MATHEMATICAL COMMUNICATIONS
Math. Commun. **20**(2015), 53–68

Pseudo almost automorphic behavior of solutions to a semi-linear fractional differential equation*

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Received March 4, 2014; accepted January 18, 2015

Abstract. In this paper, we shall deal with μ-pseudo almost automorphic solutions to a semi-linear fractional differential equation by a new concept of μ-pseudo almost automorphic functions presented recently. First we establish some new properties of μ-pseudo almost automorphic functions, and then apply the obtained results to prove some existence theorems combined with the Leray-Schauder alternative theorem.

AMS subject classifications: 34B15, 43A60, 34G20, 47A55

Key words: fractional differential equation, μ-pseudo almost automorphic function, composition theorems, Leray-Schauder alternative theorem

1. Introduction

The concept of almost automorphy was first introduced in the literature by Bochner in [8]; for more details about this topic we refer to [1, 2, 12, 13, 20, 21, 24]. Since then, there have been several interesting, natural and powerful generalizations of classical almost automorphic functions. The concept of asymptotically almost automorphic functions was introduced by N'Guérékata in [19]. In [17, 28], Liang, Xiao and Zhang presented the concept of pseudo almost automorphy. In [22], N'Guérékata and Pankov introduced the concept of Stepanov-like almost automorphy and applied this concept to investigate the existence and uniqueness of an almost automorphic solution to the autonomous semilinear equation. In [6], Blot et al. introduced the notion of weighted pseudo almost automorphic functions with values in a Banach space. In [29, 30], Chang, N'Guérékata et al. investigated some properties and new composition theorems of Stepanov-like weighted pseudo almost automorphic functions. Recently, in [7], Blot, Cieutat and Ezzinbi applied the measure theory to define an ergodic function and they investigate many interesting properties of μ-pseudo almost automorphic functions.

In recent years, great attention has been paid to different typed solutions to a fractional differential equations [3, 4, 23, 25, 26, 27]. The study of almost automorphic solutions to fractional differential equation was initiated by Araya and Lizama

*This work was supported by the NSF of China (11361032), Program for Longyuan Youth Innovative Talents of Gansu Province of China (2014-4-80).
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[5]. In their work, the authors investigated the existence and uniqueness of almost automorphic mild solutions to some fractional differential equations. For more details about fractional differential equations we refer to [4, 11, 16]. In [10], Cuevas and Lizama considered the following fractional differential equation:

$$D_t^{\alpha}u(t) = Au(t) + D_t^{\alpha-1}f(t, u(t)), \quad t \in \mathbb{R}, \quad 1 < \alpha < 2,$$

where A is a linear operator of sectorial negative type on a Banach space. Under suitable conditions of f, the authors proved the existence and uniqueness of an almost automorphic mild solution to the above problem. Mophou [18] investigated the existence and uniqueness of weighted pseudo almost automorphic mild solutions to the following fractional differential equation:

$$D_t^{\alpha}u(t) = Au(t) + D_t^{\alpha-1}f(t, u(t), Bu(t)), \quad t \in \mathbb{R}, \tag{1}$$

where $1 < \alpha < 2$, $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ is a linear densely operator of sectorial type on a complex Banach space $(\mathbb{X}, \|\cdot\|)$, $B : \mathbb{X} \to \mathbb{X}$ is a bounded linear operator and $f : \mathbb{R} \times \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ is a weighted pseudo almost automorphic function in t for each x, $y \in \mathbb{X}$ satisfying suitable conditions. The fractional derivative D_t^{α} is to be understood in Riemann-Liouville sense.

Motivated by the above mentioned works [7, 18], the purpose of this paper is to establish some existence results of μ-pseudo almost automorphic mild solutions to the problem (1) if $f : \mathbb{R} \times \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ is a μ-pseudo almost automorphic function in t for each x, $y \in \mathbb{X}$ satisfying suitable conditions. We first prove some results on composition theorems of such functions, and then establish the existence results by the Banach contraction principle and the Leray-Schauder alternative theorem.

The rest of this paper is organized as follows. In Section 2, we present some basic definitions, lemmas, and preliminary results which will be used throughout this paper. In Section 3, we first prove some composition theorems of the μ-pseudo almost automorphic functions, and then we prove some existence results of μ-pseudo almost automorphic mild solutions to semilinear fractional differential equations (1). An example is given to illustrate the obtained results.

2. Preliminaries

In this section, we list some basic properties of μ-pseudo almost automorphic functions. Throughout the paper, the notations $(\mathbb{X}, \|\cdot\|)$ and $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ are two Banach spaces and $BC(\mathbb{R}, \mathbb{X})$ denotes the Banach space of bounded continuous functions from $\mathbb{R}$ to $\mathbb{X}$, equipped with the supremum norm $\|f\|_\infty = \sup_{t\in\mathbb{R}} \|f(t)\|$. Let $\mathcal{L}(\mathbb{X})$ be the Banach space of all bounded linear operators from $\mathbb{X}$ into itself endowed with the norm:

$$\|T\|_{\mathcal{L}(\mathbb{X})} = \sup\{\|Tx\| : x \in \mathbb{X}, \|x\| \leq 1\}.$$

Throughout this paper, we denote by $\mathcal{B}$ the Lebesgue σ-field of $\mathbb{R}$ and by $\mathcal{M}$ the set of all positive measure μ on $\mathcal{B}$ satisfying $\mu(\mathbb{R}) = +\infty$ and $\mu([a, b]) < +\infty$, for all $a, b \in \mathbb{R}$ $(a < b)$.

Definition 1 (see [20]). *A continuous function $f : \mathbb{R} \to \mathbb{X}$ is said to be almost automorphic if for every sequence of real numbers $\{s'_n\}_{n\in\mathbb{N}}$ there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ such that*

$$g(t) = \lim_{n\to\infty} f(t + s_n)$$

exists for all t in $\mathbb{R}$, and

$$\lim_{n\to\infty} g(t - s_n) = f(t)$$

for all t in $\mathbb{R}$.

Definition 2 (see [18, 20]). *A continuous function $f : \mathbb{R} \times \mathbb{Y} \times \mathbb{Y} \to \mathbb{X}$ is said to be almost automorphic if $f(t, x, y)$ is almost automorphic in $t \in \mathbb{R}$ uniformly for all $(x, y) \in K$, where K is any bounded subset of $\mathbb{Y} \times \mathbb{Y}$. The collection of all such functions will be denoted by $AA(\mathbb{R} \times \mathbb{Y} \times \mathbb{Y}, \mathbb{X})$.*

Definition 3 (see [18]). *A bounded continuous function with a vanishing mean value can be defined as*

$$AA_0(\mathbb{R}, \mathbb{X}) = \left\{ \phi \in BC(\mathbb{R}, \mathbb{X}) : \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \|\phi(\sigma)\| d\sigma = 0 \right\}.$$

Similarly, by $AA_0(\mathbb{R} \times \mathbb{Y} \times \mathbb{Y}, \mathbb{X})$ we define the set of all continuous functions $f : \mathbb{R} \times \mathbb{Y} \times \mathbb{Y} \to \mathbb{X}$ which belong to $BC(\mathbb{R} \times \mathbb{Y} \times \mathbb{Y}, \mathbb{X})$ and satisfy

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \|\phi(\sigma, x, y)\| d\sigma = 0,$$

uniformly for (x, y) in any bounded subset of $\mathbb{Y} \times \mathbb{Y}$.

Definition 4 (see [18]). *A function $f \in BC(\mathbb{R} \times \mathbb{Y} \times \mathbb{Y}, \mathbb{X})$ is called pseudo almost automorphic in $t \in \mathbb{R}$ uniformly in $(x, y) \in \mathbb{Y} \times \mathbb{Y}$ if it can be written as $f = g + \phi$, where $g \in AA(\mathbb{R} \times \mathbb{Y} \times \mathbb{Y}, \mathbb{X})$ and $\phi \in AA_0(\mathbb{R} \times \mathbb{Y} \times \mathbb{Y}, \mathbb{X})$. We denote by $PAA(\mathbb{R}, \mathbb{X})$(respectively $PAA(\mathbb{R} \times \mathbb{Y} \times \mathbb{Y}, \mathbb{X})$), the set of all pseudo almost automorphic functions $f : \mathbb{R} \to \mathbb{X}$, (respectively $f : \mathbb{R} \times \mathbb{Y} \times \mathbb{Y} \to \mathbb{X}$).*

Definition 5 (see [7]). *Let $\mu \in \mathcal{M}$. A bounded continuous function $f : \mathbb{R} \to \mathbb{X}$ is said to be μ-ergodic if*

$$\lim_{r\to+\infty} \frac{1}{\mu([-r, r])} \int_{[-r,r]} \|f(t)\| d\mu(t) = 0.$$

We denote the space of all such functions by $\varepsilon(\mathbb{R}, \mathbb{X}, \mu)$.

Similarly, by $\varepsilon(\mathbb{R} \times \mathbb{Y} \times \mathbb{Y}, \mathbb{X}, \mu)$ we define the space of all continuous functions $f : \mathbb{R} \times \mathbb{Y} \times \mathbb{Y} \to \mathbb{X}$ which belong to $BC(\mathbb{R} \times \mathbb{Y} \times \mathbb{Y}, \mathbb{X})$ and satisfy

$$\lim_{r\to+\infty} \frac{1}{\mu([-r, r])} \int_{[-r,r]} \|f(t, x, y)\| d\mu(t) = 0,$$

uniformly for (x, y) in any bounded subset of $\mathbb{Y} \times \mathbb{Y}$.

Definition 6 (see [7]). *Let $\mu \in \mathcal{M}$. A continuous function $f : \mathbb{R} \to \mathbb{X}$ is said to be μ-pseudo almost automorphic if f is written in the form: $f = g + \phi$, where $g \in AA(\mathbb{R}, \mathbb{X})$ and $\phi \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu)$. We denote the space of all such functions by $PAA(\mathbb{R}, \mathbb{X}, \mu)$.*

Definition 7. *Let $\mu \in \mathcal{M}$. A continuous function $f : \mathbb{R} \times \mathbb{Y} \times \mathbb{Y} \to \mathbb{X}$ is said to be μ-pseudo almost automorphic if f is written in the form: $f = g + \phi$, where $g \in AA(\mathbb{R} \times \mathbb{Y} \times \mathbb{Y}, \mathbb{X})$ and $\phi \in \varepsilon(\mathbb{R} \times \mathbb{Y} \times \mathbb{Y}, \mathbb{X}, \mu)$. We denote the space of all such functions by $PAA(\mathbb{R} \times \mathbb{Y} \times \mathbb{Y}, \mathbb{X}, \mu)$.*

Obviously, we have $AA(\mathbb{R}\times\mathbb{Y}\times\mathbb{Y}, \mathbb{X}) \subset PAA(\mathbb{R}\times\mathbb{Y}\times\mathbb{Y}, \mathbb{X}, \mu) \subset BC(\mathbb{R}\times\mathbb{Y}\times\mathbb{Y}, \mathbb{X})$.

For $\mu \in \mathcal{M}$ and $\tau \in \mathbb{R}$, we denote μ_τ the positive measure on $(\mathbb{R}, \mathcal{B})$ defined by

$$\mu_\tau(\mathcal{A}) = \mu(a + \tau : a \in \mathcal{A}) \quad \text{for } \mathcal{A} \in \mathcal{B}.$$

From $\mu \in \mathcal{M}$, we list the following hypothesis ([7]).

(H0) For all $\tau \in \mathbb{R}$, there exist $\gamma > 0$ and a bounded interval I such that

$$\mu_\tau(\mathcal{A}) \le \gamma \mu(\mathcal{A}),$$

when $\mathcal{A} \in \mathcal{B}$ satisfies $\mathcal{A} \bigcap I = \varnothing$.

Lemma 1 (see [7]). *Let $\mu \in \mathcal{M}$ satisfy (H0). Then $\varepsilon(\mathbb{R}, \mathbb{X}, \mu)$ is translation invariant; therefore, $PAA(\mathbb{R}, \mathbb{X}, \mu)$ is also translation invariant.*

Lemma 2 (see [7, Proposition 2.13]). *Let $\mu \in \mathcal{M}$. Then $(\varepsilon(\mathbb{R}, \mathbb{X}, \mu), \|\cdot\|_\infty)$ is a Banach space.*

Lemma 3 (see [7, Theorem 4.1]). *Let $\mu \in \mathcal{M}$ and $f \in PAA(\mathbb{R}, \mathbb{X}, \mu)$ be such that $f = g + \phi$, where $g \in AA(\mathbb{R}, \mathbb{X})$ and $\phi \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu)$. If $PAA(\mathbb{R}, \mathbb{X}, \mu)$ is translation invariant, then $\{g(t) : t \in \mathbb{R}\} \subset \overline{\{f(t) : t \in \mathbb{R}\}}$, (the closure of the range of f).*

Lemma 4 (see [7, Theorem 2.14]). *Let $\mu \in \mathcal{M}$ and I be a bounded interval (eventually $I = \emptyset$). Assume that $f \in BC(\mathbb{R}, \mathbb{X})$. Then the following assertions are equivalent.*

(i) $f \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu)$.

(ii) $\lim_{r\to+\infty} \dfrac{1}{\mu([-r,r] \setminus I)} \int_{[-r,r]\setminus I} \|f(t)\| d\mu(t) = 0.$

(iii) For any $\varepsilon > 0, \lim_{r\to+\infty} \dfrac{\mu(\{t \in [-r,r] \setminus I : \|f(t)\| > \varepsilon\})}{\mu([-r,r] \setminus I)} = 0.$

Remark 1 (see [7, Remark 2.15]). *From $\mu \in \mathcal{M}$ and the fact that $\mu([-r,r]) = \mu([-r,r] \setminus I) + \mu(I)$ for r sufficiently large, we deduce that*

$$\lim_{r\to+\infty} \mu([-r,r] \setminus I) = +\infty.$$

Lemma 5 (see [7, Theorem 4.7]). *Let $\mu \in \mathcal{M}$. Assume that $PAA(\mathbb{R}, \mathbb{X}, \mu)$ is translation invariant. Then the decomposition of a μ-pseudo almost automorphic function in the form $f = g + \phi$, where $g \in AA(\mathbb{R}, \mathbb{X})$ and $\phi \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu)$, is unique.*

Lemma 6 (see [7, Theorem 4.9]). *Let $\mu \in \mathcal{M}$. Assume that $PAA(\mathbb{R}, \mathbb{X}, \mu)$ is translation invariant. Then $(PAA(\mathbb{R}, \mathbb{X}, \mu), \|\cdot\|_\infty)$ is a Banach space.*

Definition 8 (see [9]). *A closed linear operator $(A, D(A))$ with a dense domain $D(A)$ in a Banach space $\mathbb{X}$ is said to be sectorial of type ω and angle θ if there are constants $\omega \in \mathbb{R}$, $\theta \in (0, \frac{\pi}{2})$, $M > 0$ such that its resolvent exists outside the sector*

$$\omega + \Sigma_\theta := \{\lambda + \omega : \lambda \in \mathcal{C},\ |arg(-\lambda)| < \theta\}, \tag{2}$$

$$\|(\lambda - A)^{-1}\| \le \frac{M}{|\lambda - \omega|},\ \lambda \notin \omega + \Sigma_\theta. \tag{3}$$

Definition 9. *Let $1 < \alpha < 2$. Let A be a closed and linear operator with a domain $D(A)$ defined on a Banach space $\mathbb{X}$. We say that A is the generator of a solution operator if there exist $\omega \in \mathbb{R}$ and a strongly continuous function $E_\alpha : \mathbb{R}_+ \to \mathcal{L}(X)$ such that $\{\lambda^\alpha : Re\lambda > \omega\} \subset \rho(A)$ and*

$$\lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1}x = \int_0^\infty e^{-\lambda t} E_\alpha(t)x dt,\ Re\lambda > \omega,\ x \in \mathbb{X}.$$

From [9], if A is sectional of type $\omega \in \mathbb{R}$ with $0 \le \theta < \pi(1 - \alpha/2)$, then A is a generator of a solution operator given by

$$E_\alpha(t) = \int_{\mathcal{G}} e^{\lambda t}\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1} d\lambda,\ t \ge 0,$$

with $\mathcal{G}$ a suitable path lying outside the sector $\omega + \Sigma_0$. Furthermore, the following lemma holds.

Lemma 7 (see [9]). *Let $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ be a sectorial operator in a complex Banach space $\mathbb{X}$, satisfying hypotheses (2) and (3), for some $M > 0$, $\omega < 0$ and $0 \le \theta < \pi(1 - \alpha/2)$. Then there exists $C(\theta, \alpha) > 0$ depending solely on θ and α, such that*

$$\|E_\alpha(t)\|_{\mathcal{L}(\mathbb{X})} \le \frac{C(\theta, \alpha)M}{1 + |\omega|t^\alpha},\quad t \ge 0. \tag{4}$$

Now, we recall a useful compactness criterion.

Let $h' : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $h'(t) \ge 1$ for all $t \in \mathbb{R}$ and $h'(t) \to \infty$ as $|t| \to \infty$. We consider the space

$$C_{h'}(\mathbb{X}) = \left\{ u \in C(\mathbb{R}, \mathbb{X}) : \lim_{|t|\to\infty} \frac{u(t)}{h'(t)} = 0 \right\}.$$

Endowed with the norm $\|u\|_{h'} = \sup_{t\in\mathbb{R}} \frac{\|u(t)\|}{h'(t)}$, it is a Banach space (see [15]).

Lemma 8 (see [15]). *A subset $\mathcal{R} \subseteq C_{h'}(\mathbb{X})$ is a relatively compact set if it verifies the following conditions:*

(c-1) The set $\mathcal{R}(t) = \{u(t) : u \in \mathcal{R}\}$ is relatively compact in $\mathbb{X}$ for each $t \in \mathbb{R}$.

(c-2) The set $\mathcal{R}$ is equicontinuous.

(c-3) For each $\epsilon > 0$,there exists $L > 0$ such that $\|u(t)\| \le \epsilon h'(t)$ for all $u \in \mathcal{R}$ and all $|t| > L$.

Lemma 9 (Leray-Schauder Alternative Theorem, see [14])**.** *Let $\mathbb{D}$ be a closed convex subset of a Banach space $\mathbb{X}$ such that $0 \in \mathbb{D}$. Let $\mathbb{F} : \mathbb{D} \to \mathbb{D}$ be a completely continuous map. Then the set $\{x \in \mathbb{D} : x = \lambda\mathbb{F}(x),\ 0 < \lambda < 1\}$ is unbounded or the map $\mathbb{F}$ has a fixed point in $\mathbb{D}$.*

3. Main results

In this section, we first prove some composition theorems for μ-pseudo almost automorphic functions under suitable conditions, and then apply these composition theorems to establish some existence results for problem (1).

Theorem 1. *Let $\mu \in \mathcal{M}$ and $f = g + h \in PAA(\mathbb{R} \times \mathbb{X} \times \mathbb{X}, \mathbb{X}, \mu)$. Assume that*

(H1) $f(t, x, y)$ is uniformly continuous on any bounded subset $K \subset \mathbb{X} \times \mathbb{X}$ uniformly in $t \in \mathbb{R}$.

(H2) $g(t, x, y)$ is uniformly continuous on any bounded subset $K \subset \mathbb{X} \times \mathbb{X}$ uniformly in $t \in \mathbb{R}$.

Then the function defined by $F(\cdot) := f(\cdot, \phi(\cdot), \varphi(\cdot)) \in PAA(\mathbb{R}, \mathbb{X}, \mu)$ if $\phi,\ \varphi \in PAA(\mathbb{R}, \mathbb{X}, \mu)$.

Proof. Let $f = g + h$ with $g \in AA(\mathbb{R} \times \mathbb{X} \times \mathbb{X}, \mathbb{X})$, $h \in \varepsilon(\mathbb{R} \times \mathbb{X} \times \mathbb{X}, \mathbb{X}, \mu)$, and $\phi = u + v,\ \varphi = x + y$, with $u,\ x \in AA(\mathbb{R}, \mathbb{X})$, and $v,\ y \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu)$. Now we define

$$\begin{aligned} F(t) &= g(t, u(t), x(t)) + f(t, \phi(t), \varphi(t)) - g(t, u(t), x(t)) \\ &= g(t, u(t), x(t)) + f(t, \phi(t), \varphi(t)) - f(t, u(t), x(t)) + h(t, u(t), x(t)). \end{aligned}$$

Let us rewrite

$$G(t) = g(t, u(t), x(t)),\ \Phi(t) = f(t, \phi(t), \varphi(t)) - f(t, u(t), x(t)),\ H(t) = h(t, u(t), x(t)).$$

Thus, we have $F(t) = G(t) + \Phi(t) + H(t)$.

In view of [17, Lemma 2.2], $G(t) \in AA(\mathbb{R}, \mathbb{X})$. Next we prove that $\Phi(t) \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu)$. Clearly , $\Phi(t) \in BC(\mathbb{R}, \mathbb{X})$. For Φ to be in $\varepsilon(\mathbb{R}, \mathbb{X}, \mu)$, it is enough to show that

$$\lim_{r\to+\infty} \frac{1}{\mu([-r, r])} \int_{[-r,r]} \|\Phi(t)\| d\mu(t) = 0.$$

By Lemma 3, $\underline{u(\mathbb{R})} \times \underline{x(\mathbb{R})} \subset \overline{\phi(\mathbb{R})} \times \overline{\varphi(\mathbb{R})}$ which is a bounded set. From assumption (H1) with $K = \overline{\phi(\mathbb{R})} \times \overline{\varphi(\mathbb{R})}$, we conclude that for each $\varepsilon > 0$, there exists a constant $\delta > 0$ such that for all $t \in \mathbb{R}$,

$$\|\phi - u\| + \|\varphi - x\| \le 2\delta \Rightarrow \|f(t, \phi(t), \varphi(t)) - f(t, u(t), x(t))\| \le \varepsilon.$$

Denote the following set by $A_{r,\varepsilon} = \{t \in [-r, r] : \|f(t)\| > \varepsilon\}$. Thus we obtain

$$\begin{aligned} A_{r,\varepsilon}(\Phi) = A_{r,\varepsilon}(f(t, \phi(t), \varphi(t)) - f(t, u(t), x(t))) \subset A_{r,\delta}(\phi(t) - u(t)) \\ \cup A_{r,\delta}(\varphi(t) - x(t)) = A_{r,\delta}(v) \cup A_{r,\delta}(y). \end{aligned}$$

Therefore, the following inequality holds

$$\frac{\mu\{t \in [-r,r] : \|f(t,\phi(t),\varphi(t)) - f(t,u(t),x(t))\| > \varepsilon\}}{\mu([-r,r])} \le \frac{\mu\{t \in [-r,r] : \|\phi(t) - u(t)\| > \delta\}}{\mu([-r,r])} + \frac{\mu\{t \in [-r,r] : \|\varphi(t) - x(t)\| > \delta\}}{\mu([-r,r])}.$$

Since $\phi(t) = u(t) + v(t)$, $\varphi(t) = x(t) + y(t)$ and $v,\ y \in \varepsilon(\mathbb{R},\mathbb{X},\mu)$, Lemma 4 yields that for the above-mentioned δ we have

$$\lim_{r\to+\infty} \frac{\mu\{t \in [-r,r] : \|\phi(t) - u(t)\| > \delta\}}{\mu([-r,r])} = \lim_{r\to+\infty} \frac{\mu\{t \in [-r,r] : \|\varphi(t) - x(t)\| > \delta\}}{\mu([-r,r])} = 0,$$

and then we obtain

$$\lim_{r\to+\infty} \frac{\mu\{t \in [-r,r] : \|f(t,\phi(t),\varphi(t)) - f(t,u(t),x(t))\| > \varepsilon\}}{\mu([-r,r])} = 0. \tag{5}$$

From Lemma 4 and relation (5), we draw a conclusion that $\Phi(t) \in \varepsilon(\mathbb{R},\mathbb{X},\mu)$.

Finally, it remains only to show that $H(t) = h(t,u(t),x(t)) \in \varepsilon(\mathbb{R},\mathbb{X},\mu)$. We have that the set $u([-r,r]) \times x([-r,r])$ is compact since u and x are continuous on $\mathbb{R}$ as almost automorphic functions. So, the function g belongs to $AA(\mathbb{R}\times\mathbb{Y}\times\mathbb{Y},\mathbb{X})$, and g is uniformly continuous on $[-r,r] \times u([-r,r]) \times x([-r,r])$. Then it follows from (H1) that $h(t,a,b)$ is uniformly continuous with $(a,b) \in u([-r,r]) \times x([-r,r])$ uniformly in $t \in [-r,r]$. Thus for any $\varepsilon > 0$, there exists a constant $\delta > 0$ such that for $(a_1,b_1),\ (a_2,b_2) \in u([-r,r]) \times x([-r,r])$ with $\|a_1 - a_2\| + \|b_1 - b_2\| < \delta$ we have

$$\|h(t,a_1,b_1) - h(t,a_2,b_2)\| < \frac{\varepsilon}{2},\ \forall t \in [-r,r]. \tag{6}$$

On the other hand, since the set $u([-r,r]) \times x([-r,r])$ is compact, there exist finite balls O_k with center $(\beta_k,\gamma_k) \in u([-r,r]) \times x([-r,r])$, $k = 1,\cdots,m$, and radius less than δ such that $u([-r,r]) \times x([-r,r]) \subset \cup_{k=1}^m O_k$. Then the sets $U_k := \{t \in [-r,r] : (u(t),x(t)) \in O_k\}$, $k = 1,\cdots,m$ are open in $[-r,r]$ and $[-r,r] = \cup_{k=1}^m U_k$.

Define V_k by

$$V_1 = U_1, V_k = U_k - \cup_{i=1}^{k-1} U_i,\ 2 \le k \le m.$$

Then it is obvious that $V_i \cap V_j = \emptyset$, if $i \ne j,\ 1 \le i,\ j \le m$. So we get

$$\begin{aligned}\Lambda : &= \{t \in [-r,r] : \|H(t)\| \ge \varepsilon\} = \{t \in [-r,r] : \|h(t,u(t),x(t))\| \ge \varepsilon\} \\ &\subset \cup_{k=1}^m \{t \in V_k : \|h(t,u(t),x(t)) - h(t,\beta_k,\gamma_k)\| + \|h(t,\beta_k,\gamma_k)\| \ge \varepsilon\} \\ &\subset \cup_{k=1}^m \left(\left\{t \in V_k : \|h(t,u(t),x(t)) - h(t,\beta_k,\gamma_k)\| \ge \frac{\varepsilon}{2}\right\}\right. \\ &\quad \left.\cup \left\{t \in V_k : \|h(t,\beta_k,\gamma_k)\| \ge \frac{\varepsilon}{2}\right\}\right).\end{aligned}$$

It follows from relation (6) that

$$\Big\{t \in V_k : \|h(t,u(t),x(t)) - h(t,\beta_k,\gamma_k)\| \geq \frac{\varepsilon}{2}\Big\} = \emptyset, \quad k = 1,\ldots,m.$$

Thus, if we set $A_{r,\frac{\varepsilon}{2}}(h_k) := A_{r,\frac{\varepsilon}{2}}(h(t,\beta_k,\gamma_k))$, then $A_{r,\varepsilon}(H) \subset \cup_{k=1}^{m} A_{r,\frac{\varepsilon}{2}}(h_k)$ and

$$\frac{1}{\mu([-r,r])}\int_{[-r,r]}\|H(t)\|d\mu(t) \leq \sum_{k=1}^{m}\frac{1}{\mu([-r,r])}\int_{[-r,r]}\|h_k(t)\|d\mu(t).$$

And since $h \in \varepsilon(\mathbb{R}\times\mathbb{X}\times\mathbb{X},\mathbb{X},\mu)$, we have

$$\lim_{r\to+\infty}\frac{1}{\mu([-r,r])}\int_{[-r,r]}\|h_k(t)\|d\mu(t) = 0,\ k = 1,\cdots,m.$$

It follows that $\lim_{r\to+\infty}\frac{1}{\mu([-r,r])}\int_{[-r,r]}\|H(t)\|d\mu(t) = 0$. According to Lemma 4, we deduce that $H(t) = h(t,u(t),x(t)) \in \varepsilon(\mathbb{R},\mathbb{X},\mu)$. The proof is completed. □

From the above theorem, we have the following results.

Corollary 1. *Let $\mu \in \mathcal{M}$. Suppose that $f = g + h \in PAA(\mathbb{R}\times\mathbb{X}\times\mathbb{X},\mathbb{X},\mu)$ with $g \in AA(\mathbb{R}\times\mathbb{X}\times\mathbb{X},\mathbb{X})$, $h \in \varepsilon(\mathbb{R}\times\mathbb{X}\times\mathbb{X},\mathbb{X},\mu)$, and both f and g are Lipschitzian with $(x,y) \in \mathbb{X}\times\mathbb{X}$ uniformly in $t \in \mathbb{R}$. Then the function defined by $F(\cdot) := f(\cdot,\phi(\cdot),\varphi(\cdot)) \in PAA(\mathbb{R},\mathbb{X},\mu)$ if $\phi,\ \varphi \in PAA(\mathbb{R},\mathbb{X},\mu)$.*

Lemma 10. *Let $\mu \in \mathcal{M}$ and $f = g+h \in PAA(\mathbb{R}\times\mathbb{X}\times\mathbb{X},\mathbb{X},\mu)$. Assume that f and g satisfy condition $(H1),(H2)$. Then the function defined by $F(\cdot) := f(\cdot,u(\cdot),Bu(\cdot)) \in PAA(\mathbb{R},\mathbb{X},\mu)$ if $u \in PAA(\mathbb{R},\mathbb{X},\mu)$.*

Proof. First we note that if $u \in PAA(\mathbb{R},\mathbb{X},\mu)$, then $u = x + y$ with $x \in AA(\mathbb{R},\mathbb{X})$ and $y \in \varepsilon(\mathbb{R},\mathbb{X},\mu)$. Since B is a bounded linear operator on $\mathbb{X}$, it is easy to show that $Bu = Bx + By$ are also bounded and $By(\cdot) \in \varepsilon(\mathbb{R},\mathbb{X},\mu)$. Therefore, $Bx(\cdot) \in AA(\mathbb{R},\mathbb{X})$([20, Corollary 2.16]), we deduce that $Bu(\cdot) \in PAA(\mathbb{R},\mathbb{X},\mu)$. Hence, in view of Theorem 1, we have $F(\cdot) \in PAA(\mathbb{R},\mathbb{X},\mu)$. □

Definition 10 (see [3])**.** *Assume that A generates an integrable solution operator $E_\alpha(t)$. A continuous function $u : \mathbb{R} \to \mathbb{X}$ satisfying the integral equation*

$$u(t) = \int_{-\infty}^{t} E_\alpha(t-s)f(s,u(s),Bu(s))ds, \quad t \in \mathbb{R}$$

is called a mild solution on $\mathbb{R}$ to problem (1).

Let us list the following assumptions:

(H3) A is a sectorial operator of type $\omega < 0$.

(H4) There exist positive constants L_f, L'_f such that

$$\|f(t,x,u) - f(t,y,v)\| \leq L_f\|x-y\| + L'_f\|u-v\|,\ \forall x,\ y,\ u,\ v \in \mathbb{X}.$$

(H5) There exists a continuous nondecreasing function $W : [0, \infty) \to (0, \infty)$ such that

$$\|f(t, x, y)\| \leq W(\|x\| + \|y\|) \quad for \ \ all \ \ t \in \mathbb{R} \ \ and \ \ x, \ y \in \mathbb{X}.$$

Lemma 11. *Let $\mu \in \mathcal{M}$. Let also $f = g + h \in PAA(\mathbb{R}, \mathbb{X}, \mu)$ with $g \in AA(\mathbb{R}, \mathbb{X})$ and $h \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu)$. Then*

$$F(t) := \int_{-\infty}^{t} E_\alpha(t - s) f(s) ds \in PAA(\mathbb{R}, \mathbb{X}, \mu).$$

Proof. Let $F(t) = G(t) + H(t)$ with

$$G(t) := \int_{-\infty}^{t} E_\alpha(t - s) g(s) ds,$$
$$H(t) := \int_{-\infty}^{t} E_\alpha(t - s) h(s) ds.$$

Applying Lemma 3.1 in [5], we obtain that $G \in AA(\mathbb{R}, \mathbb{X})$, and by equation (4), the operator $E_\alpha(t)$ is bounded above by $\frac{C(\theta,\alpha)M}{1+|\omega|t^\alpha}$ which belong to $\mathcal{L}^1(\mathbb{R}_+)$.

Now let us show that $H(t) \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu)$. For $r > 0$, we notice that

$$\begin{aligned}
\frac{1}{\mu([-r, r])} \int_{[-r,r]} \|H(t)\| d\mu(t) &= \frac{1}{\mu([-r, r])} \int_{[-r,r]} \left\| \int_{-\infty}^{t} E_\alpha(t - s) h(s) ds \right\| d\mu(t) \\
&= \frac{1}{\mu([-r, r])} \int_{[-r,r]} \left\| \int_{0}^{\infty} E_\alpha(s) h(t - s) ds \right\| d\mu(t) \\
&\leq \frac{1}{\mu([-r, r])} \int_{[-r,r]} \int_{0}^{\infty} \|E_\alpha(s)\| \|h(t - s)\| ds d\mu(t) \\
&\leq C(\theta, \alpha) M \int_{0}^{\infty} \frac{1}{1 + |\omega| s^\alpha} \left(\frac{1}{\mu([-r, r])} \int_{[-r,r]} \|h(t - s)\| d\mu(t) \right) ds \\
&= C(\theta, \alpha) M \int_{0}^{\infty} \frac{\Omega_r(s)}{1 + |\omega| s^\alpha} ds,
\end{aligned}$$

where $\Omega_r(s) = \frac{1}{\mu([-r,r])} \int_{[-r,r]} \|h(t - s)\| d\mu(t)$.

By the fact that the space $\varepsilon(\mathbb{R}, \mathbb{X}, \mu)$ is translation invariant, it follows that $t \to h(t - s)$ belongs to $\varepsilon(\mathbb{R}, \mathbb{X}, \mu)$ for each $s \in \mathbb{R}$ and hence $\Omega_r(s) \to 0$ as $r \to +\infty$.

Next, since Ω_r is bounded ($\|\Omega_r\| \leq \|h\|_\infty$) and $\frac{1}{1+|\omega|s^\alpha}$ is integrable in $[0, \infty]$, using the Lebesgue dominated convergence theorem it follows that

$$\lim_{r \to +\infty} \int_{0}^{\infty} \frac{\Omega_r(s)}{1 + |\omega| s^\alpha} ds = 0.$$

The proof is now completed. □

Theorem 2. *Let $\mu \in \mathcal{M}$. Let also $f = g + h \in PAA(\mathbb{R} \times \mathbb{X} \times \mathbb{X}, \mathbb{X}, \mu)$ with $g \in AA(\mathbb{R} \times \mathbb{X} \times \mathbb{X}, \mathbb{X})$ and $h \in \varepsilon(\mathbb{R} \times \mathbb{X} \times \mathbb{X}, \mathbb{X}, \mu)$. Assume that (H3)-(H4) hold, then (1) has a unique mild solution in $PAA(\mathbb{R}, \mathbb{X}, \mu)$ provided*

$$(L_f + L'_f \|B\|_{\mathcal{L}(\mathbb{X})}) C(\theta, \alpha) M \frac{|\omega|^{(-\frac{1}{\alpha})}\pi}{\alpha \sin(\pi/\alpha)} < 1, \qquad (*)$$

where the constants $C(\theta, \alpha)$ and M are those defined in Lemma 7.

Proof. Consider the operator $Q : PAA(\mathbb{R}, \mathbb{X}, \mu) \to PAA(\mathbb{R}, \mathbb{X}, \mu)$ such that

$$(Qu)(t) := \int_{-\infty}^{t} E_\alpha(t - \sigma) f(\sigma, u(\sigma), Bu(\sigma)) d\sigma, \quad t \in \mathbb{R}.$$

In view of Corollary 1, Lemmas 10-11, the operator (Qu) is well defined.

Now for $u, v \in PAA(\mathbb{R}, \mathbb{X}, \mu)$. From relation (4) and condition (H4), we have

$$\begin{aligned}
&\|(Qu)(t) - (Qv)(t)\| \\
&\quad = \left\| \int_{-\infty}^{t} E_\alpha(t-\sigma)(f(\sigma, u(\sigma), Bu(\sigma)) - f(\sigma, v(\sigma), Bv(\sigma))) d\sigma \right\| \\
&\quad \le \int_{-\infty}^{t} \|E_\alpha(t-\sigma)\|_{\mathcal{L}(\mathbb{X})} \|f(\sigma, u(\sigma), Bu(\sigma)) - f(\sigma, v(\sigma), Bv(\sigma))\| d\sigma \\
&\quad \le \int_{-\infty}^{t} \frac{C(\theta,\alpha)M}{1 + |\omega|(t-\omega)^\alpha} (L_f \|u(\sigma) - v(\sigma)\| + L'_f \|Bu(\sigma) - Bv(\sigma)\|) d\sigma \\
&\quad \le \left(L_f + L'_f \|B\|_{\mathcal{L}(\mathbb{X})}\right) \sup_{t \in \mathbb{R}} \|u(t) - v(t)\| \int_0^\infty \frac{C(\theta,\alpha)M}{1 + |\omega|\sigma^\alpha} d\sigma \\
&\quad \le \left(L_f + L'_f \|B\|_{\mathcal{L}(\mathbb{X})}\right) C(\theta,\alpha) M \frac{|\omega|^{-\frac{1}{\alpha}}\pi}{\alpha \sin(\pi/\alpha)} \|u - v\|_\infty, \quad \forall t \in \mathbb{R}.
\end{aligned}$$

Thus

$$\|Qu - Qv\|_\infty \le (L_f + L'_f \|B\|_{\mathcal{L}(\mathbb{X})}) C(\theta, \alpha) M \frac{|\omega|^{-\frac{1}{\alpha}}\pi}{\alpha \sin(\pi/\alpha)} \|u - v\|_\infty.$$

From the equation $(*)$ and the Banach contraction principle, we can complete the proof. □

The following existence result is based upon the nonlinear Leray-Schauder alternative theorem.

Theorem 3. *Assume that $\mu \in \mathcal{M}$ and A is a sectorial of type $\omega < 0$. Let $f \in PAA(\mathbb{R} \times \mathbb{X} \times \mathbb{X}, \mathbb{X}, \mu)$ satisfying (H1), (H2) and (H5) and the following additional conditions:*

(i) For each $\mathbb{C} \ge 0$

$$\lim_{|t| \to \infty} \frac{1}{h'(t)} \int_{-\infty}^{t} \frac{W((1 + \|B\|_{\mathcal{L}(\mathbb{X})})\mathbb{C} h'(s))}{1 + |\omega|(t-s)^\alpha} ds = 0,$$

where h' is the function applied to define the space $C_{h'}(\mathbb{X})$.
We set

$$\beta(\mathbb{C}) := C(\theta,\alpha)M \left\| \int_{-\infty}^{t} \frac{W\left((1+\|B\|_{\mathcal{L}(\mathbb{X})})\mathbb{C}h'(s)\right)}{1+|\omega|(t-s)^{\alpha}} ds \right\|_{h'},$$

where $C(\theta,\alpha)$ and M are constants given in inequality (4).

(ii) *For each $\varepsilon > 0$, there is $\delta > 0$ such that for every u, $v \in C_{h'}(\mathbb{X})$, $\|u-v\|_{h'} \leq \delta$ implies that*

$$C(\theta,\alpha)M \int_{-\infty}^{t} \frac{\|f(s,u(s),Bu(s)) - f(s,v(s),Bv(s))\|}{1+|\omega|(t-s)^{\alpha}} ds \leq \varepsilon, \quad for \quad all \quad t \in \mathbb{R}.$$

(iii) $\liminf_{\xi\to\infty} \frac{\xi}{\beta(\xi)} > 1$.

(iv) *For all a, $b \in \mathbb{R}$, $a < b$ and $\Lambda > 0$, the set $\{f(s,x,Bx) : a \leq s \leq b, \ x \in C_{h'}(\mathbb{X}), \ \|x\|_{h'} \leq \Lambda\}$ is relatively compact in $\mathbb{X}$.*

Then equation (1) admits at least one μ-pseudo almost automorphic mild solution.

Proof. We define the operator $\Gamma : C_{h'}(\mathbb{X}) \to C_{h'}(\mathbb{X})$ by

$$(\Gamma x)(t) := \int_{-\infty}^{t} E_{\alpha}(t-s) f(s,x(s),Bx(s)) ds, \quad t \in \mathbb{R}.$$

We will show that Γ has a fixed point in $PAA(\mathbb{R},\mathbb{X},\mu)$. For the sake of convenience, we divide the proof into several steps.
Step 1: For $x \in C_{h'}(\mathbb{X})$, we have that

$$\begin{aligned} \|\Gamma x(t)\| &\leq C(\theta,\alpha)M \int_{-\infty}^{t} \frac{W(\|x(s)\| + \|B\|_{\mathcal{L}(\mathbb{X})}\|x(s)\|)}{1+|\omega|(t-s)^{\alpha}} ds \\ &\leq C(\theta,\alpha)M \int_{-\infty}^{t} \frac{W((1+\|B\|_{\mathcal{L}(\mathbb{X})})\|x\|_{h'}h'(s))}{1+|\omega|(t-s)^{\alpha}} ds. \end{aligned}$$

It follows from condition (i) that Γ is well defined.
Step 2: The operator Γ is continuous.

In fact, for any $\varepsilon > 0$, we take $\delta > 0$ involved in condition (ii). If x, $y \in C_{h'}(\mathbb{X})$ and $\|x-y\|_{h'} \leq \delta$, then

$$\begin{aligned} \|(\Gamma x)(t) - (\Gamma y)(t)\| &\leq C(\theta,\alpha)M \int_{-\infty}^{t} \frac{\|f(s,x(s),Bx(s)) - f(s,y(s),By(s))\|}{1+|\omega|(t-s)^{\alpha}} ds \\ &\leq \varepsilon, \end{aligned}$$

which shows the assertion.
Step 3: The operator Γ is completely continuous.

We define $B_{\Lambda}(\mathbb{X})$ for the closed ball with center at 0 and radius Λ in the space $\mathbb{X}$. Let $V'(t) = \Gamma(B_{\Lambda}(C_{h'}(\mathbb{X})))$ and $v'(t) = \Gamma(x)$ for $x \in B_{\Lambda}(C_{h'}(\mathbb{X}))$. Firstly, we will prove that $V'(t)$ is a relatively compact subset of $\mathbb{X}$ for each $t \in \mathbb{R}$. It follows from condition (i) that the function $s \to \dfrac{W((1+\|B\|_{\mathcal{L}(\mathbb{X})})\Lambda h'(t-s))}{1+|\omega|s^\alpha}$ is integrable on $[0,\infty)$. Hence, for $\varepsilon > 0$, we can choose $a \geq 0$ such that

$$C(\theta,\alpha)M\int_a^\infty \frac{W((1+\|B\|_{\mathcal{L}(\mathbb{X})})\Lambda h'(t-s))}{1+|\omega|s^\alpha}ds \leq \varepsilon.$$

Since

$$v'(t) = \int_0^a E_\alpha(s)f(t-s,x(t-s),Bx(t-s))ds + \int_a^\infty E_\alpha(s)f(t-s,x(t-s),Bx(t-s))ds$$

and

$$\begin{aligned}&\left\|\int_a^\infty E_\alpha(s)f(t-s,x(t-s),Bx(t-s))ds\right\|\\ &\qquad\leq C(\theta,\alpha)M\int_a^\infty \frac{W((1+\|B\|_{\mathcal{L}(\mathbb{X})})\Lambda h'(t-s))}{1+|\omega|s^\alpha}ds \leq \varepsilon,\end{aligned}$$

we get $v'(t) \in \overline{aco(\mathbb{N})} + B_\varepsilon(\mathbb{X})$, where $co(\mathbb{N})$ denotes the convex hull of $\mathbb{N}$ and $\mathbb{N} = \{E_\alpha(s)f(\xi,x,Bx) : 0 \leq s \leq a, t-a \leq \xi \leq t, \|x\|_h \leq \Lambda\}$. Just as the proofs in [4, Theorem 3.5(ii)] and [15, Theorem 4.9(iii)], using the continuity of $E_\alpha(s)$ and property (iv) of f, we can infer that $\mathbb{N}$ is a relatively compact set and $V'(t) \subset \overline{aco(\mathbb{N})} + B_\varepsilon(\mathbb{X})$, which establishes our assertion.

Secondly, we show that the set $V'(t)$ is equicontinuous. In fact, we can decompose

$$\begin{aligned}v'(t+s)-v'(t) &= \int_0^s E_\alpha(\sigma)f(t+s-\sigma,x(t+s-\sigma),Bx(t+s-\sigma))d\sigma\\ &\quad+\int_0^a [E_\alpha(\sigma+s)-E_\alpha(\sigma)]f(t-\sigma,x(t-\sigma),Bx(t-\sigma))d\sigma\\ &\quad+\int_a^\infty [E_\alpha(\sigma+s)-E_\alpha(\sigma)]f(t-\sigma,x(t-\sigma),Bx(t-\sigma))d\sigma.\end{aligned}$$

For each $\varepsilon > 0$, we can choose $a > 0$ and $\delta_1 > 0$ such that

$$\begin{aligned}&\left\|\int_0^s E_\alpha(\sigma)f(t+s-\sigma,x(t+s-\sigma),Bx(t+s-\sigma))d\sigma\right.\\ &\qquad\left.+\int_a^\infty [E_\alpha(\sigma+s)-E_\alpha(\sigma)]f(t-\sigma,x(t-\sigma),Bx(t-\sigma))d\sigma\right\|\\ &\quad\leq C(\theta,\alpha)M\left[\int_0^s \frac{W((1+\|B\|_{\mathcal{L}(\mathbb{X})})\Lambda h'(t+s-\sigma))}{1+|\omega|\sigma^\alpha}d\sigma\right.\\ &\qquad\left.+2\int_a^\infty \frac{W((1+\|B\|_{\mathcal{L}(\mathbb{X})})\Lambda h'(t-\sigma))}{1+|\omega|\sigma^\alpha}d\sigma\right]\\ &\quad\leq \varepsilon/2\end{aligned}$$

for $s \leq \delta_1$. Moreover, since $\{f(t-\sigma,x(t-\sigma),Bx(t-\sigma)) : 0 \leq \sigma \leq a, x \in B_\Lambda(C_{h'}(\mathbb{X}))\}$ is a relatively compact set and E_α is strongly continuous, we can choose $\delta_2 > 0$ such that $\|[E_\alpha(\sigma+s)-E_\alpha(\sigma)]f(t-\sigma,x(t-\sigma),Bx(t-\sigma))\| \le \frac{\varepsilon}{2a}$ for $s<\delta_2$. Combining these estimates, we get $\|v'(t+s)-v'(t)\| \le \varepsilon$ for s small enough and independent of $x \in B_\Lambda(C_{h'}(\mathbb{X}))$.

Finally, applying condition (i), we can see that

$$\frac{\|v'(t)\|}{h'(t)} \le \frac{C(\theta,\alpha)M}{h'(t)} \int_{-\infty}^{t} \frac{W((1+\|B\|_{\mathcal{L}(\mathbb{X})})\Lambda h'(s))}{1+|\omega|(t-s)^\alpha} ds \to 0, |t| \to \infty$$

and this convergence is independent of $x \in B_\Lambda(C_{h'}(\mathbb{X}))$. Hence, by Lemma 8, V' is a relatively compact set in $C_{h'}(\mathbb{X})$.

Step 4: The set $\{x^\lambda : x^\lambda = \lambda\Gamma(x^\lambda), \lambda \in (0,1)\}$ is bounded.

Let us assume that $x^\lambda(\cdot)$ is a solution of equation $x^\lambda = \lambda\Gamma(x^\lambda)$ for some $0<\lambda<1$.

We can estimate

$$\begin{aligned} \|x^\lambda(t)\| &= \lambda \left\| \int_{-\infty}^{t} E_\alpha(t-s) f(s,x^\lambda(s),Bx^\lambda(s))ds \right\| \\ &\le C(\theta,\alpha)M \int_{-\infty}^{t} \frac{W((1+\|B\|_{\mathcal{L}(\mathbb{X})})\|x^\lambda\|_{h'}h'(s))}{1+|\omega|(t-s)^\alpha} ds \\ &\le \beta(\|x^\lambda\|_{h'})h'(t). \end{aligned}$$

Hence, we get

$$\frac{\|x^\lambda\|_{h'}}{\beta(\|x^\lambda\|_{h'})} \le 1,$$

and combining with condition (iii), we conclude that the set $\{x^\lambda : x^\lambda = \lambda\Gamma(x^\lambda), \lambda \in (0,1)\}$ is bounded.

Step 5: It follows from hypotheses (H1)-(H2) and Lemma 10 that the function $t \to f(t,x(t),Bx(t))$ belongs to $PAA(\mathbb{R},\mathbb{X},\mu)$ whenever $x \in PAA(\mathbb{R},\mathbb{X},\mu)$. Hence, using Lemma 11, we obtain $\Gamma(PAA(\mathbb{R},\mathbb{X},\mu)) \subset PAA(\mathbb{R},\mathbb{X},\mu)$ noting that $PAA(\mathbb{R},\mathbb{X},\mu)$ is a closed subspace of $C_{h'}(\mathbb{X})$. Consequently, we can consider $\Gamma : PAA(\mathbb{R},\mathbb{X},\mu) \to PAA(\mathbb{R},\mathbb{X},\mu)$. Using Steps 1-3, we deduce that this map is completely continuous. Applying Lemma 9, we infer that Γ has a fixed point $x \in PAA(\mathbb{R},\mathbb{X},\mu)$, which completes the proof. □

Remark 2. *Similar conditions (ii)-(iv) in Theorem 3 are applied in [15] to investigate the compact almost automorphic solutions of a semilinear integral equation, and to consider pseudo almost periodic solutions of a fractional order differential equation in [4].*

From [3, 15] and Theorem 3, we deduce the following corollary.

Corollary 2. *Assume that $\mu \in \mathcal{M}$ and A is a sectorial of type $\omega < 0$. Let $(H2)$ hold and $f \in PAA(\mathbb{R},\mathbb{X},\mu)$ satisfying $(H1)$ and inequality (4) and the following Holder type condition:*

$$\|f(t,x_1,x_2)-f(t,y_1,y_2)\| \le \gamma\left[\|x_1-y_1\|^\beta + \|x_2-y_2\|^\beta\right], \quad 0<\beta<1,$$

for all $t \in \mathbb{R}$ and x_i, $y_i \in \mathbb{X}$ for $i=1,2$, where $\gamma>0$ is a constant. Moreover, assume the following conditions:

(a) $f(t,0,0) = q$.

(b) $\sup_{t\in\mathbb{R}} C(\theta,\alpha)M \int_{-\infty}^{t} \frac{(1+\|B\|_{\mathcal{L}(\mathbb{X})})h'(s)^\beta}{1+|\omega|(t-s)^\alpha} ds = \gamma_2 < \infty$.

(c) For all $a, b \in \mathbb{R}$, $a < b$ *and* $p > 0$, *the set* $\{f(s,x,Bx) : a \le s \le b, x \in C_{h'}(\mathbb{X}), \|x\|_{h'} \le p\}$ *is relatively compact in* $\mathbb{X}$.

Then equation (1) admits a μ*-pseudo almost automorphic mild solution.*

Proof. Let $\gamma_0 = \|q\|$, $\gamma_1 = \gamma$. We take $W(\xi_1 + \xi_2) = \gamma_0 + \gamma_1 \left[\xi_1^\beta + \xi_2^\beta\right]$. Then condition (H5) is satisfied. It follows from (b); we can see that function f satisfies (i) in Theorem 3. Note that for each $\varepsilon > 0$ there is $0 < \delta^\beta < \frac{\varepsilon}{\gamma_1\gamma_2}$ such that for every $x, y \in C_{h'}(\mathbb{X})$, $\|x-y\|_{h'} \le \delta$ implies that

$$C(\theta,\alpha)M \int_{-\infty}^{t} \frac{\|f(s,x(s),Bx(s)) - f(s,y(s),By(s))\|}{1+|\omega|(t-s)^\alpha} ds \le \varepsilon$$

for all $t \in \mathbb{R}$. Assumption (iii) in Theorem 3 can be easily verified by the definition of W. So from Theorem 3 we can prove that problem (1) has a μ-pseudo almost automorphic mild solution. □

Corollary 3. *Let* $\mu \in \mathcal{M}$. *Let also* $f = g + h \in PAA(\mathbb{R}\times\mathbb{X}\times\mathbb{X},\mathbb{X},\mu)$ *with* $g \in AA(\mathbb{R}\times\mathbb{X}\times\mathbb{X},\mathbb{X},\mu)$ *and* $h \in \varepsilon(\mathbb{R}\times\mathbb{X}\times\mathbb{X},\mathbb{X},\mu)$. *Assume that (H3)-H(4) hold. Then*

$$D_t^\alpha u(t) = Au(t) + D_t^{\alpha-1} f(t,u(t),u(t-\tau)), \quad \tau,\ t \in \mathbb{R},$$

has a unique mild solution in $PAA(\mathbb{R}\times\mathbb{X}\times\mathbb{X},\mathbb{X},\mu)$ *provided*

$$(L_f + L_f')C(\theta,\alpha)M \frac{|\omega|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)} < 1.$$

Proof. Since $\forall t \in \mathbb{R},\ u(t) \in \mathbb{X}$, it suffices to consider shift operators defined as $\forall t \in \mathbb{R},\ Bu(t) := u(t-\tau)$. Thus $\|B\|_{\mathcal{L}(\mathbb{X})} = 1$. □

Example 1. *To illustrate Theorem 2, we consider the following fractional differential equation:*

$$D_t^\alpha u(t,x) = \frac{\partial^2}{\partial x^2} u(t) - au(t,x) + D_t^{\alpha-1} f(u(t,x), Bu(t,x)), \tag{7}$$

$t \in \mathbb{R}, x \in [0,\pi]$, *with boundary conditions* $u(t,0) = (t,\pi) = 0, \quad t \in \mathbb{R}$, *where* $1 < \alpha < 2$, $B = \zeta I_d$,

$$f(u(t,x), Bu(t,x)) = \left(\sin\frac{1}{2+\cos t + \cos\sqrt{2}t} + \sin^-(-|t|)\right)(\sin(u(t,x)) + \zeta u(t,x))$$

for each $t \in \mathbb{R}$, a, $\zeta > 0$ *(for* $\sin^-(-|t|)$*, we can see [7]).*

Set $(\mathbb{X}, \|\cdot\|_{\mathbb{X}}) = (L^2([0,\pi]), \|\cdot\|_2)$ *and define*

$$D(A) = \{u \in L^2([0,\pi]) : u'' \in L^2([0,\pi]), u(0) = u(\pi) = 0\},$$

$$Au = \Delta u = u'', \quad \forall u \in D(A).$$

It is well-known that A is the infinitesimal generator of an analytic semigroup on $L^2([0,\pi])$. Thus A is a sectorial of type $\omega = -a < 0$. Set the measure whose Radon-Nikodym derivative ϱ is defined by $\varrho(t) = \sin^+(-|t|) + \exp(-|t|)$ (see [7]) for $t \in \mathbb{R}$. Then $\varepsilon(\mathbb{R}, \mathbb{X}, \mu)$ is translation invariant. We have

$$\begin{aligned} &\|f(t, u(t,\cdot), \zeta u(t,\cdot)) - f(t, v(t,\cdot), \zeta v(t,\cdot))\|_2 \\ \le\ &\|u(t,\cdot) - v(t,\cdot)\|_2 + \zeta\|u(t,\cdot) - v(t,\cdot)\|_2 \\ \le\ &(1+\zeta)\|u(t,\cdot) - v(t,\cdot)\|_2 \end{aligned}$$

for all $u(t,\cdot)$, $v(t,\cdot) \in L^2([0,\pi])$, $t \in \mathbb{R}$. Furthermore, one can easily check that $t \mapsto \sin \frac{1}{2+\cos t+\cos\sqrt{2}t} + \sin^-(-|t|)$ belongs to $PAA(\mathbb{R}, \mathbb{X}, \mu)$ with $\sin^-(-|t|)$ as a μ-ergodic component and $\sin \frac{1}{2+\cos t+\cos\sqrt{2}t}$ as its almost automorphic component. Consequently, f is a μ-pseudo almost automorphic function with Radon-Nikodym derivative $\varrho(t) = \sin^+(-|t|) + \exp(-|t|)$ for $t \in \mathbb{R}$. Hence by choosing ζ and a such that

$$(1+\zeta)a^{1/\alpha} < \frac{\alpha \sin(\pi/\alpha)}{C(\theta,\alpha)M}.$$

Assumptions of Theorem 2 are satisfied and (4.1) *has a unique solution in $PAA(\mathbb{R}, \mathbb{X}, \mu)$.*

Acknowledgement

The author would like to thank the referees for their helpful suggestions.

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