$ *be such that for* $n\geq 2$, $\lambda^{(n)}$ *is absolutely continuous and let* $\lambda^{(n+1)}\geq 0$ *on* $[\alpha,\beta]$ *with* $\mathfrak{O}$, $\widehat{\mathfrak{O}}$ *defined in* (4.3) *and* (4.4) *respectively. Then the representation* (4.7) *and the remainder* $\mathfrak{K}_n(\alpha,\beta;\lambda)$ *satisfies the estimation*

$$|\mathfrak{K}_n(\alpha,\beta;\lambda)|\leq\frac{(\beta-\alpha)||\mathfrak{O}'||_{\infty}}{(n-1)!}\left[\frac{\lambda^{(n-1)}(\beta)+\lambda^{(n-1)}(\alpha)}{2}-\frac{\lambda^{(n-2)}(\beta)-\lambda^{(n-2)}(\alpha)}{\beta-\alpha}\right], \tag{4.14}$$

where as the representation (4.8) *and the remainder* $\widehat{\mathfrak{K}}_n(\alpha,\beta;\lambda)$ *satisfies the estimation*

$$|\widehat{\mathfrak{K}}_n(\alpha,\beta;\lambda)|\leq\frac{(\beta-\alpha)||\widehat{\mathfrak{O}}'||_{\infty}}{(n-1)!}\left[\frac{\lambda^{(n-1)}(\beta)+\lambda^{(n-1)}(\alpha)}{2}-\frac{\lambda^{(n-2)}(\beta)-\lambda^{(n-2)}(\alpha)}{\beta-\alpha}\right]. \tag{4.15}$$

PROOF. Applying Theorem 4.2 for $f \mapsto \mathfrak{O}$ and $h \mapsto \lambda^{(n)}$, we get

$$\left|\frac{1}{\beta-\alpha}\int_\alpha^\beta \mathfrak{O}(v)\lambda^{(n)}(v)dv - \frac{1}{\beta-\alpha}\int_\alpha^\beta \mathfrak{O}(v)dv.\frac{1}{\beta-\alpha}\int_\alpha^\beta \lambda^{(n)}(v)dv\right| \le \frac{1}{2(\beta-\alpha)}||\mathfrak{O}'||_\infty \int_\alpha^\beta (v-\alpha)(\beta-v)\lambda^{(n+1)}(v)dv. \tag{4.16}$$

Since

$$\int_\alpha^\beta (v-\alpha)(\beta-v)\lambda^{(n+1)}(v)dv = \int_\alpha^\beta [2v-(\alpha+\beta)]\lambda^{(n)}(v)dv = (\beta-\alpha)\big[\lambda^{(n-1)}(\beta)+\lambda^{(n-1)}(\alpha)\big] - 2\big(\lambda^{(n-2)}(\beta)-\lambda^{(n-2)}(\alpha)\big).$$

Therefore, using identity (2.1) and the inequality (4.16), we deduce (4.14). Similarly using (3.1) instead of (2.1), we get (4.15) □

Now we intend to give the Ostrowski type inequalities related to generalizations of Popoviciu's inequality.

THEOREM 4.5. *Suppose all the assumptions of Theorem 2.1 be satisfied. Moreover, assume (p,q) is a pair of conjugate exponents, that is $p,q \in [1,\infty]$ such that $1/p+1/q=1$. Let $|\lambda^{(n)}|^p : [\alpha,\beta] \to \mathbb{R}$ be a R-integrable function for some $n \ge 2$. Then, we have*

$$\Bigg|\mathbf{P}(\mathbf{x},\mathbf{p};\lambda(x)) - \tfrac{1}{\beta-\alpha}\textstyle\sum_{l=0}^{n-2}\left(\tfrac{1}{l!(l+2)}\right) \times\Bigg(\lambda^{(l+1)}(\alpha)\,\mathbf{P}(\mathbf{x},\mathbf{p};(x-\alpha)^{l+2}) - \lambda^{(l+1)}(\beta)\,\mathbf{P}(\mathbf{x},\mathbf{p};(x-\beta)^{l+2})\Bigg)\Bigg| \le \tfrac{1}{(n-1)!}||\lambda^{(n)}||_p\Bigg(\int_\alpha^\beta\Big|\mathbf{P}(\mathbf{x},\mathbf{p};R_n(x,v))\Big|^q dv\Bigg)^{1/q}, \tag{4.17}$$

$$\Bigg|\mathbf{P}(\mathbf{x},\mathbf{p};\lambda(x)) - \tfrac{1}{\beta-\alpha}\textstyle\sum_{l=0}^{n-2}\left(\tfrac{1}{(l+2)!}\right) \times\Bigg(\mathbf{P}(\mathbf{x},\mathbf{p};\lambda^{(l+1)}(x)\,(\alpha-x)^{l+2}) - \mathbf{P}(\mathbf{x},\mathbf{p};\lambda^{(l+1)}(x)\,(\beta-x)^{l+2})\Bigg)\Bigg| \le \tfrac{1}{(n-1)!}||\lambda^{(n)}||_p\Bigg(\int_\alpha^\beta\Big|\mathbf{P}(\mathbf{x},\mathbf{p};\widehat{R}_n(x,v))\Big|^q dv\Bigg)^{1/q}. \tag{4.18}$$

The constant on the R.H.S. of (4.17) and (4.18) are sharp for $1 < p \le \infty$ and the best possible for $p = 1$.

PROOF. Let us denote

$$\mathfrak{I} = \frac{1}{(n-1)!}\Big(\mathbf{P}(\mathbf{x},\mathbf{p};R_n(x,v))\Big), \quad v \in [\alpha,\beta].$$

Using identity (2.1), we obtain

$$\left|\mathbf{P}(\mathbf{x},\mathbf{p};\lambda(x)) - \tfrac{1}{\beta-\alpha}\sum_{l=0}^{n-2}\left(\tfrac{1}{l!(l+2)}\right)\right. \tag{4.19}$$

$$\left.\times\left(\lambda^{(l+1)}(\alpha)\,\mathbf{P}(\mathbf{x},\mathbf{p};(x-\alpha)^{l+2}) - \lambda^{(l+1)}(\beta)\,\mathbf{P}(\mathbf{x},\mathbf{p};(x-\beta)^{l+2})\right)\right|$$

$$= \left|\int_\alpha^\beta \mathfrak{I}(v)\lambda^{(n)}(v)dv\right|.$$

Apply Hölder's inequality for integrals on the right hand side of (4.19), we have

$$\left|\int_\alpha^\beta \mathfrak{I}(v)\lambda^{(n)}(v)dv\right| \leq \left(\int_\alpha^\beta \left|\lambda^{(n)}(v)\right|^p dv\right)^{\frac{1}{p}}\left(\int_\alpha^\beta |\mathfrak{I}(v)|^q\, dv\right)^{\frac{1}{q}},$$

which combine together with (4.19) gives (4.17).

For the proof of the sharpness of the constant $\left(\int_\alpha^\beta |\mathfrak{I}(v)|^q dv\right)^{1/q}$, let us define the function λ for which the equality in (4.17) is obtained.

For $1 < p \leq \infty$ take λ to be such that

$$\lambda^{(n)}(v) = \operatorname{sgn}\mathfrak{I}(v)|\mathfrak{I}(v)|^{\frac{1}{p-1}}.$$

For $p=\infty$ take $\lambda^{(n)}(v) = \operatorname{sgn}\mathfrak{I}(v)$.

For $p=1$, we prove that

$$\left|\int_\alpha^\beta \mathfrak{I}(v)\lambda^{(n)}(v)dv\right| \leq \max_{v\in[\alpha,\beta]}|\mathfrak{I}(v)|\left(\int_\alpha^\beta \lambda^{(n)}(v)dv\right) \tag{4.20}$$

is the best possible inequality. Suppose that $|\mathfrak{I}(v)|$ attains its maximum at $v_0 \in [\alpha,\beta]$. To start with first we assume that $\mathfrak{I}(v_0) > 0$. For δ small enough we define $\lambda_\delta(v)$ by

$$\lambda_\delta(v) = \begin{cases} 0\,, & \alpha \leq v \leq t_0\,,\\ \frac{1}{\delta n!}(v-v_0)^n\,, & v_o \leq v \leq v_0+\delta\,,\\ \frac{1}{n!}(v-v_0)^{n-1}\,, & v_0+\delta \leq v \leq \beta\,. \end{cases}$$

Then for δ small enough

$$\left|\int_\alpha^\beta \mathfrak{I}(v)\lambda^{(n)}(v)dv\right| = \left|\int_{v_0}^{v_0+\delta}\mathfrak{I}(v)\frac{1}{\delta}dv\right| = \frac{1}{\delta}\int_{v_0}^{v_0+\delta}\mathfrak{I}(v)dv.$$

Now from inequality (4.20), we have

$$\frac{1}{\delta}\int_{v_0}^{v_0+\delta}\mathfrak{I}(v)dv \leq \mathfrak{I}(v_0)\int_{v_0}^{v_0+\delta}\frac{1}{\delta}dv = \mathfrak{I}(v_0).$$

Since

$$\lim_{\delta\to 0}\frac{1}{\delta}\int_{v_0}^{v_0+\delta}\mathfrak{I}(v)dv = \mathfrak{I}(v_0),$$

the statement follows. The case when $\Im(v_0) < 0$, we define $\lambda_\delta(v)$ by

$$\lambda_\delta(v) = \begin{cases} \frac{1}{n!}(v - v_0 - \delta)^{n-1}, & \alpha \le v \le v_0, \\ \frac{-1}{\delta n!}(v - v_0 - \delta)^{n}, & v_o \le v \le v_0 + \delta, \\ 0, & v_0 + \delta \le v \le \beta, \end{cases}$$

and rest of the proof is the same as above.

The proof of (4.18) is also similar, but we use (3.1) instead of (2.1). □

5. Mean Value Theorems and $n-$exponential convexity

We recall some definitions and basic results from [5], [8] and [15] which are required in sequel.

Definition 5.1. *A function $\lambda : I \to \mathbb{R}$ is $n-$exponentially convex in the Jensen sense on I if*

$$\sum_{i,j=1}^{n} \xi_i \xi_j \, \lambda\left(\frac{x_i + x_j}{2}\right) \ge 0,$$

hold for all choices $\xi_1, \ldots, \xi_n \in \mathbb{R}$ and all choices $x_1, \ldots, x_n \in I$. A function $\lambda : I \to \mathbb{R}$ is $n-$exponentially convex if it is $n-$exponentially convex in the Jensen sense and continuous on I.

Definition 5.2. *A function $\lambda : I \to \mathbb{R}$ is exponentially convex in the Jensen sense on I if it is $n-$exponentially convex in the Jensen sense for all $n \in \mathbb{N}$.*

A function $\lambda : I \to \mathbb{R}$ is exponentially convex if it is exponentially convex in the Jensen sense and continuous.

Proposition 5.1. *If $\lambda : I \to \mathbb{R}$ is an $n-$exponentially convex in the Jensen sense, then the matrix $\left[\lambda\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{m}$ is a positive semi-definite matrix for all $m \in \mathbb{N}, m \le n$. Particularly,*

$$\det\left[\lambda\left(\frac{x_i + x_j}{2}\right)\right]_{i,j=1}^{m} \ge 0$$

for all $m \in \mathbb{N}$, $m = 1, 2, ..., n$.

Remark 5.2. It is known that $\lambda : I \to \mathbb{R}$ is a log-convex in the Jensen sense if and only if

$$\alpha^2 \lambda(x) + 2\alpha\beta\lambda\left(\frac{x+y}{2}\right) + \beta^2 \lambda(y) \ge 0,$$

holds for every $\alpha, \beta \in \mathbb{R}$ and $x, y \in I$. It follows that a positive function is log-convex in the Jensen sense if and only if it is $2-$exponentially convex in the Jensen sense.

A positive function is log-convex if and only if it is $2-$exponentially convex.

Remark 5.3. *By the virtue of Theorem 2.2, we define the positive linear functionals with respect to $n-$convex function λ as follows*

$$\Omega_1(\lambda) := \mathbf{P}(\mathbf{x},\mathbf{p};\lambda(x)) - \frac{1}{\beta-\alpha}\sum_{l=0}^{n-2}\left(\frac{1}{l!(l+2)}\right) \times\left(\lambda^{(l+1)}(\alpha)\,\mathbf{P}(\mathbf{x},\mathbf{p};(x-\alpha)^{l+2}) - \lambda^{(l+1)}(\beta)\,\mathbf{P}(\mathbf{x},\mathbf{p};(x-\beta)^{l+2})\right) \geq 0, \tag{5.1}$$

and

$$\Omega_2(\lambda) := \mathbf{P}(\mathbf{x},\mathbf{p};\lambda(x)) - \frac{1}{\beta-\alpha}\sum_{l=0}^{n-2}\left(\frac{1}{(l+2)!}\right) \times\left(\mathbf{P}(\mathbf{x},\mathbf{p};\lambda^{(l+1)}(x)\,(\alpha-x)^{l+2}) - \mathbf{P}(\mathbf{x},\mathbf{p};\lambda^{(l+1)}(x)\,(\beta-x)^{l+2})\right) \geq 0. \tag{5.2}$$

Lagrange and Cauchy type mean value theorems related to defined functional is given in the following theorems.

Theorem 5.4. *Let $\lambda : [\alpha,\beta] \to \mathbb{R}$ be such that $\lambda \in C^n[\alpha,\beta]$. If the inequalities in* (2.3) *and* (3.3) *are valid, then there exist $\xi_i \in [\alpha,\beta]$ such that*

$$\Omega_i(\lambda) = \lambda^{(n)}(\xi)\Omega_i(\varphi); \quad i = 1,2,$$

where $\varphi(x) = \frac{x^n}{n!}$ and $\Omega_i(\cdot)$ are defined in Remark 5.3.

Proof. Similar to the proof of Theorem 4.1 in [9]. □

Theorem 5.5. *Let $\lambda,\psi : [\alpha,\beta] \to \mathbb{R}$ be such that $\lambda,\psi \in C^n[\alpha,\beta]$. If the inequalities in* (2.3) *and* (3.3) *are valid, then there exist $\xi_i \in [\alpha,\beta]$ such that*

$$\frac{\Omega_i(\lambda)}{\Omega_i(\psi)} = \frac{\lambda^{(n)}(\xi)}{\psi^{(n)}(\xi)}; \quad i = 1,2,$$

provided that the denominators are non-zero and $\Omega_i(\cdot)$ are defined in Remark 5.3.

Proof. Similar to the proof of Corollary 4.2 in [9]. □

Theorem 5.5 enables us to define Cauchy means, because if

$$\xi_i = \left(\frac{\lambda^{(n)}}{\psi^{(n)}}\right)^{-1}\left(\frac{\Omega_i(\lambda)}{\Omega_i(\psi)}\right),$$

which show that ξ_i $(i = 1,2)$ are means of α, β for given functions λ and ψ.

Next we construct the non trivial examples of $n-$exponentially and exponentially convex functions from positive linear functionals $\Omega_i(\cdot)$ $(i = 1,2)$. We use the idea given in [15]. In the sequel I and J are intervals in $\mathbb{R}$.

THEOREM 5.6. *Let $\Theta = \{\lambda_t : t \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval I in $\mathbb{R}$ such that the function $t \mapsto [x_0, \ldots, x_n; \lambda_t]$ is $n-$exponentially convex in the Jensen sense on J for every $(n+1)$ mutually different points $x_0, \ldots, x_n \in I$. Then for the linear functionals $\Omega_i(\lambda_t)$ $(i = 1, 2)$ as defined in Remark 5.3, the following statements are valid for each $i = 1, 2$:*

(i) The function $t \mapsto \Omega_i(\lambda_t)$ is $n-$exponentially convex in the Jensen sense on J and the matrix $[\Omega_i(\lambda_{\frac{t_j+t_l}{2}})]_{j,l=1}^{m}$ is a positive semi-definite for all $m \in \mathbb{N}, m \leq n$, $t_1, .., t_m \in J$. Particularly,

$$\det[\Omega_i(\lambda_{\frac{t_j+t_l}{2}})]_{j,l=1}^{m} \geq 0 \text{ for all } m \in \mathbb{N},\ m = 1, 2, ..., n.$$

(ii) If the function $t \mapsto \Omega_i(\lambda_t)$ is continuous on J, then it is $n-$exponentially convex on J.

PROOF. Fix $i = 1, 2$.
(i) For $\xi_j \in \mathbb{R}$ and $t_j \in J$, $j = 1, \ldots, n$, we define the function

$$h(x) = \sum_{j,l=1}^{n} \xi_j \xi_l \lambda_{\frac{t_j+t_l}{2}}(x).$$

Using the assumption that the function $t \mapsto [x_0, \ldots, x_n; \lambda_t]$ is $n-$exponentially convex in the Jensen sense, we have

$$[x_0, \ldots, x_n, h] = \sum_{j,l=1}^{n} \xi_j \xi_l [x_0, \ldots, x_n; \lambda_{\frac{t_j+t_l}{2}}] \geq 0,$$

which in turn implies that h is a $n-$convex function on J, therefore from Remark 5.3 we have $\Omega_i(h) \geq 0$. The linearity of $\Omega_i(\cdot)$ gives

$$\sum_{j,l=1}^{n} \xi_j \xi_l \Omega_i(\lambda_{\frac{t_j+t_l}{2}}) \geq 0.$$

We conclude that the function $t \mapsto \Omega_i(\lambda_t)$ is $n-$exponentially convex on J in the Jensen sense.

The remaining part follows from Proposition 5.1.

(ii) If the function $t \mapsto \Omega_i(\lambda_t)$ is continuous on J, then it is $n-$exponentially convex on J by definition. ∎

The following corollary is an immediate consequence of the above theorem.

COROLLARY 5.7. *Let $\Theta = \{\lambda_t : t \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval I in $\mathbb{R}$, such that the function $t \mapsto [x_0, \ldots, x_n; \lambda_t]$ is exponentially convex in the Jensen sense on J for every $(n+1)$ mutually different points $x_0, \ldots, x_n \in I$. Then for the linear functional $\Omega_i(\lambda_t)$ $(i = 1, 2)$, the following statements hold:*

(i) The function $t \mapsto \Omega_i(\lambda_t)$ *is exponentially convex in the Jensen sense on* J *and the matrix* $[\Omega_i(\lambda_{\frac{t_j+t_l}{2}})]_{j,l=1}^m$ *is a positive semi-definite for all* $m \in \mathbb{N}, m \leq n,\ t_1, .., t_m \in J$*. Particularly,*

$$\det[\Omega_i(\lambda_{\frac{t_j+t_l}{2}})]_{j,l=1}^m \geq 0 \text{ for all } m \in \mathbb{N},\ m = 1, 2, ..., n.$$

(ii) If the function $t \mapsto \Omega_i(\lambda_t)$ *is continuous on* J*, then it is exponentially convex on* J*.*

Corollary 5.8. *Let* $\Theta = \{\lambda_t : t \in J\}$*, where* J *is an interval in* $\mathbb{R}$*, be a family of functions defined on an interval* I *in* $\mathbb{R}$*, such that the function* $t \mapsto [x_0, \ldots, x_n; \lambda_t]$ *is* $2-$*exponentially convex in the Jensen sense on* J *for every* $(n+1)$ *mutually different points* $x_0, \ldots, x_n \in I$*. Let* $\Omega_i(\cdot)$ $(i = 1, 2)$ *be linear functionals, then the following statements hold:*

(i) If the function $t \mapsto \Omega_i(\lambda_t)$ *is continuous on* J*, then it is* $2-$*exponentially convex function on* J*. If* $t \mapsto \Omega_i(\lambda_t)$ *is additionally strictly positive, then it is also log-convex on* J*. Furthermore, the following inequality holds true:*

$$[\Omega_i(\lambda_s)]^{t-r} \leq [\Omega_i(\lambda_r)]^{t-s} [\Omega_i(\lambda_t)]^{s-r},$$

for every choice $r, s, t \in J$*, such that* $r < s < t$*.*

(ii) If the function $t \mapsto \Omega_i(\lambda_t)$ *is strictly positive and differentiable on* J*, then for every* $p, q, u, v \in J$*, such that* $p \leq u$ *and* $q \leq v$*, we have*

$$\mu_{p,q}(\Omega_i, \Theta) \leq \mu_{u,v}(\Omega_i, \Theta), \tag{5.3}$$

where

$$\mu_{p,q}(\Omega_i, \Theta) = \begin{cases} \left(\frac{\Omega_i(\lambda_p)}{\Omega_i(\lambda_q)}\right)^{\frac{1}{p-q}}, & p \neq q, \\ \exp\left(\frac{\frac{d}{dp}\Omega_i(\lambda_p)}{\Omega_i(\lambda_p)}\right), & p = q, \end{cases} \tag{5.4}$$

for $\lambda_p, \lambda_q \in \Theta$*.*

Proof. Fix $i = 1, 2$.

(i) This is an immediate consequence of Theorem 5.6 and Remark 5.2.

(ii) Since $p \mapsto \Omega_i(\lambda_t)$ is positive and continuous, by (i) we have that $t \mapsto \Omega_i(\lambda_t)$ is log-convex on J, that is, the function $t \mapsto \log \Omega_i(\lambda_t)$ is convex on J. Hence we get

$$\frac{\log \Omega_i(\lambda_p) - \log \Omega_i(\lambda_q)}{p-q} \leq \frac{\log \Omega_i(\lambda_u) - \log \Omega_i(\lambda_v)}{u-v}, \tag{5.5}$$

for $p \leq u, q \leq v, p \neq q, u \neq v$. So, we conclude that

$$\mu_{p,q}(\Omega_i, \Theta) \leq \mu_{u,v}(\Omega_i, \Theta).$$

Cases $p = q$ and $u = v$ follow from (5.5) as limit cases.

∎

6. Examples

In this section, we present some families of functions which fulfil the conditions of Theorem 5.6, Corollary 5.7 and Corollary 5.8. This enables us to construct a large families of functions which are exponentially convex. Explicit form of this functions is obtained after we calculate explicit action of functionals on a given family.

Example 6.1. Let us consider a family of functions

$$\Theta_1 = \{\lambda_t : \mathbb{R} \to \mathbb{R} : t \in \mathbb{R}\}$$

defined by

$$\lambda_t(x) = \begin{cases} \frac{e^{tx}}{t^n}, & t \neq 0, \\ \frac{x^n}{n!}, & t = 0. \end{cases}$$

Since $\frac{d^n \lambda_t}{dx^n}(x) = e^{tx} > 0$, the function λ_t is $n-$convex on $\mathbb{R}$ for every $t \in \mathbb{R}$ and $t \mapsto \frac{d^n \lambda_t}{dx^n}(x)$ is exponentially convex by definition. Using analogous arguing as in the proof of Theorem 5.6 we also have that $t \mapsto [x_0, \ldots, x_n; \lambda_t]$ is exponentially convex (and so exponentially convex in the Jensen sense). Now, using Corollary 5.7 we conclude that $t \mapsto \Omega_i(\lambda_t)$ $(i = 1, 2)$ are exponentially convex in the Jensen sense. It is easy to verify that this mapping is continuous (although the mapping $t \mapsto \lambda_t$ is not continuous for $t = 0$), so it is exponentially convex. For this family of functions, $\mu_{t,q}(\Omega_i, \Theta_1)$ from (5.4), becomes

$$\mu_{t,q}(\Omega_i, \Theta_1) = \begin{cases} \left(\frac{\Omega_i(\lambda_t)}{\Omega_i(\lambda_q)}\right)^{\frac{1}{t-q}}, & t \neq q, \\ \exp\left(\frac{\Omega_i(id \cdot \lambda_t)}{\Omega_i(\lambda_t)} - \frac{n}{t}\right), & t = q \neq 0, \\ \exp\left(\frac{1}{n+1} \frac{\Omega_i(id \cdot \lambda_0)}{\Omega_i(\lambda_0)}\right), & t = q = 0, \end{cases} \qquad i = 1, 2$$

where "id" is the identity function. By Corollary 5.8 $\mu_{t,q}(\Omega_i, \Theta_1)$ is a monotone function in parameters t and q.

Since

$$\left(\frac{\frac{d^n f_t}{dx^n}}{\frac{d^n f_q}{dx^n}}\right)^{\frac{1}{t-q}} (\log x) = x,$$

using Theorem 5.5 it follows that

$$M_{t,q}(\Omega_i, \Theta_1) = \log \mu_{t,q}(\Omega_i, \Theta_1), \quad i = 1, 2$$

satisfies

$$\alpha \leq M_{t,q}(\Omega_i, \Theta_1) \leq \beta, \quad i = 1, 2.$$

Hence $M_{t,q}(\Omega_i, \Theta_1)$ $(i = 1, 2)$ are monotonic means.

Example 6.2. Let us consider a family of functions

$$\Theta_2 = \{g_t : (0, \infty) \to \mathbb{R} : t \in \mathbb{R}\}$$

defined by

$$g_t(x) = \begin{cases} \frac{x^t}{t(t-1)\cdots(t-n+1)}, & t \notin \{0,1,\ldots,n-1\}, \\ \frac{x^j \log x}{(-1)^{n-1-j} j!(n-1-j)!}, & t = j \in \{0,1,\ldots,n-1\}. \end{cases}$$

Since $\frac{d^n g_t}{dx^n}(x) = x^{t-n} > 0$, the function g_t is $n-$convex for $x > 0$ and $t \mapsto \frac{d^n g_t}{dx^n}(x)$ is exponentially convex by definition. Arguing as in Example 6.1 we get that the mappings $t \mapsto \Omega_i(g_t)$ is exponentially convex for each $i = 1, 2$. Hence, for this family of functions $\mu_{p,q}(\Omega_i, \Theta_2)$ $(i = 1, 2)$, from (5.4), are equal to

$$\mu_{t,q}(\Omega_i, \Theta_2) =$$

$$\begin{cases} \left(\frac{\Omega_i(g_t)}{\Omega_i(g_q)}\right)^{\frac{1}{t-q}}, & t \neq q, \\ \exp\left((-1)^{n-1}(n-1)!\frac{\Omega_i(g_0 g_t)}{\Omega_i(g_t)} + \sum\limits_{k=0}^{n-1} \frac{1}{k-t}\right), & t = q \notin \{0,1,\ldots,n-1\}, \\ \exp\left((-1)^{n-1}(n-1)!\frac{\Omega_i(g_0 g_t)}{2\Omega_i(g_t)} + \sum\limits_{\substack{k=0 \\ k\neq t}}^{n-1} \frac{1}{k-t}\right), & t = q \in \{0,1,\ldots,n-1\}. \end{cases}$$

Again, using Theorem 5.5 we conclude that

$$\alpha \leq \left(\frac{\Omega_i(g_t)}{\Omega_i(g_q)}\right)^{\frac{1}{t-q}} \leq \beta, \quad i = 1, 2.$$

Hence $\mu_{t,q}(\Omega_i, \Theta_2)$ $(i = 1, 2)$ are means and their monotonicity follows by (5.3).

Example 6.3. Let

$$\Theta_3 = \{\zeta_t : (0,\infty) \to \mathbb{R} : t \in (0,\infty)\}$$

be a family of functions defined by

$$\zeta_t(x) = \begin{cases} \frac{t^{-x}}{(-\log t)^n}, & t \neq 1; \\ \frac{x^n}{(n)!}, & t = 1. \end{cases}$$

Since $\frac{d^n \zeta_t}{dx^n}(x) = t^{-x}$ is the Laplace transform of a non-negative function (see [19]) it is exponentially convex. Obviously ζ_t are $n-$convex functions for every $t > 0$.

For this family of functions, $\mu_{t,q}(\Omega_i, \Theta_3)$, in this case for $[\alpha, \beta] \subset \mathbb{R}^+$, from (5.4) become

$$\mu_{t,q}(\Omega_i, \Theta_3) = \begin{cases} \left(\frac{\Omega_i(\zeta_t)}{\Omega_i(\zeta_q)}\right)^{\frac{1}{t-q}}, & t \neq q; \\ \exp\left(-\frac{\Omega_i(id.\zeta_t)}{t\Omega_i(\zeta_t)} - \frac{n}{t \log t}\right), & t = q \neq 1; \\ \exp\left(-\frac{1}{n+1}\frac{\Omega_i(id.\zeta_1)}{\Omega_i(\zeta_1)}\right), & t = q = 1, \end{cases} \quad i = 1, 2$$

where "*id*" is the identity function. By Corollary 5.8 $\mu_{p,q}(\Omega_i, \Theta_3)$ $(i = 1, 2)$ are monotone functions in parameters t and q.
Using Theorem 5.5 it follows that

$$M_{t,q}(\Omega_i, \Theta_3) = -L(t,q) \log \mu_{t,q}(\Omega_i, \Theta_3), \quad i = 1, 2$$

satisfy

$$\alpha \leq M_{t,q}(\Omega_i, \Theta_3) \leq \beta, \quad i = 1, 2.$$

This shows that $M_{t,q}(\Omega_i, \Theta_3)$ is a mean for each $i = 1, 2$. Because of the inequality (5.3), these means are monotonic. Furthermore, $L(t, q)$ is logarithmic mean defined by

$$L(t,q) = \begin{cases} \frac{t-q}{\log t - \log q}, & \text{t} \neq \text{q}; \\ \\ t, & \text{t} = \text{q}. \end{cases}$$

Example 6.4. Let

$$\Theta_4 = \{\gamma_t : (0, \infty) \to \mathbb{R} : t \in (0, \infty)\}$$

be a family of functions defined by

$$\gamma_t(x) = \frac{e^{-x\sqrt{t}}}{\left(-\sqrt{t}\right)^n}.$$

Since $\frac{d^n \gamma_t}{dx^n}(x) = e^{-x\sqrt{t}}$ is the Laplace transform of a non-negative function (see [19]) it is exponentially convex. Obviously γ_t are $n-$convex function for every $t > 0$.
For this family of functions, $\mu_{t,q}(\Omega_i, \Theta_4)$ $(i = 1, 2)$, in this case for $[\alpha, \beta] \subset \mathbb{R}^+$, from (5.4) become

$$\mu_{t,q}(\Omega_i, \Theta_4) = \begin{cases} \left(\frac{\Omega_i(\gamma_t)}{\Omega_i(\gamma_q)}\right)^{\frac{1}{t-q}}, & t \neq q; \\ \exp\left(-\frac{\Omega_i(id.\gamma_t)}{2\sqrt{t}\Omega_i(\gamma_t)} - \frac{n}{2t}\right), & t = q; \end{cases} \quad i = 1, 2.$$

By Corollary 5.8, these are monotone functions in parameters t and q.
Using Theorem 5.5 it follows that

$$M_{t,q}(\Omega_i, \Theta_4) = -\left(\sqrt{t} + \sqrt{q}\right) \log \mu_{t,q}(\Omega_i, \Theta_4), i = 1, 2$$

satisfy

$$\alpha \leq M_{t,q}(\Omega_i, \Theta_4) \leq \beta, i = 1, 2.$$

This shows that $M_{t,q}(\Omega_i, \Theta_4)$ $(i = 1, 2)$ are means. Because of the above inequality (5.3), these means are monotonic.

Remark 6.5. The examples presented in this section are analogous to the examples given in [1, Section 5] and [2, Section 5].

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Težinske nejednakosti Popoviciuovog tipa preko poopćenih Montgomeryevih identiteta

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