MATHEMATICAL COMMUNICATIONS
Math. Commun. **20**(2015), 83–95

k-type null slant helices in Minkowski space-time

Emilija Nešović[1], Esra Betül Koç Öztürk[2,*] and Ufuk Öztürk[2]

[1] *Department of Mathematics and Informatics, Faculty of Science, University of Kragujevac, SRB-34 000 Kragujevac, Serbia*
[2] *Department Mathematics, Faculty of Science, University of Cankiri Karatekin, TR-18 100 Cankiri, Turkey*

Received July 11, 2014; accepted January 30, 2015

Abstract. In this paper, we introduce the notion of a k-type null slant helices in Minkowski space-time, where $k \in \{0, 1, 2, 3\}$. We give necessary and sufficient conditions for null Cartan curves to be k-type null slant helices in terms of their curvatures κ_2 and κ_3. In particular, we characterize k-type null slant helices lying on the pseudosphere $S_1^3(r)$. We find the relationships between 0-type and 1-type null slant helices, as well as between 1-type and 2-type null slant helices. Moreover, we prove that there are no 3-type null slant helices in Minkowski space-time.

AMS subject classifications: 53C40, 53C50

Key words: null Cartan curve, k-type null slant helix, Minkowski space-time

1. Introduction

In the Euclidean 3-space, it is well-known that a general helix (or a curve of constant slope) is a curve whose tangent makes a constant angle with a fixed direction, which is called the axis of the helix. The ratio of the curvature and the torsion of such curve is constant, which is the necessary and sufficient condition for a curve to be a general helix. Several authors introduced different types of helices and investigated their properties.

In [8], a slant helix is defined as a curve having the property that its principal normal vector makes a constant angle with a fixed direction (see also [9]). In [2] and [6], some characterizations of slant helices in Minkowski 3-space are given. In Lorentz–Minkowski spaces, null generalized helices are studied in [7]. In [3], k-type spacelike slant helices in Minkowski space-time are defined and characterized.

In this paper, we introduce the notion of k-type null slant helices in Minkowski space-time, where $k \in \{0, 1, 2, 3\}$. We give necessary and sufficient conditions for null Cartan curves to be k-type null slant helices in terms of their curvatures κ_2 and κ_3. In particular, we characterize k-type null slant helices lying on the pseudosphere $S_1^3(r)$. We find the relationships between 0-type and 1-type null slant helices, as well as between 1-type and 2-type null slant helices. Moreover, we prove that there are no 3-type null slant helices in Minkowski space-time.

*Corresponding author. *Email addresses:* `nesovickg@sbb.rs` (E. Nešović), `e.betul.e@gmail.com` (E. B. Koç Öztürk), `ozturkufuk06@gmail.com` (U. Öztürk)

2. Preliminaries

The Minkowski space-time $\mathbb{E}_1^4$ is a 4-dimensional affine space endowed with an indefinite flat metric g with signature $(-,+,+,+)$. This means that there are affine coordinates (x_1, x_2, x_3, x_4) such that metric bilinear form can be written as

$$g(x,y) = -x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4, \tag{1}$$

for any two $x = (x_1, x_2, x_3, x_4)$ and $y = (y_1, y_2, y_3, y_4)$ in $\mathbb{E}_1^4$. Since g is an indefinite metric, recall that a vector $v \in \mathbb{E}_1^4 \backslash \{0\}$ can be *spacelike* if $g(v,v) > 0$, *timelike* if $g(v,v) < 0$ and *null (lightlike)* if $g(v,v) = 0$. The vector $v = 0$ is said to be spacelike. The norm of a vector v is given by $||v|| = \sqrt{|g(v,v)|}$. Two vectors v and w are said to be orthogonal, if $g(v,w) = 0$. An arbitrary curve $\alpha(s)$ in $\mathbb{E}_1^4$ can locally be *spacelike*, *timelike* or *null* (*lightlike*) if all its velocity vectors $\alpha'(s)$ are spacelike, timelike or null, respectively ([4]). A non-null curve α is parametrized by the arc-length parameter s (or has the unit speed) if $g(\alpha'(s), \alpha'(s)) = \pm 1$. In particular, a null curve α is said to be parameterized by the *pseudo-arc* s if $g\left(\alpha''\left(s\right),\ \alpha''\left(s\right)\right) = 1$, where a *pseudo-arc function* s is defined by $s(t) = \int_0^t (g(\alpha''(u), \alpha''(u)))^{\frac{1}{4}}\, du$ in [4].

Consider a null curve $\alpha = \alpha(s)$ in E_1^4 parameterized by the pseudo-arc s. Then there exists a unique pseudo-orthonormal frame $\{T, N, B_1, B_2\}$ along α, where $T(s) = \alpha'(s)$, $N(s) = \alpha''(s)$, $B_1(s)$ and $B_2(s)$ are the tangent, the principal normal, the first binormal and the second binormal vector of α, respectively, satisfying the conditions

$$\begin{aligned} &g(T,T) = g(B_1,B_1) = 0, \quad g(N,N) = g(B_2,B_2) = 1, \\ &g(T,N) = g(T,B_2) = g(N,B_1) = g(N,B_2) = g(B_1,B_2) = 0, \quad g(T,B_1) = 1. \end{aligned}$$

The frame $\{T, N, B_1, B_2\}$ is called the *Cartan frame* of α, and it consists of two linearly independent null vectors T and B_1 of two spacelike vectors N and B_2. Any two vectors of the Cartan frame up to the pair (T, B_1) are orthogonal. It is positively oriented if $det(T, N, B_1, B_2) = 1$.

Definition 1. *A non-geodesic null curve $\alpha : I \mapsto \mathbb{E}_1^4$ parameterized by the pseudo-arc s is called a Cartan curve if there exists a unique positively oriented Cartan frame $\{T, N, B_1, B_2\}$ along α and three smooth functions κ_1, κ_2 and κ_3 satisfying Cartan equations [5]*

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & 0 & 0 \\ \kappa_2 & 0 & -\kappa_1 & 0 \\ 0 & -\kappa_2 & 0 & \kappa_3 \\ -\kappa_3 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}. \tag{2}$$

The functions $\kappa_1(s) = 1, \kappa_2(s)$ and $\kappa_3(s)$ are called the first, the second and the third Cartan curvature of α.

Let us set $T(s) = V_1(s)$, $N(s) = V_2(s)$, $B_1(s) = V_3(s)$ and $B_2 = V_4(s)$.

Definition 2. *A null Cartan curve α with the Cartan frame $\{V_1, V_2, V_3, V_4\}$ in the Minkowski space-time $\mathbb{E}_1^4$ is called a k-type null slant helix if there exists a non-zero fixed direction $U \in E_1^4$ such that there holds*

$$g\left(V_{k+1}, U\right) = const,$$

for $0 \leq k \leq 3$. The fixed direction U is called an axis of the helix.

In particular, 0-type null slant helices are generalized null helices and 1-type null slant helices are null slant helices.

3. k-type null slant helices in Minkowski space-time

In this section, we will consider only non-geodesic null Cartan curves lying in $\mathbb{E}_1^4$, i.e., the null Cartan curves with the first curvature $\kappa_1(s) = 1$ and the third curvature $\kappa_3(s) \neq 0$ for each s. When the third curvature $\kappa_3(s)$ is a non-zero function, note that the second curvature $\kappa_2(s)$ can be equal to zero or different from zero. Let $\mathbb{R}_0$ denote $\mathbb{R}\backslash\{0\}$.

Theorem 1. *Let $\alpha : I \to \mathbb{E}_1^4$ be a null Cartan curve in $\mathbb{E}_1^4$. Then α is a 0-type null slant helix if and only if its curvature functions $\kappa_2 \neq const$ and $\kappa_3 \neq 0$ satisfy the relation*

$$\left(\frac{\kappa_2'}{\kappa_3}\right)' + \kappa_3 = 0. \tag{3}$$

Proof. Assume that α is a 0-type null slant helix parameterized by the pseudo-arc s. Then there exists a non-zero fixed direction $U \in \mathbb{E}_1^4$ such that

$$g(T, U) = c, \quad c \in \mathbb{R}_0. \tag{4}$$

Differentiating equation (4) with respect to s and using Cartan equations (2), we easily get

$$g(N, U) = 0.$$

With respect to the Cartan frame $\{T, N, B_1, B_2\}$, the fixed direction U can be decomposed as

$$U = u_1 T + cB_1 + u_4 B_2, \tag{5}$$

where u_1 and u_4 are some differentiable functions of s. Differentiating equation (5) with respect to s and using Cartan equations (2), we obtain the following system of differential equations

$$\begin{cases} u_1' - \kappa_3 u_4 = 0, \\ u_1 - c\kappa_2 = 0, \\ u_4' + c\kappa_3 = 0. \end{cases} \tag{6}$$

If $\kappa_2 = 0$, relation (6) implies $c = 0$ or $\kappa_3 = 0$, which is a contradiction. Thus $\kappa_2 \neq 0$. From the second and the third equation of (6) we get

$$\begin{cases} u_1 = c\kappa_2, \\ u_4 = -c\int \kappa_3 ds. \end{cases} \tag{7}$$

Substituting (7) in the first equation of (6) we obtain

$$\kappa_2' + \kappa_3 \int \kappa_3 ds = 0. \tag{8}$$

If $\kappa_2 = const$, the previous relation implies $\kappa_3 = 0$, which is a contradiction. Hence $\kappa_2 \neq const$. Differentiating (8) with respect to s, it follows that (3) holds. Conversely, assume that curvature functions $\kappa_2 \neq const$ and $\kappa_3 \neq 0$ satisfy relation (3). Consider the vector U given by

$$U = c\kappa_2 T + cB_1 + c\left(\frac{\kappa_2'}{\kappa_3}\right) B_2.$$

where $c \in \mathbb{R}_0$. Differentiating the previous equation with respect to s and using Cartan equations (2), we find $U' = 0$. Hence U is a fixed direction. It can be easily checked that

$$g(T, U) = c.$$

According to Definition 2, the curve α is a 0-type null slant helix with the axis U. $\square$

Corollary 1. *The axis of a* 0*-type null slant helix* α *in* $\mathbb{E}_1^4$ *with the curvatures* $\kappa_2 \neq const$ *and* $\kappa_3 \neq 0$ *is given by*

$$U = c\kappa_2 T + cB_1 + c\left(\frac{\kappa_2'}{\kappa_3}\right) B_2, \tag{9}$$

where $c \in \mathbb{R}_0$.

Putting $c = 0$ in relation (4) and differentiating relation (4) three times with respect to s, we get $\kappa_3 g(B_2, U) = 0$. Since $\kappa_3 \neq 0$, it follows that $g(B_2, U) = 0$. Then relation (5) reads $U = 0$, which is a contradiction. Therefore, we obtain the next corollary.

Corollary 2. *There are no* 0*-type null slant helices in* $\mathbb{E}_1^4$ *with curvatures* $\kappa_2 \neq const$ *and* $\kappa_3 \neq 0$*, whose axis* U *is orthogonal to the tangent vector of the helix.*

Theorem 2. *Let* $\alpha : I \to \mathbb{E}_1^4$ *be a null Cartan curve in* $\mathbb{E}_1^4$*. Then* α *is a* 0*-type null slant helix if and only if for its curvature functions* $\kappa_2 \neq const$ *and* $\kappa_3 \neq 0$ *it holds*

$$2\kappa_2 + \left(\frac{\kappa_2'}{\kappa_3}\right)^2 = const. \tag{10}$$

Proof. Assume that α is a 0-type null slant helix parameterized by the pseudo-arc s. According to Corollary 1, the axis of α is given by (9). By using (9) and the condition

$$g(U, U) = const,$$

it follows that (10) holds. Conversely, assume that (10) holds. Differentiating relation (10) with respect to s, we obtain

$$2\kappa_2' + 2\left(\frac{\kappa_2'}{\kappa_3}\right)\left(\frac{\kappa_2'}{\kappa_3}\right)' = 0.$$

It follows that

$$\kappa_3 + \left(\frac{\kappa_2'}{\kappa_3}\right)' = 0.$$

According to Theorem 1, α is a 0-type null slant helix. $\square$

If the axis U of 0-type null slant helix with the curvature $\kappa_3 \neq 0$ is a null direction, by using relation (9) we obtain

$$2\kappa_2 + \left(\frac{\kappa_2'}{\kappa_3}\right)^2 = 0.$$

Integrating the last equation, we get the next corollary.

Corollary 3. *Let α be a 0-type null slant helix in $\mathbb{E}_1^4$ with the curvature $\kappa_3 \neq 0$. If the axis of α is a null direction given by (9), then curvature functions of α satisfy the relation*

$$\kappa_2(s) = -\frac{1}{2}\Big(\int \kappa_3(s)\, ds + C_1\Big)^2,$$

where $C_1 \in \mathbb{R}$.

Theorem 3. *Let α be a 0-type null slant helix with curvatures $\kappa_2 \neq const$ and $\kappa_3 \neq 0$, lying on a pseudosphere $S_1^3(r)$ in $\mathbb{E}_1^4$. Then its curvature functions are given by*

$$\kappa_2(s) = -\frac{1}{2}A^2 s^2 + Bs + C, \quad \kappa_3(s) = A, \tag{11}$$

where $A \in \mathbb{R}_0$ and $B, C \in \mathbb{R}$.

Proof. Assume that a 0-type null slant helix α parameterized by pseudo-arc s lies on the pseudosphere $S_1^3(r)$ with the center at the origin and of radius $r \in \mathbb{R}^+$. Then

$$g(\alpha, \alpha) = r^2.$$

Differentiating the previous relation four times with respect to s and using (2), we obtain

$$g(\alpha, B_2) = -\frac{1}{\kappa_3}.$$

Differentiating the last relation with respect to s and using (2), we get

$$\left(\frac{1}{\kappa_3}\right)' = 0.$$

It follows that

$$\kappa_3 = const = A, \quad A \in \mathbb{R}_0. \tag{12}$$

According to Theorem 1, the curvatures of α satisfy relation (3). Substituting (12) in relation (3), we obtain

$$\kappa_2 = -\frac{1}{2}A^2 s^2 + Bs + C, \tag{13}$$

where $A \in \mathbb{R}_0$ and $B, C \in \mathbb{R}$. By using (12) and (13), it follows that (11) holds. □

Corollary 4. *Let α be a 0-type null slant helix with curvatures $\kappa_2 \neq const$ and $\kappa_3 \neq 0$ lying on a pseudosphere $S_1^3(r)$ in $\mathbb{E}_1^4$. Then its position vector satisfies the differential equation*

$$\alpha^{(5)} + \left(A^2 s^2 - 2Bs - 2C\right) \alpha''' + 3\left(A^2 s - B\right) \alpha'' = 0,$$

where $A \in \mathbb{R}_0$ and $B, C \in \mathbb{R}$.

Let us consider now 1-type null slant helices in E_1^4.

Theorem 4. *Let $\alpha : I \to \mathbb{E}_1^4$ be a null Cartan curve in $\mathbb{E}_1^4$. Then α is a 1-type null slant helix if and only if its curvature functions $\kappa_2 \neq 0$ and $\kappa_3 \neq 0$ satisfy the relation*

$$s\kappa_2'(s) + 2\kappa_2(s) + \kappa_3(s) \int s\kappa_3(s)\, ds = 0. \tag{14}$$

Proof. Assume that α is a 1-type null slant helix parameterized by the pseudo-arc s in $\mathbb{E}_1^4$. Then there exists a non-zero constant vector field $U \in \mathbb{E}_1^4$ such that there holds

$$g(N, U) = c, \quad c \in \mathbb{R}_0. \tag{15}$$

By using the Cartan frame of α, the fixed direction U can be decomposed as

$$U = u_1 T + cN + u_3 B_1 + u_4 B_2, \tag{16}$$

where u_1, u_3 and u_4 are some differentiable functions of s. Differentiating equation (16) with respect to s and using Cartan equations (2), we obtain the following system of differential equations

$$\begin{cases} u_1' - \kappa_3 u_4 + c\kappa_2 = 0, \\ u_1 - \kappa_2 u_3 = 0, \\ u_3' - c = 0, \\ u_4' + \kappa_3 u_3 = 0. \end{cases} \tag{17}$$

If $\kappa_2(s) = 0$, relation (17) implies $c = 0$, which is a contradiction. Hence $\kappa_2(s) \neq 0$. From the last three equations of (17), we get

$$\begin{cases} u_1 = cs\kappa_2, \\ u_3 = cs, \\ u_4 = -c\int s\kappa_3 ds. \end{cases} \tag{18}$$

Substituting (18) in the first equation of (17), we obtain that curvature functions of α satisfy the relation (14). Conversely, assume that (14) holds. Consider the vector U given by

$$U = cs\kappa_2 T + cN + csB_1 - c\left(\int s\kappa_3\, ds\right) B_2,$$

where $c \in \mathbb{R}_0$. Differentiating the previous equation with respect to s and using Cartan equations (2) and (17), we find $U' = 0$. Hence U is a fixed direction. It can be easily checked that

$$g(N, U) = c, \quad c \in \mathbb{R}_0.$$

According to Definition 2, the curve α is a 1-type null slant helix with the axis U. $\square$

Corollary 5. *The axis of a* 1*-type null slant helix* α *in* $\mathbb{E}_1^4$ *with curvatures* $\kappa_2 \neq 0$ *and* $\kappa_3 \neq 0$ *is given by*

$$U = cs\kappa_2 T + cN + csB_1 - c\left(\int s\kappa_3\, ds\right) B_2,$$

where $c \in \mathbb{R}_0$.

If α is 1-type null slant helix whose principal normal is orthogonal to the axis U, substituting $c = 0$ in relation (17), we get that (3) holds, which means that α is 0-type null slant helix. Conversely, every 0-type null slant helix is a 1-type null slant helix with respect to the same axis whose principal normal is orthogonal to the axis. This proves the next corollary.

Corollary 6. *Let* α *be a null Cartan curve with the curvature* $\kappa_3 \neq 0$ *in* $\mathbb{E}_1^4$. *Then* α *is a* 0*-type null slant helix if and only if* α *is a* 1*-type null slant helix whose principal normal* N *is orthogonal to the axis* U *of the helix.*

Example 1. *Let* α *be a* 0*-type null slant helix in* E_1^4. *By using Cartan equations (2), it follows that its tangent vector satisfies the fourth order linear differential equation*

$$T^{(4)} = (\kappa_2'' + \kappa_3^2 - \frac{\kappa_2'\kappa_3'}{\kappa_3})T + (3\kappa_2' - \frac{2\kappa_2\kappa_3'}{\kappa_3})T' + 2\kappa_2 T'' + \frac{\kappa_3'}{\kappa_3}T'''. \tag{19}$$

By choosing one of the curvatures κ_2 *and* κ_3 *to be an arbitrary differentiable function and using relation (3), equation (19) is very difficult to solve in a general case. Only in some special cases, it can be solved in such way to give a nice parametrization of the curve* α. *Let us choose*

$$\kappa_3(s) = \frac{\sqrt{12}}{s^2}.$$

Relation (3) implies

$$\kappa_2(s) = -\frac{c_0}{s^2} - \frac{6}{s^2} + c_1, \quad c_0, c_1 \in \mathbb{R}_0.$$

Putting $c_0 = c_1 = 0$ *and substituting* κ_2 *and* κ_3 *in relation (19), we get the fourth order linear differential equation with non-constant coefficients*

$$s^3 T'''' + 2s^2 T''' + 12sT'' - 12T' = 0.$$

Putting $T' = N$ *in the last equation, we obtain the third order Euler differential equation*

$$s^3 N''' + 2s^2 N'' + 12sN' - 12N = 0,$$

whose general solution reads

$$N = C_1 s + C_2 \sin(2\sqrt{3}\,\ln s) + C_3 \cos(2\sqrt{3}\,\ln s),$$

where C_1, C_2, C_3 *are constant vectors in* E_1^4. *Up to isometries of* E_1^4, *we may choose* C_1, C_2 *and* C_3 *such that*

$$N(s) = (s, s, \sin(2\sqrt{3}\,\ln s), \cos(2\sqrt{3}\,\ln s)).$$

Integrating the last equation two times, we find that α has a parameter equation

$$\alpha(s) = (\frac{s^3}{6} + \frac{s}{26}, \frac{s^3}{6} - \frac{s}{26}, \frac{s^2}{104}(-5\sin(2\sqrt{3}\ln s) - 3\sqrt{3}\cos(2\sqrt{3}\ln s)),$$
$$\frac{s^2}{104}(-5\cos(2\sqrt{3}\ln s) + 3\sqrt{3}\sin(2\sqrt{3}\ln s))).$$

A straightforward calculation shows that the Cartan frame of α is given by

$$\begin{aligned}
T(s) &= (\frac{s^2}{2} + \frac{1}{26}, \frac{s^2}{2} - \frac{1}{26}, \frac{s}{13}(\sin(2\sqrt{3}\ln s) - 2\sqrt{3}\cos(2\sqrt{3}\ln s)),\\
&\quad \frac{s}{13}(\cos(2\sqrt{3}\ln s) + 2\sqrt{3}\sin(2\sqrt{3}\ln s))),\\
N(s) &= (s, s, \sin(2\sqrt{3}\ln s), \cos(2\sqrt{3}\ln s)),\\
B_1(s) &= (-4 - \frac{3}{13s^2}, -4 + \frac{3}{13s^2}, \frac{1}{13s}(-6\sin(2\sqrt{3}\ln s) - 14\sqrt{3}\cos(2\sqrt{3}\ln s)),\\
&\quad \frac{1}{13s}(-6\cos(2\sqrt{3}\ln s) + 14\sqrt{3}\sin(2\sqrt{3}\ln s))),\\
B_2(s) &= (\frac{\sqrt{3}}{13s}(1 - 13s^2), -\frac{\sqrt{3}}{13s}(13s^2 + 1), \frac{1}{13}(\cos(2\sqrt{3}\ln s) + 2\sqrt{3}\sin(2\sqrt{3}\ln s)),\\
&\quad \frac{1}{13}(2\sqrt{3}\cos(2\sqrt{3}\ln s) - \sin(2\sqrt{3}\ln s))).
\end{aligned}$$

According to Corollary 1, the axis of α reads

$$U = c\kappa_2 T + cB_1 + c\left(\frac{\kappa_2'}{\kappa_3}\right)B_2, \quad c \in \mathbb{R}_0.$$

Substituting κ_2, κ_3, T, B_1 and B_2 in the previous equation, we find

$$U = (-13c, -13c, 0, 0).$$

Consequently, the axis is a lightlike vector. It can be easily checked that

$$g(T, U) = c, \quad g(N, U) = 0,$$

which means that α is also a 1-type null slant helix whose axis U is orthogonal to the principal normal.

The following Theorem 5 can be proved similarly to Theorem 2, so we omit its proof.

Theorem 5. *Let α be a null Cartan curve in $\mathbb{E}_1^4$ with curvatures $\kappa_2 \neq 0$ and $\kappa_3 \neq 0$. Then α is a 1-type null slant helix if and only if it holds*

$$2s^2\kappa_2 + \left(\int s\kappa_3\, ds\right)^2 + 1 = const. \tag{20}$$

Theorem 6. *Let α be a 1-type null slant helix with curvatures $\kappa_2 \neq 0$ and $\kappa_3 \neq 0$ lying on a pseudosphere $S_1^3(r)$ in E_1^4. Then its curvature functions are given by*

$$\kappa_2(s) = \frac{B}{s^2} - \frac{1}{8}A^2 s^2, \quad \kappa_3(s) = A, \tag{21}$$

where $A \in \mathbb{R}_0$, $B \in \mathbb{R}$.

Proof. By assumption, the curve α lies on the pseudosphere $S_1^3(r)$. This implies $\kappa_3 = A$, $A \in \mathbb{R}_0$. Substituting $\kappa_3 = A$ in relation (20), we get

$$2s^2\kappa_2 + \left(A\int s\,ds\right)^2 + 1 = C, \quad C \in R.$$

It follows that the curvature κ_2 is given by (21), which proves the theorem. □

Corollary 7. *Let α be a 1-type null slant helix with curvatures $\kappa_2 \neq 0$ and $\kappa_3 \neq 0$ lying on a pseudosphere $S_1^3(r)$ in E_1^4. Then its position vector satisfies the differential equation*

$$4s^4\alpha^{(5)} - (8Bs^2 - A^2s^6)\alpha''' + (24Bs - 3A^2s^5)\alpha'' - (24B + 3A^2s^4)\alpha' = 0,$$

where $A \in \mathbb{R}_0$, $B \in \mathbb{R}$.

In the next theorem, we obtain the relationship between 1-type null slant helices and 2-type null slant helices.

Theorem 7. *Let α be a 1-type null slant helix with the curvature $\kappa_3 \neq 0$ in $\mathbb{E}_1^4$. Then α is a 2-type null slant helix with respect to the same axis if and only if its curvature functions satisfy the relations*

$$\kappa_2(s) = \frac{d}{cs+e}, \quad (cs+e)\kappa_3 + \left(\frac{cd}{(cs+e)\kappa_3}\right)' = 0, \tag{22}$$

where $c, d \in \mathbb{R}_0$ and $e \in \mathbb{R}$.

Proof. Assume that a 1-type null slant helix α parameterized by the pseudo-arc s is a 2-type null slant helix with respect to the same axis U in $\mathbb{E}_1^4$. Then it holds

$$g(N, U) = c, \quad g(B_1, U) = d, \quad c, d \in R_0. \tag{23}$$

By using the Cartan frame of α, a non-zero fixed direction U can be decomposed as

$$U = dT + cN + u_3B_1 + u_4B_2, \tag{24}$$

where u_3 and u_4 are some differentiable functions of s. Differentiating equation (24) with respect to s and using Cartan equations (2), we obtain the system of differential equations

$$\begin{cases} -\kappa_3 u_4 + c\kappa_2 = 0, \\ d - \kappa_2 u_3 = 0, \\ u_3' - c = 0, \\ u_4' + \kappa_3 u_3 = 0. \end{cases} \tag{25}$$

From the first two equations of (25) we obtain

$$u_3 = \frac{d}{\kappa_2}, \quad u_4 = \frac{c\kappa_2}{\kappa_3}.$$

Substituting this in the last two equations of (25), it follows that (22) holds. Conversely, assume that α is a 1-type null slant helix parameterized by the pseudo-arc s whose curvature functions satisfy relation (22). Consider the vector $U \in \mathbb{E}^4_1$ given by

$$U = dT + cN + \frac{d}{\kappa_2}B_1 + \frac{c\kappa_2}{\kappa_3}B_2, \tag{26}$$

where $c, d \in \mathbb{R}_0$. Differentiating relation (26) with respect to s and using Cartan equations (2), we find $U' = 0$. Hence U is a fixed direction. It can be easily checked that

$$g(N, U) = c, \quad g(B_1, U) = d, \quad d \in \mathbb{R}_0.$$

According to Definition 2, the curve α is a 2-type null slant helix with respect to the same axis, which proves the theorem. □

Theorem 8. *Let α be a null Cartan curve in $\mathbb{E}^4_1$. Then α is a 2-type null slant helix if only if its curvatures $\kappa_2 \neq const$ and $\kappa_3 \neq const$ satisfy the relation*

$$u_3'' = \kappa_2 u_3 - c, \tag{27}$$

where u_3 is given by

$$u_3 = ce^{-\int \frac{\kappa_3(\kappa_2^2+\kappa_3^2)}{\kappa_2'\kappa_3-\kappa_2\kappa_3'}ds} \int \frac{\kappa_2\kappa_3}{\kappa_2'\kappa_3 - \kappa_2\kappa_3'} e^{\int \frac{\kappa_3(\kappa_2^2+\kappa_3^2)}{\kappa_2'\kappa_3-\kappa_2\kappa_3'}ds} ds$$

and $c \in \mathbb{R}_0$.

Proof. Assume that α is a 2-type null slant helix parameterized by the pseudo-arc s. Then there exists a non-zero fixed direction $U \in \mathbb{E}^4_1$ such that there holds

$$g(B_1, U) = c, \quad c \in \mathbb{R}_0. \tag{28}$$

By using the Cartan frame of α, the fixed direction U can be decomposed as

$$U = cT + u_2N + u_3B_1 + u_4B_2, \tag{29}$$

where u_2, u_3 and u_4 are some differentiable functions of s. Differentiating equation (29) with respect to s and using Cartan equations (2), we obtain the system of differential equations

$$\begin{cases} \kappa_2u_2 - \kappa_3u_4 = 0, \\ u_2' - \kappa_2u_3 + c = 0, \\ u_3' - u_2 = 0, \\ u_4' + \kappa_3u_3 = 0. \end{cases} \tag{30}$$

If $\kappa_2(s) = const$ and $\kappa_3 = const \neq 0$, relation (30) implies $c = 0$, which is a contradiction. Consequently, $\kappa_2(s) \neq const$ and $\kappa_3(s) \neq const$. From the first and the third equation of (30) we get

$$u_4 = \frac{\kappa_2}{\kappa_3} u_3'. \tag{31}$$

Substituting (31) in the fourth equation of (30), we find

$$\left(\frac{\kappa_2}{\kappa_3}\right)' u_3' + \frac{\kappa_2}{\kappa_3} u_3'' + \kappa_3 u_3 = 0. \tag{32}$$

From the second and the third equation of (30) we obtain

$$u_3'' = \kappa_2 u_3 - c. \tag{33}$$

Substituting (33) in (32), we obtain the first order linear differential equation

$$u_3' + \frac{\kappa_3(\kappa_2^2 + \kappa_3^2)}{\kappa_2'\kappa_3 - \kappa_2\kappa_3'} u_3 = \frac{c\kappa_2\kappa_3}{\kappa_2'\kappa_3 - \kappa_2\kappa_3'}.$$

A general solution of the previous differential equation reads

$$u_3 = e^{-\int \frac{\kappa_3(\kappa_2^2+\kappa_3^2)}{\kappa_2'\kappa_3-\kappa_2\kappa_3'} ds} (c_0 + c \int \frac{\kappa_2\kappa_3}{\kappa_2'\kappa_3 - \kappa_2\kappa_3'} e^{\int \frac{\kappa_3(\kappa_2^2+\kappa_3^2)}{\kappa_2'\kappa_3-\kappa_2\kappa_3'} ds} ds),$$

where $c_0 \in \mathbb{R}$. Taking $c_0 = 0$, we find

$$u_3 = ce^{-\int \frac{\kappa_3(\kappa_2^2+\kappa_3^2)}{\kappa_2'\kappa_3-\kappa_2\kappa_3'} ds} \int \frac{\kappa_2\kappa_3}{\kappa_2'\kappa_3 - \kappa_2\kappa_3'} e^{\int \frac{\kappa_3(\kappa_2^2+\kappa_3^2)}{\kappa_2'\kappa_3-\kappa_2\kappa_3'} ds} ds. \tag{34}$$

Relations (33) and (34) imply that relation (27) holds. Conversely, assume that (27) holds. Consider the vector U given by

$$U = cT + u_2 N + u_3 B_1 + u_4 B_2, \tag{35}$$

where

$$\begin{cases} u_2 = u_3', \\ u_3 = ce^{-\int \frac{\kappa_3(\kappa_2^2+\kappa_3^2)}{\kappa_2'\kappa_3-\kappa_2\kappa_3'} ds} \int \frac{\kappa_2\kappa_3}{\kappa_2'\kappa_3-\kappa_2\kappa_3'} e^{\int \frac{\kappa_3(\kappa_2^2+\kappa_3^2)}{\kappa_2'\kappa_3-\kappa_2\kappa_3'} ds} ds, \\ u_4 = \frac{\kappa_2}{\kappa_3} u_3'. \end{cases}$$

and $c \in \mathbb{R}_0$. Differentiating relation (35) with respect to s and using Cartan equations (2), we find $U' = 0$. Hence U is a fixed direction. It can be easily checked that

$$g(B_1, U) = c, \quad c \in R_0.$$

According to Definition 2, the curve α is a 2-type null slant helix. $\square$

If α is a null Cartan curve lying on a pseudosphere $S_1^3(r)$, then $\kappa_3 = const \neq 0$. According to Theorem 8, if α is a 2-type null slant helix, then $\kappa_3 \neq const$. Hence the next corollary holds.

Corollary 8. *There are no* 2*-type null slant helices with curvatures* $\kappa_2 \neq const$ *and* $\kappa_3 \neq const$*, lying on a pseudosphere* $S_1^3(r)$ *in* $\mathbb{E}_1^4$*.*

Corollary 9. *The axis of* 2*-type null slant helix* α *in* $\mathbb{E}_1^4$ *with curvatures* $\kappa_2 \neq const$ *and* $\kappa_3 \neq const$ *is given by*

$$U = cT + u_2 N + u_3 B_1 + u_4 B_2,$$

where

$$\begin{cases} u_2 = u_3', \\ u_3 = ce^{-\int \frac{\kappa_3(\kappa_2^2+\kappa_3^2)}{\kappa_2'\kappa_3-\kappa_2\kappa_3'}ds} \int \frac{\kappa_2\kappa_3}{\kappa_2'\kappa_3-\kappa_2\kappa_3'} e^{\int \frac{\kappa_3(\kappa_2^2+\kappa_3^2)}{\kappa_2'\kappa_3-\kappa_2\kappa_3'}ds} ds, \\ u_4 = \frac{\kappa_2}{\kappa_3} u_3'. \end{cases}$$

and $c \in \mathbb{R}_0$*.*

Theorem 9. *There are no* 3*-type null slant helices in* $\mathbb{E}_1^4$ *with the curvature* $\kappa_3 \neq 0$*.*

Proof. Assume that there exists a 3-type null slant helix α with the curvature $\kappa_3 \neq 0$, parameterized by the pseudo-arc s in Minkowski space-time. Then there also exists a constant vector field $U \neq 0$ in $\mathbb{E}_1^4$ such that it holds

$$g\,(B_2, U) = c, \quad c \in \mathbb{R}_0. \tag{36}$$

Differentiating relation (36) with respect to s and using Cartan equations (2), we get

$$g\,(T, U) = 0.$$

Differentiating the last equation three times with respect to s, we get

$$\kappa_3 g(B_2, U) = 0,$$

which is a contradiction. □

Acknowledgement

The authors would like to thank the referees for their helpful suggestions.

References

[1] A. T. Ali, R. López, *Timelike B_2-slant helices in Minkowski space* $\mathbb{E}_1^4$, Arch. Math. (Brno) **46**(2010), 39–46.
[2] A. T. Ali, R. López, *Slant helices in Minkowski space* $\mathbb{E}_1^3$, J. Korean Math. Soc. **48**(2011), 159–167.

[3] A. T. Ali, R. López, M. Turgut, *k-type partially null and pseudo null slant helices in Minkowski* 4-*space*, Math. Commun. **17**(2012), 93–103.

[4] W. B. Bonnor, *Null curves in a Minkowski space-time*, Tensor **20**(1969), 229–242.

[5] K. L. Duggal, A. Bejancu, *Lightlike submanifolds of semi-Riemannian manifolds and applications, Mathematics and its Applications, 364*, Kluwer Academic Publishers Group, Dordrecht, 1996.

[6] M. Ergüt, H. B. Öztekin, S. Aykurt, *Non-null k-slant helices and their spherical indicatrices in Minkowski 3-space*, J. Adv. Res. Dyn. Control Syst. **2**(2010), 1–12.

[7] A. Ferrández, A. Giménez, P. Lucas, *Null generalized helices in Lorentz-Minkowski spaces*, J. Phys. A **35**(2002), 8243–8251.

[8] S. Izumiya, N. Tkeuchi, *New special curves and developable surfaces*, Turk J. Math. **28**(2004), 153–163.

[9] L. Kula, Y. Yayli, *On slant helix and its spherical indicatrix*, Appl. Math. Comput. **169**(2005), 600–607.

[10] M. Turgut, S. Yilmaz, *Some characterizations of type-3 slant helices in Minkowski space-time*, Involve **2**(2009), 115–120.