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A Newton two-stage waveform relaxation method for solving systems of nonlinear algebraic equations

Davod Khojasteh Salkuyeh[1,*] and Zeinab Hassanzadeh[1]

[1] *Faculty of Mathematical Sciences, University of Guilan, P.O. Box 41 335-1914 Guilan, Rasht, Iran*

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Abstract. In this paper, a Newton two-stage waveform relaxation method is introduced to solve systems of nonlinear algebraic equations. The proposed method is derived from the Newton waveform relaxation method by adding further a splitting function and inner iterations. Sufficient conditions for the convergence of the method have been provided. Some numerical examples are given to show the effectiveness of the presented method and to compare it with two available methods.
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1. Introduction

Waveform relaxation (WR) iterative method was first introduced by Lelarasmee in his Ph.D. dissertation on the time domain analysis of large scale nonlinear dynamical systems in 1982 [10]. After that, the WR method has been investigated in order to numerically solve dynamical linear and nonlinear systems of ordinary differential equations (ODEs), partial differential equations (PDEs) and differential-algebraic equations (DAEs); see for example [3, 7, 8, 11] and references therein. An interesting feature of the WR method is the fact that it can be implemented on parallel computations. In [5], the WR method was coupled with a two-stage iterations strategy for solving initial value problems (IVPs) of ODEs.

The Newton method is one of the basic and important numerical methods for solving systems of nonlinear equations, which converges locally and quadratically (see [2, 4, 9]). In [12], the Newton method in conjunction with the WR method, which is reviewed here, has been used to solve a system of nonlinear equations. Assume that we are solving the system of equations

$$f(x) = 0, \tag{1}$$

with an initial guess x^0 of the unknown solution x^*, where $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$. The key idea of the Newton waveform relaxation method (NWR) for solving system (1) is to choose a splitting function $F : D \times D \to \mathbb{R}^n$, such that

$$F(x, x) = f(x), \tag{2}$$

*Corresponding author. *Email addresses:* `khojasteh@guilan.ac.ir` (D. K. Salkuyeh), `hassanzadeh.83@gmail.com` (Z. Hassanzadeh)

for any $x \in \mathbb{R}^n$, which is called the consistency condition. The consistency condition (2) is useful to prove the convergence of the NWR iterative method. Denoting the kth iterate by x^k, this then leads to the following iteration scheme

$$F(x^k, x^{k+1}) = 0. \tag{3}$$

Consistency condition (2) ensures that the solution to (1) is a fixed point of (3). In [12], by utilizing the Newton method a procedure of the following form has been stated to solve Eq. (1).

Algorithm 1:

1. **for** $k = 0, 1, \ldots$
2. with a given initial approximation $\bar{x}^0$ of x^{k+1}
3. **for** $m = 0, 1, \ldots, M-1$
4. Solve $F_2(x^k, \bar{x}^m)\Delta\bar{x}^m = -F(x^k, \bar{x}^m)$ for $\Delta\bar{x}^m$
5. $\bar{x}^{m+1} := \bar{x}^m + \Delta\bar{x}^m$
6. **end for**
7. $x^{k+1} := \bar{x}^M$
8. **end for**

In the above procedure, $F_2(x^k, \bar{x}^m)$ is the Fréchet derivative [2] of $F(x, y)$ with respect to y at $(x^k, \bar{x}^m)$, i.e.,

$$F_2(x^k, \bar{x}^m) = \frac{\partial F(x, y)}{\partial y}|_{(x^k, \bar{x}^m)}, \tag{4}$$

which is the $n \times n$ Jacobian matrix evaluated at $(x^k, \bar{x}^m)$. Moreover, it is assumed that the matrix $F_2(x^k, \bar{x}^m)$ is nonsingular for all $m = 0, 1, \ldots, M-1$, and $k = 0, 1, \ldots$. By setting $\bar{x}^0 = x^k$ and $M = 1$ in the above method, we obtain the iterative method

$$\begin{cases} F_2(x^k, x^k)\Delta x^k = -f(x^k), \\ x^{k+1} = x^k + \Delta x^k, \end{cases} \quad k = 0, 1, \ldots, \tag{5}$$

which is called the *Newton waveform relaxation (NWR)* method. The importance of the splitting function F is that a special choice of F can make the matrix $F_2(x, x)$ diagonal and nonsingular. Therefore, NWR method (5) can be processed stably by computer in parallel computations with a multi-processor.

In this paper, we establish the *Newton two-stage waveform relaxation* (NTSWR) method for solving system (1), which is generated by selecting splitting for the function F and adding another nested loop within the inner iteration of Algorithm 1. The advantage of this procedure is its rapid convergence, which is studied in the following sections.

The rest of the paper is organized as follows. After introducing the NTSWR method in Section 2, we investigate the convergence analysis of the method in Section 3. Finally, some numerical examples are given in Section 4.

2. Newton two-stage waveform relaxation method

As we mentioned in the previous section, to solve system (1), the NWR method is generated in the form (5). We choose the splitting function $G : D \times D \times D \to \mathbb{R}^n$ for F such that

$$G(x, y, y) = F(x, y), \quad G(x, x, x) = F(x, x) = f(x), \tag{6}$$

for any $x, y \in \mathbb{R}^n$, which is called the consistency condition for the splitting function G. By replacing G in (3), the inner iterations are generated. Hence we have

$$G(x^k, z^v, z^{v+1}) = 0, \tag{7}$$

where v denotes the inner iterations for the kth outer iteration. This procedure can be summarized as Algorithm 2.

Algorithm 2:

1. **for** $k = 0, 1, \ldots$
2. with a given initial approximation $\bar{x}^0$ of x^{k+1}
3. $z^0 := x^k$
4. **for** $v = 0, 1, \ldots, s-1$
5. **for** $m = 0, 1, \ldots, M-1$
6. Solve $G_3(x^k, z^v, \bar{z}^m)\Delta\bar{z}^m = -G(x^k, z^v, \bar{z}^m)$ for $\Delta\bar{z}^m$
7. $\bar{z}^{m+1} := \bar{z}^m + \Delta\bar{z}^m$
8. **end for**
9. $z^{v+1} := \bar{z}^M$
10. **end for**
11. $x^{k+1} := z^s$
12. **end for**

In Algorithm 2, $G_3(x^k, z^v, \bar{z}^m)$ is the Fréchet derivative [2] of $G(x, y, z)$ with respect to z at $(x^k, z^v, \bar{z}^m)$, i.e.,

$$G_3(x^k, z^v, z^{v+1}) = \frac{\partial G(x, y, z)}{\partial z}|_{(x^k, z^v, z^{v+1})}. \tag{8}$$

which is the $n \times n$ Jacobian matrix evaluated at $(x^k, z^v, \bar{z}^m)$.

Similarly to the NWR method, by setting $\bar{z}^0 = z^v$, $M = 1$, and using consistency condition (6) we obtain the following iterative method

$$\begin{cases} G_3(x^k, z^v, z^v)\Delta z^v = -F(x^k, z^v), \quad z^{v+1} = z^v + \Delta z^v, \\ z^0 = x^k, \quad x^{k+1} = z^s, \quad k = 0, 1, \ldots, \quad v = 0, 1, \ldots, s-1, \\ \text{with a given initial approximation } x^0 \text{ of } x^{k+1}. \end{cases} \tag{9}$$

Hereafter, we call this procedure the NTSWR method. In the next section, we give some sufficient conditions for the convergence of this method.

3. Convergence analysis

At first, we recall theorems concerning the convergence of the NWR method which is needed in the next section.

Theorem 1 ([12]). *Let $F : D \times D \to \mathbb{R}^n$ be a continuous mapping satisfying consistency condition (2) with $D \subseteq \mathbb{R}^n$ open and convex. In addition, assume that the function $F(x, y)$ is Fréchet differentiable with the second variable y and matrix $F_2(x, y)$ is nonsingular for any $x, y \in D$. If*

$$\|F_2^{-1}(x,x)(F(y,y) - F(x,y))\| \leq \alpha\|x - y\| \quad \forall x, y \in D, \tag{10}$$

$$\|F_2^{-1}(x,x)(F_2(x,x) - F_2(x,y))\| \leq \beta\|x - y\| \quad \forall x, y \in D, \tag{11}$$

$$\begin{aligned} \|x^0 - x^*\| = \epsilon_0, &\quad \varrho = \alpha + \frac{1}{2}\beta\epsilon_0 < 1, \\ B(x^*, r) \subset D, &\quad r = \|x^0 - x^*\|, \end{aligned} \tag{12}$$

where $B(x^*, r)$ is a ball around x^* with radius r, then the sequence $\{x^k\}$ obtained from the NWR method is well defined, remains in the open ball $B(x^*, r)$ and converges to x^* with $F(x^*, x^*) = 0$ (i.e., $f(x^*) = 0$).

Corollary 1 ([12]). *If the conditions concerning the splitting function F in Lemma 1 are fulfilled, and moreover, $\alpha < 1$ in (10) and $\beta = 0$ in (11), i.e., the Jacobian matrix $F_2(x, y)$ is independent of the second variable y, then the NWR method converges globally.*

By exerting extensive affine covariant Lipschitz condition [6] for functions of three variables we prove the convergence of the NTSWR method in the following theorems.

Theorem 2. *Let $G : D \times D \times D \to \mathbb{R}^n$ be a continuous function with D an open and convex set in $\mathbb{R}^n$ satisfying consistency condition (6). Let also the function $G(x, y, z)$ be Fréchet differentiable with respect to the third variable z and let the matrix $G_3(x, y, z)$ be nonsingular for any $x, y, z \in D$. Assume that*

$$\|G_3^{-1}(y,x,x)(G(y,z,z) - G(z,z,z))\| \leq \alpha\|y - z\| \quad \forall x, y, z \in D, \tag{13}$$

$$\|G_3^{-1}(x,y,y)(G(x,y,z) - G(x,z,z))\| \leq \beta\|y - z\| \quad \forall x, y, z \in D, \tag{14}$$

$$\|G_3^{-1}(x,y,y)(G_3(x,y,z) - G_3(x,y,y))\| \leq \gamma\|y - z\| \quad \forall x, y, z \in D, \tag{15}$$

and

$$L = \alpha + \beta + \frac{1}{2}\gamma\epsilon^0 < 1, \quad \epsilon^1 \leq L\epsilon^0 \quad \textit{and} \quad B(x^*, \epsilon^0) \subset D, \tag{16}$$

where $\|x^j - x^*\| = \epsilon^j$ for $j = 0, 1$, and $B(x^*, \epsilon^0)$ is a ball about x^* with radius ϵ^0. Then the sequence $\{x^k\}$ obtained from the NTSWR method is well defined, remains in the open ball $B(x^*, \epsilon^0)$ and converges to x^* with $G(x^*, x^*, x^*) = 0$ (i.e., $f(x^*) = 0$).

Proof. Set $e^v = z^v - x^*$ and $e^k = x^k - x^*$. From Eq. (8), $f(x^*) = 0$ and consistency condition (6), we have

$$\begin{aligned}\|e^v + \Delta z^v\| &= \|e^v + G_3^{-1}(x^k, z^v, z^v)(-G(x^k, z^v, z^v)) + G_3^{-1}(x^k, z^v, z^v)f(x^*)\| \\ &= \|G_3^{-1}(x^k, z^v, z^v)\left(G(x^*, x^*, x^*) - G(x^k, z^v, z^v) + G_3(x^k, z^v, z^v)e^v\right)\| \\ &\leq \|G_3^{-1}(x^k, z^v, z^v)\left(G(x^*, x^*, x^*) - G(x^k, x^*, x^*)\right)\| \\ &\quad + \|G_3^{-1}(x^k, z^v, z^v)\left(G(x^k, x^*, x^*) - G(x^k, z^v, z^v) + G_3(x^k, z^v, z^v)e^v\right)\|.\end{aligned}$$

By applying condition (13), we derive

$$\begin{aligned}\|e^v + \Delta z^v\| &\leq \|G_3^{-1}(x^k, z^v, z^v)\left(G(x^k, x^*, x^*) - G(x^k, z^v, z^v) + G_3(x^k, z^v, z^v)e^v\right)\| \\ &\quad + \alpha\|e^k\| \\ &\leq \alpha\|e^k\| + \|G_3^{-1}(x^k, z^v, z^v)\left(G(x^k, x^*, x^*) - G(x^k, z^v, x^*)\right)\| \\ &\quad + \|G_3^{-1}(x^k, z^v, z^v)\left(G(x^k, z^v, x^*) - G(x^k, z^v, z^v) + G_3(x^k, z^v, z^v)e^v\right)\|.\end{aligned}$$

From condition (14) we can write

$$\begin{aligned}\|e^v + \Delta z^v\| &\leq \|G_3^{-1}(x^k, z^v, z^v)\left(G(x^k, z^v, x^*) - G(x^k, z^v, z^v) + G_3(x^k, z^v, z^v)e^v\right)\| \\ &\quad + \alpha\|e^k\| + \beta\|e^v\| \\ &\leq \|G_3^{-1}(x^k, z^v, z^v)\left(\int_0^1 \left(G_3(x^k, z^v, z^v) - G_3(x^k, z^v, x^* + se^v)\right)e^v ds\right)\| \\ &\quad + \alpha\|e^k\| + \beta\|e^v\|.\end{aligned}$$

By putting condition (15), it is deduced that

$$\|e^v + \Delta z^v\| \leq \alpha\|e^k\| + \beta\|e^v\| + \int_0^1 \gamma\|z^v - x^* - se^v\|\|e^v\|ds.$$

Therefore,

$$\|e^v + \Delta z^v\| \leq \alpha\|e^k\| + \|e^v\|\left(\beta + \frac{\gamma}{2}\|e^v\|\right). \tag{17}$$

As we mentioned, for a fixed number s of inner iterations, (17) is equivalent to

$$\|z^s - x^* + \Delta z^s\| \leq \alpha\|e^k\| + \|z^s - x^*\|\left(\beta + \frac{\gamma}{2}\|z^s - x^*\|\right). \tag{18}$$

By substituting $z^s = x^{k+1}$, (18) becomes

$$\|e^{k+2}\| \leq \alpha\|e^k\| + \|e^{k+1}\|\left(\beta + \frac{\gamma}{2}\|e^{k+1}\|\right). \tag{19}$$

For the brevity of notation, let $\epsilon^k = \|e^k\|$, $k = 0, 1, \ldots$. Then Eq. (19) takes the following form

$$\epsilon^{k+2} \leq \epsilon^{k+1}\left(\beta + \frac{\gamma}{2}\epsilon^{k+1}\right) + \alpha\epsilon^k. \tag{20}$$

By the mathematical principle of induction, we show that

$$\begin{cases} \epsilon^{2m-1} \le L^m \epsilon^0, \\ \quad \epsilon^{2m} \le L^m \epsilon^0, \end{cases} \quad m = 1, 2, \ldots . \tag{21}$$

For $m = 1$, we should prove

$$\begin{cases} \epsilon^1 \le L\epsilon^0, \\ \epsilon^2 \le L\epsilon^0. \end{cases}$$

The first inequality follows from (16), and to prove the second inequality, invoking Eq. (20) yields

$$\begin{aligned} \epsilon^2 &\le \epsilon^1 \left(\beta + \frac{\gamma}{2}\epsilon^1\right) + \alpha\epsilon^0 \\ &\le L\epsilon^0 \left(\beta + \frac{\gamma}{2}L\epsilon^0\right) + \alpha\epsilon^0 \\ &\le \epsilon^0 \left(L\left(\beta + \frac{\gamma}{2}L\epsilon^0\right) + \alpha\right) \\ &\le \epsilon^0 \left(\beta + \frac{\gamma}{2}\epsilon^0 + \alpha\right) \qquad \text{(from } L < 1) \\ &= L\epsilon^0. \end{aligned}$$

Now assume that Eq. (21) holds for fixed m. Then, for $m + 1$ we have

$$\begin{aligned} \epsilon^{2(m+1)-1} = \epsilon^{2m+1} &\le \epsilon^{2m} \left(\beta + \frac{\gamma}{2}\epsilon^{2m}\right) + \alpha\epsilon^{2m-1} \\ &\le L^m\epsilon^0 \left(\beta + \frac{\gamma}{2}\epsilon^0\right) + \alpha L^m \epsilon^0 \quad \text{(from (21) and } \epsilon^{2m} \le L^m\epsilon^0 \le \epsilon^0) \\ &= L^{m+1}\epsilon^0, \end{aligned} \tag{22}$$

and

$$\begin{aligned} \epsilon^{2(m+1)} &= \epsilon^{2m+2} \\ &\le \epsilon^{2m+1} \left(\beta + \frac{\gamma}{2}\epsilon^{2m+1}\right) + \alpha\epsilon^{2m} \\ &\le L^{m+1}\epsilon^0 \left(\beta + \frac{\gamma}{2}\epsilon^0\right) + \alpha L^m\epsilon^0 \quad \text{(from (21), (22) and } \epsilon^{2m+1} \le L^{m+1}\epsilon^0 \le \epsilon^0) \\ &\le L^m\epsilon^0 \left(L\left(\beta + \frac{\gamma}{2}\epsilon^0\right) + \alpha\right) \\ &= L^{m+1}\epsilon^0, \end{aligned}$$

which completes the induction.

Now, since $0 \le L < 1$, from (21) we see that

$$\lim_{k \to +\infty} x^k = x^*,$$

which completes the proof. □

Corollary 2. *Assume that the conditions concerning the splitting function G in Theorem 2 are fulfilled and, moreover, $\alpha + \beta < 1$, $\epsilon^1 \le L\epsilon^0$ and the Jacobian matrix $G_3(x, y, z)$ is independent of the third variable z, i.e., $\gamma = 0$. Then the NTSWR method converges to the solution.*

Proof. The proof is obviously deduced from Theorem 2. □

4. Numerical results

In this section, we give three numerical examples to validate the theoretical results and to numerically compare the Newton, the NWR and the NTSWR methods. All numerical experiments were computed in double precision using some MATLAB codes on a Pentium 4 PC, with a 3.00 GHz CPU and 1.96GB of RAM. In general, for an arbitrary given system of nonlinear equations, it is time consuming and difficult to verify sufficient conditions for the convergence given in theorems of the previous section as well as the convergence conditions of the Newton method given in the literature. However, to illustrate the presented theoretical results we verify sufficient conditions for the convergence of the NWR and the NTSWR method for Example 1.

Example 1. *Consider the system of nonlinear equations*

$$f(x) = \begin{pmatrix} 2\sin x_1 + e^{\sin x_2} \\ x_1 + \sin x_2 + 3 \end{pmatrix} = 0. \tag{23}$$

We solve this example by the Newton, the NWR and the NTSWR method. Let

$$D = [-3, 0] \times [-0.3, -0.2].$$

The splitting function F is supposed to be

$$F(x^k, x^{k+1}) = \begin{pmatrix} 2\sin x_1^k + (a(x_2^{k+1} - x_2^k) + 1)e^{\sin x_2^k} \\ ax_1^{k+1} + (1-a)x_1^k + \sin x_2^k + 3 \end{pmatrix}. \tag{24}$$

It is clear that $F(x,x) = f(x)$; therefore consistency condition (2) is satisfied for any $0 \neq a \in \mathbb{R}$. We consider the splitting function G for F in the following form

$$G(x^k, z^v, z^{v+1}) = \begin{pmatrix} 2\sin x_1^k + (ab(z_2^{v+1} - z_2^v) + az_2^v - ax_2^k + 1)e^{\sin x_2^k} \\ ab(z_1^{v+1} - z_1^v) + az_1^v + (1-a)x_1^k + \sin x_2^k + 3 \end{pmatrix}. \tag{25}$$

Obviously, we have $G(x,z,z) = F(x,z)$ and $G(x,x,x) = F(x,x) = f(x)$, and hence G satisfies consistency conditions (6) for any $a, b \in \mathbb{R}$ and $a, b \neq 0$. We first investigate the conditions of Corollary 1 to apply the NWR method. According to the definition of the Jacobian matrix, we have

$$F_2(x^k, x) = \begin{pmatrix} 0 & ae^{\sin x_2^k} \\ a & 0 \end{pmatrix}. \tag{26}$$

Thus the matrix function $F_2(x^k, x)$ is nonsingular for any $x \in \mathbb{R}^2$ and $a \neq 0$, and Lipschitz condition (11) holds with the constant $\beta = 0$. From (24) and (26), we can write

$$\begin{aligned} &\left\| F_2^{-1}(x^k, x^k)(F(x^{k+1}, x^{k+1}) - F(x^k, x^{k+1})) \right\|_1 \\ &= \left\| \begin{pmatrix} \frac{(1-a)(x_1^{k+1} - x_1^k) + \sin x_2^{k+1} - \sin x_2^k}{a} \\ \frac{e^{\sin x_2^{k+1}} - e^{\sin x_2^k} + 2(\sin x_1^{k+1} - \sin x_1^k) - ae^{\sin x_2^k}(x_2^{k+1} - x_2^k)}{ae^{\sin x_2^k}} \end{pmatrix} \right\|_1. \end{aligned} \tag{27}$$

By applying the mean value theorem, we have

$$\begin{cases} \sin x_2^{k+1} - \sin x_2^k = (\cos\gamma_1)(x_2^{k+1} - x_2^k), & \gamma_1 \in (x_2^k, x_2^{k+1}), \\ e^{\sin x_2^{k+1}} - e^{\sin x_2^k} = (\cos\gamma_2)e^{\sin\gamma_2}(x_2^{k+1} - x_2^k), & \gamma_2 \in (x_2^k, x_2^{k+1}), \\ \sin x_1^{k+1} - \sin x_1^k = (\cos\gamma_3)(x_1^{k+1} - x_1^k), & \gamma_3 \in (x_1^k, x_1^{k+1}). \end{cases} \tag{28}$$

Substituting relations (28) in (27) yields

$$\begin{aligned} &\left\|F_2^{-1}(x^k, x^k)(F(x^{k+1}, x^{k+1}) - F(x^k, x^{k+1}))\right\|_1 \\ &= \left\| \begin{pmatrix} \frac{(1-a)(x_1^{k+1}-x_1^k)+(\cos\gamma_1)(x_2^{k+1}-x_2^k)}{a} \\ \frac{((\cos\gamma_2)e^{\sin\gamma_2}-ae^{\sin x_2^k})(x_2^{k+1}-x_2^k)+2(\cos\gamma_3)(x_1^{k+1}-x_1^k)}{ae^{\sin x_2^k}} \end{pmatrix} \right\|_1 \leq \alpha(a) \left\|x^{k+1} - x^k\right\|_1, \end{aligned}$$

where $\alpha(a)$ is given by

$$\begin{aligned} \alpha(a) &= \max\left\{ \left|\frac{1-a}{a}\right|, \max_{\gamma_1}\left|\frac{\cos\gamma_1}{a}\right|, \max_{\gamma_2, x_2^k}\left|\frac{(\cos\gamma_2)e^{\sin\gamma_2} - ae^{\sin x_2^k}}{ae^{\sin x_2^k}}\right|, \max_{\gamma_3, x_2^k}\left|\frac{2\cos\gamma_3}{ae^{\sin x_2^k}}\right| \right\} \\ &\leq \max\left\{ \left|\frac{1-a}{a}\right|, \frac{1}{|a|}, \max_{\gamma_2, x_2^k}\left|\frac{1}{a}\cos\gamma_2 e^{\sin\gamma_2 - \sin x_2^k} - 1\right|, \frac{2e}{|a|} \right\} =: P. \end{aligned}$$

We set $a = 6$. In this case it is easy to see that $P \leq 0.91$. Hence, conditions of Corollary 1 are satisfied and therefore the NWR method (5) can be applied.

In continuation, we peruse the conditions of Corollary 2 for splitting function G to use the NTSWR method. Obviously the matrix

$$G_3(x^k, z^v, z) = \begin{pmatrix} 0 & abe^{\sin x_2^k} \\ ab & 0 \end{pmatrix}, \tag{29}$$

is nonsingular for any $z \in \mathbb{R}^2$ and with the Lipschitz constant $\gamma = 0$. Similarly to the NWR method, by (25), (29) and the mean value theorem, we have

$$\begin{aligned} &\left\|G_3^{-1}(z^v, x^k, x^k)(G(z^v, z^{v+1}, z^{v+1}) - G(z^{v+1}, z^{v+1}, z^{v+1}))\right\|_1 \\ &= \left\| \begin{pmatrix} \frac{\sin z_2^{v+1} - \sin z_2^v + (1-a)(z_1^{v+1} - z_1^v)}{ab} \\ \frac{2(\sin z_1^{v+1} - \sin z_1^v) + e^{\sin z_2^{v+1}} - e^{\sin z_2^v} - ae^{\sin z_2^v}(z_2^{v+1} - z_2^v)}{abe^{\sin x_2^k}} \end{pmatrix} \right\|_1 \\ &= \left\| \begin{pmatrix} \frac{\cos\gamma_4(z_2^{v+1} - z_2^v) + (1-a)(z_1^{v+1} - z_1^v)}{ab} \\ \frac{2(\cos\gamma_5)(z_1^{v+1} - z_1^v) + ((\cos\gamma_6)e^{\sin\gamma_6} - ae^{\sin z_2^v})(z_2^{v+1} - z_2^v)}{abe^{\sin x_2^k}} \end{pmatrix} \right\|_1 \\ &\leq \alpha(a, b) \left\|z^{v+1} - z^v\right\|_1. \end{aligned}$$

Therefore

$$\alpha(a,b) = \max\left\{\left|\frac{1-a}{ab}\right|, \max_{\gamma_1}\left|\frac{\cos\gamma_4}{ab}\right|, \max_{\gamma_6,z_2^{\nu},x_2^k}\left|\frac{(\cos\gamma_6)e^{\sin\gamma_6}-ae^{\sin z_2^{\nu}}}{abe^{\sin x_2^k}}\right|,\right.$$
$$\left.\max_{\gamma_5,x_2^k}\left|\frac{2\cos\gamma_5}{abe^{\sin x_2^k}}\right|\right\}$$
$$\leq \max\left\{\left|\frac{1-a}{ab}\right|, \frac{1}{|ab|}, \max_{\gamma_6,z_2^{\nu},x_2^k}\left|\frac{(\cos\gamma_6)e^{\sin\gamma_6}-ae^{\sin z_2^{\nu}}}{abe^{\sin x_2^k}}\right|, \frac{2e}{|ab|}\right\} =: Q_1.$$

We set $a = 6$ *and* $b = 1.1$. *It can be seen that* $Q_1 \leq 0.86$. *Similarly,*

$$\left\|G_3^{-1}(x^k,z^v,z^v)(G(x^k,z^v,z^{v+1})-G(x^k,z^{v+1},z^{v+1}))\right\|_1$$
$$= \left\|\begin{pmatrix} \frac{a(1-b)(z_1^{v+1}-z_1^v)}{ab} \\ \frac{a(1-b)e^{\sin x_2^k}(z_2^{v+1}-z_2^v)}{abe^{\sin x_2^k}} \end{pmatrix}\right\|_1 \leq \beta(a,b)\|z^{v+1}-z^v\|_1.$$

Hence

$$\beta(a,b) = \left|\frac{a(1-b)}{ab}\right| = \frac{1}{11} =: Q_2.,$$

Therefore, we have

$$\alpha(a,b)+\beta(a,b) \leq Q_1 + Q_2 = 0.96 < 1.$$

Thereupon the NTSWR method (8) can be applied to solve the given system of nonlinear equations. We compare numerical results of the Newton, the NWR and the NTSWR method. In doing so, the number of inner iterations are set to be $s = 1$ *and the outer iterations is terminated as soon as the stopping criterion* $\|f(x^k)\|_\infty \leq 10^{-14}$ *holds. We consider* $x^0 = (-1,-0.28)^T$ *and* $x^0 = (1,1)^T$ *as the initial vectors. The number of iterations of the method together with the CPU times (in parenthesis and in seconds) for the convergence of the method are given in Table 1. It is noted that each code is executed ten times and the minimum time obtained is quoted in the tables. The convergence history of the methods is displayed in Figure 1 and Figure 2. As seen for both of the starting points* $x^0 = (-1,-0.28)^T$ *and* $x^0 = (1,1)^T$, *the Newton method fails to converge. In fact, the Newton method has not converged in 400 iterations and a stagnation has occurred. In terms of the number of iterations, the NTSWR method is superior to the NWR method, and it is roughly half that of the NWR method. However, the CPU time of the NTSWR method is slightly greater than that of the NWR method. It is necessary to mention that the initial guess* $x^0 = (1,1)^T$ *is not in* D. *Nevertheless, both the NWR and the NTSWR method are convergent.*

In continuation, we investigate the effect of the number of inner iterations on the outer iterations. In doing so, we report the number of outer iterations for different values of s *in Table 2. This table shows that for small values of* s *(e.g.,* $s = 1,2$ *) the NTSWR method provides quite suitable results in comparison with the NWR method. An interesting observation is that the results of the NTSWR method with* $s = 1$ *are better than those of the NWR method.*

Initial guess	Newton	NWR	NTSWR
$x^0 = (-1, -0.28)^T$	Fail	245(0.078)	133(0.109)
$x^0 = (\ 1, \ \ \ 1 \ \ \ )^T$	Fail	251(0.078)	134(0.109)

Table 1: *Numerical results for Example 1 with* $s = 1$

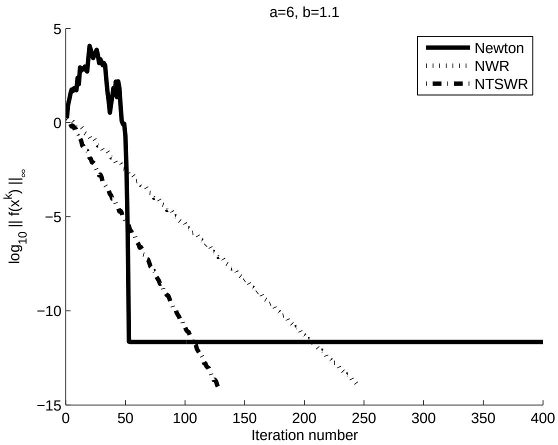

Figure 1: $\log_{10} \|f(x^k)\|_\infty$ *with initial guess* $x^0 = (-1, -0.28)^T$ *for Example 1*

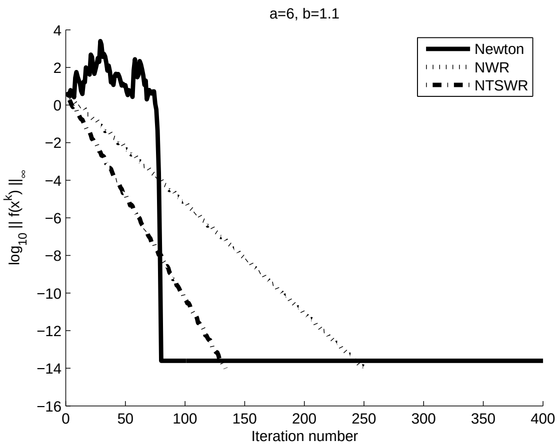

Figure 2: $log_{10}\|f(x^k)\|_\infty$ *with initial approximation* $x^0 = (1, 1)^T$ *for Example 1*

Initial guess	$s = 1$	$s = 2$	$s = 3$	$s = 4$	$s = 5$
$x^0 = (-1, -0.28)^T$	133	128	127	127	127
$x^0 = (\ 1, \ \ \ 1 \ \ \ )^T$	134	130	130	130	130

Table 2: *Effect of the number of inner iterations on outer iterations for Example 1*

Example 2. *Consider the following nonlinear equations*

$$f(x) = \begin{pmatrix} 10^9 \arctan(\frac{x_1}{10^9}) + e^{\sin x_2} + 3 \\ x_1 + x_2 - \sin(3x_1) \end{pmatrix} = 0. \tag{30}$$

This system was examined by the NWR method in [12] by using the splitting function

$$F(x^k, x^{k+1}) = \begin{pmatrix} 10^9 \arctan(\frac{x_1^k}{10^9}) + [a(x_1^{k+1} - x_1^k) + 1]e^{\sin x_2^k} + 3 \\ x_1^k + a x_2^{k+1} + (1-a)x_2^k - \sin(3x_1^k) \end{pmatrix}.$$

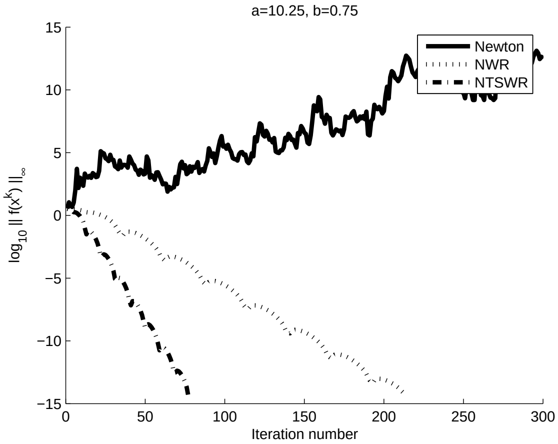

Figure 3: $\log_{10} \|f(x^k)\|_\infty$ *with initial guess* $x^0 = (-2, 2)^T$ *for Example 2*

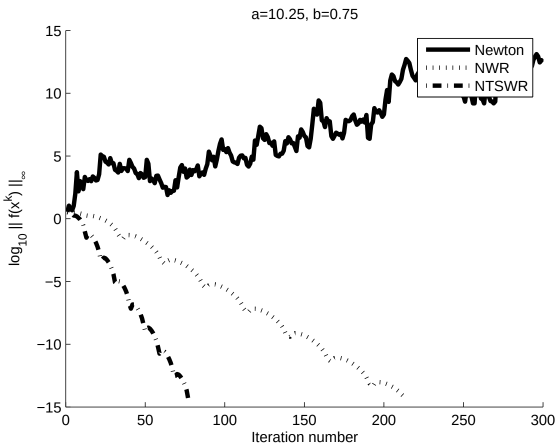

Figure 4: $log_{10}\|f(x^k)\|_\infty$ *with initial approximation* $x^0 = (2, -2)^T$ *for Example 2*

We use

$$G(x^k, z^v, z^{v+1}) = \begin{pmatrix} 10^9 \arctan(\frac{x_1^k}{10^9}) + [ab(z_1^{v+1} - z_1^v) + az_1^v - ax_1^k + 1]e^{\sin x_2^k} + 3 \\ x_1^k + ab(z_2^{v+1} - z_2^v) + (1-a)x_2^k + az_2^v - \sin(3x_1^k) \end{pmatrix},$$

as the splitting function in the NTSWR method. In this example, the number of inner iterations is set to be $s = 1$ *and the stopping criterion* $\|f(x^k)\|_\infty \leq 10^{-14}$ *is used for the outer iterations. We also set* $a = 10.25$ *and* $b = 0.75$. *We have checked two initial vectors* $x^0 = (-2, 2)^T$ *and* $x^0 = (2, -2)^T$ *and the corresponding numerical results are given in Table 3. For both starting vectors the Newton method fails to converge. As can be observed, the NTSWR method is superior to the NWR method in terms of the number of iterations. In fact, the number of iterations of the NTSWR method has been roughly reduced by a factor of three in both cases. On the other hand, the CPU times for both methods are the same. The convergence history is displayed in Figures 3 and 4.*

Initial guess	Newton	NWR	NTSWR
$x^0 = (-2, 2)^T$	Fail	212(0.078)	77(0.078)
$x^0 = (2, -2)^T$	Fail	218(0.078)	79(0.078)

Table 3: *Numerical results for Example 2 with* $s = 1$.

Similarly to the previous example, we report the number of outer iterations with respect to the inner iteration numbers. As can be seen, the NTSWR method provides quite suitable results for small values of s *in comparison with the NWR method.*

Initial guess	$s = 1$	$s = 2$	$s = 3$	$s = 4$	$s = 5$
$x^0 = (-2, 2)^T$	77	94	88	90	90
$x^0 = (2, -2)^T$	79	97	90	92	92

Table 4: *Effect of the number of inner iterations on outer iterations for Example 2*

Example 3. *Let* $f : \mathbb{R}^5 \to \mathbb{R}^5$ *with the following nonlinear equations*

$$f(x) = x + \phi(x) = 0, \quad \phi_i(x) = e^{\cos(\sum_{j=1}^{i} x_j)}, \quad i = 1, \ldots, 5. \tag{31}$$

We consider the splitting functions F *and* G *as follows*

$$F(x^k, x^{k+1}) = ax^{k+1} + (1-a)x^k + \phi(x^k),$$

and

$$G(x^k, z^v, z^{v+1}) = ab(z^{v+1} - z^v) + az^v + (1-a)x^k + \phi(x^k).$$

We set $a = 14$, $b = 0.5$ *and* $s = 1$. *The stopping criterion for the outer iterations is set to be* $\|f(x^k)\|_\infty \leq 10^{-14}$. *We use* $x^0 = (-5, \ldots, -5)^T$ *and* $x^0 = (2, \ldots, 2)^T$ *as the initial vectors and the numerical results are given in Table 5. As can be seen, for* *both initial vectors the Newton method fails to converge, but both the NWR and the NTSWR methods converge properly. Roughly speaking, the number of iterations of the NWR method is four times that of the NTSWR method for both initial vectors. Moreover, the CPU time for the NWR method is greater than that of the NTSWR method. The convergence history is displayed in Figures 5 and 6.*

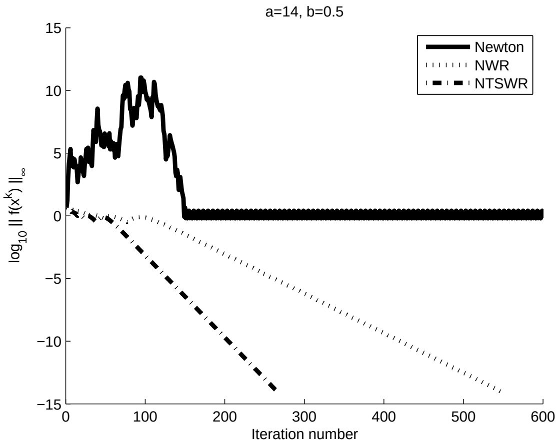

Figure 5: $\log_{10}\|f(x^k)\|_\infty$ *with initial guess* $x^0 = (-5, \ldots, -5)^T$ *for Example 3*

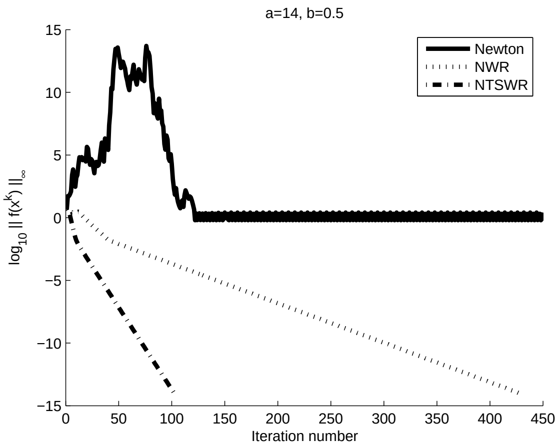

Figure 6: $log_{10}\|f(x^k)\|_\infty$ *with initial approximation* $x^0 = (2, \ldots, 2)^T$ *for Example 3*

Similarly to the previous examples, we report the number of outer iterations with respect to the inner iterations in Table 6.

Initial guess	Newton	NWR	NTSWR
$x^0 = (-5, \dots, -5)^T$	Fail	548(0.359)	133(0.218)
$x^0 = (2, \dots, 2)^T$	Fail	428(0.281)	103(0.172)

Table 5: *Numerical results for Example 3 with* $s = 1$

Initial guess	$s = 1$	$s = 2$	$s = 3$	$s = 4$	$s = 5$
$x^0 = (-5, \dots, 5)^T$	133	266	133	266	133
$x^0 = (2, \dots, 2)^T$	103	205	103	205	103

Table 6: *Effect of the number of inner iterations on outer iterations for Example 3*

5. Conclusion

In this paper, we have presented the Newton two-stage waveform relaxation method to solve systems of nonlinear algebraic equations. Then, we have given some sufficient conditions for the convergence of the method. Next, we have compared numerical results of the proposed method with those of the Newton waveform relaxation method and the classical Newton method. Numerical results show that the proposed method can be considered as an efficient method to solve a system of nonlinear equations.

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