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A NEW FORM OF EQUATIONS FOR RIGID BODY ROTATIONAL DYNAMICS

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Original scientific paper

In the paper, a new form of differential equations for rigid body attitude dynamics is obtained. Three *s*-parameters (modified Rodrigues-Hamilton parameters) and three angular velocity parameters are used as unknown variables. Built equations are particularly useful for analytical and numerical study of rotational motion of a rigid body. The topological structure of configurational *s*-manifold for a balanced rigid body is investigated. An example of the use of constructed equations to describe the rotational motion of a rigid body in a resisting medium is considered.

Keywords: *dynamics, modified Rodriguez-Hamilton parameters, rigid body, rotational motion*

Novi oblik jednadžbi rotacijskog gibanja krutog tijela

Izvorni znanstveni članak

U ovom je radu izveden novi oblik diferencijalnih jednadžbi koje opisuju dinamiku rotacijskog gibanja krutog (čvrstog) tijela oko središta njegove mase. Kao varijable koriste se tri *s*-parametra (modificirani parametri Rodriga-Hamiltona) i tri parametra kutne brzine tijela. Izvedene jednadžbe su osobito korisne za analitičko i numeričko izučavanje rotacijskog gibanja čvrstog tijela. Istražena je topološka struktura konfiguracijskog *s*-prostranstva za uravnoteženo kruto tijelo. Razmotren je primjer uporabe izvedenih jednadžbi koje opisuju rotacijsko gibanje krutog tijela u otpornoj sredini.

Ključne riječi: *dinamika, kruto tijelo, modificirani parametri Rodriga-Hamiltona, rotacijsko gibanje*

1 Introduction

The studying of attitude motion of rigid body under the action of principle moment $\vec{M}$ of external forces is usually realized in terms of Euler angles, various variants of "aircraft" angles or direction cosines. The wide range of mechanical problems is connected with the studying of rotational motion of rigid body that is such attitude motion of the body with respect to the centre of mass C, when the work of moment $\vec{M}$ of external disturbing forces is small with respect to the kinetic energy of the body's rotational motion.

These problems are usually investigated with the use of special variables. For example, the Andoyer (1862 ÷ 1929) dynamical equations of rigid body attitude motion are used widely in Beletsky-Chernousko variables $L, \rho, \sigma, \varphi, \psi, \vartheta$ [1]:

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = L\cos\vartheta\left(\frac{1}{C} - \frac{\sin^2\varphi}{A} - \frac{\cos^2\varphi}{B}\right) + \frac{M_1\cos\psi + M_2\sin\psi}{L\sin\vartheta},$$

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = L\left(\frac{\sin^2\varphi}{A} + \frac{\cos^2\varphi}{B}\right) - \frac{M_1\cos\psi + M_2\sin\psi}{L}\mathrm{ctan}\,\vartheta - \frac{M_2}{L}\mathrm{ctan}\rho,$$

$$\frac{\mathrm{d}\vartheta}{\mathrm{d}t} = L\left(\frac{1}{A} - \frac{1}{B}\right)\sin\vartheta\sin\varphi\cos\varphi + \frac{M_2\cos\psi - M_1\sin\psi}{L},$$

$$\frac{\mathrm{d}L}{\mathrm{d}t} = M_3, \frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{M_1}{L}, \frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{M_2}{L\sin\rho}.$$

Here A, B, C are the body's principal central moments of inertia, L is the absolute value of kinetic moment $\vec{L}$, the angles ρ, σ determine the position of vector $\vec{L}$ and connected with it orthogonal Cartesian coordinate system $CL_1L_2L_3$ with respect to the Koenig coordinate system $CXYZ$ according to Fig. 1, the Euler angles φ, ψ, ϑ determine the orientation of the body's principal central axes of inertia x, y, z with respect to the coordinate system $CL_1L_2L_3$, and M_1, M_2, M_3 are the projections of the moment $\vec{M}$ on the axes of moving coordinate system $CL_1L_2L_3$.

The usability of these equations is connected with the constancy of vector $\vec{L}$ (that is the values L, ρ, σ) in undisturbed motion of the body. But the computer simulation of mathematical models constructed with the use of Euler angles meets the following difficulties:

1) the kinematic equations degenerate at certain values of angles;
2) the right parts of differential equations occur complex trigonometric functions of angles, so the integration of these equations leads to significant increasing of computing time.

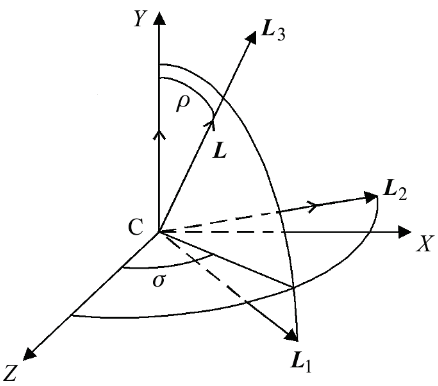

Figure 1 The coordinate system connected with vector $\vec{L}$

The Euler-Poisson model based on the use of direction cosines, is free of these disadvantages, but the closed system of equations in these variables has much higher order, so it is not convenient for computer modelling.

The above mentioned difficulties may be avoided by introducing new parameters. Among all kinematic parameters the significant place belongs to the Rodriguez-Hamilton parameters $\lambda_i (i = 0, 1, 2, 3)$ [2, 3]. They represent the components of 4-dimensional number $\Lambda = (\lambda_0, \lambda_1, \lambda_2, \lambda_3)$, called quaternion, and connected by the normalization condition $\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1$. Choosing these parameters and dimensionless projections of angular velocity on principal central axes of inertia as variables we obtain in the normal form the system of seven differential equations (3 dynamical and 4 kinematic equations) for finding seven unknowns.

But the Rodriguez-Hamilton parameters are not minimal in number. The new parameters s_1, s_2, s_3 (s-parameters) are introduced in the paper [4] for construction of the mathematical model of the body's attitude motion. They possess the same advantages as the Rodriguez-Hamilton parameters and at the same time they are minimal in number. S-parameters may be considered as the result of stereographic projection of 4-dimensional sphere of quaternions, normalized with the condition $\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1$, on 3-dimensional hyper plane, orthogonal to the axis $O\lambda_0$ (Fig. 2).

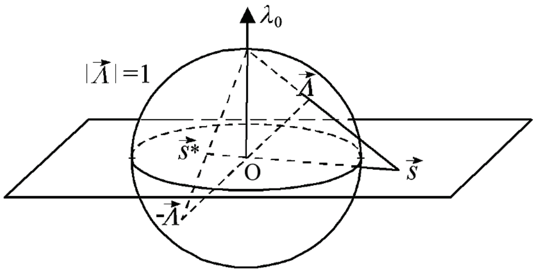

Figure 2 *S*-vector as a result of stereographic projection

At the same time parameters s_i may be expressed in terms of Rodriguez-Hamilton parameters according to the formula $s_i = \lambda_i/(1-\lambda_0)$. The differential equations of the body's attitude motion suitable for the investigation of the rotary motion were constructed in the paper [5] with the use of s-parameters. Later these equations were successfully used for the investigation of secular evolution of rotary motion of a charged satellite in a decaying orbit [6].

In this paper the new equations of the body's attitude motion are deduced. They differ from analogous equations obtained in [5] with the use of variables $L, \rho, \sigma, s_1, s_2, s_3$ because instead of variables L, ρ, σ we use the variables ω, α, β which determine the value and direction of the body's angular velocity vector and more obvious in interpretation of the results of investigation.

2 The equations of motion

Let us determine the orientation of the body with the use of moving coordinates $C\omega_1\omega_2\omega_3$ (Fig. 3), where the axis $C\omega_3$ is directed along the vector $\vec{\omega}$ of body's angular velocity and $CXYZ$ is the Koenig coordinate system.

Let us introduce the direction cosines matrix $\boldsymbol{K}$ of the axes ω_1, ω_2, ω_3 with respect to the axes X, Y, Z:

	ω_1	ω_2	ω_3
X	$\sin\alpha\cos\beta$	$\cos\alpha$	$\sin\alpha\sin\beta$
Y	$-\sin\beta$	0	$\cos\beta$
Z	$\cos\alpha\cos\beta$	$-\sin\alpha$	$\cos\alpha\sin\beta$

,

and the direction cosines matrix $\boldsymbol{A}$ of the body's principle central axes of inertia *x, y, z* with respect to the axes ω_1, ω_2, ω_3.

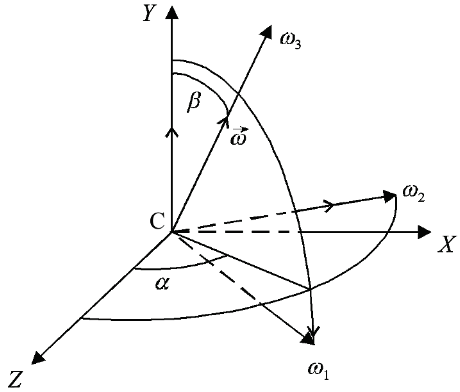

Figure 3 The coordinate system connected with vector $\vec{\omega}$

The elements of matrix $\boldsymbol{A}$ can be expressed in terms of s-parameters s_1, s_2, s_3 [4]:

$$\begin{array}{cccc} & x & y & z \\ \omega_1 & \dfrac{u_0 - 8(s_2^2+s_3^2)}{u_0} & \dfrac{4(2s_1s_2 - s_3u_1)}{u_0} & \dfrac{4(2s_1s_3 + s_2u_1)}{u_0} \\ \omega_2 & \dfrac{4(2s_1s_2 + s_3u_1)}{u_0} & \dfrac{u_0 - 8(s_1^2+s_3^2)}{u_0} & \dfrac{4(2s_2s_3 - s_1u_1)}{u_0} \\ \omega_3 & \dfrac{4(2s_1s_3 - s_2u_1)}{u_0} & \dfrac{4(2s_2s_3 + s_1u_1)}{u_0} & \dfrac{u_0 - 8(s_1^2+s_2^2)}{u_0} \end{array}, \quad (1)$$

where

$$|s|^2 = s_1^2 + s_2^2 + s_3^2, \;\; u_0 = (|s|^2+1)^2, \;\; u_1 = |s|^2 - 1.$$

The following equations are valid:

$$(0,\, 0,\, \omega)^{\mathrm{T}} = \boldsymbol{A}(\omega_x,\, \omega_y,\, \omega_z)^{\mathrm{T}}, \quad (2)$$
$$(M_X,\, M_Y,\, M_Z)^{\mathrm{T}} = \boldsymbol{K}(M_{\omega 1},\, M_{\omega 2},\, M_{\omega 3})^{\mathrm{T}}.$$

The absolute angular velocity vector we represent in the form:

$$\vec{\omega} = \vec{\omega}_{\mathrm{e}} + \vec{\omega}_{\mathrm{r}}, \quad (3)$$

where vector $\vec{\omega}_{\mathrm{e}} = \vec{\dot{\alpha}} + \vec{\dot{\beta}}$ is the angular velocity of the coordinate system $C\omega_1\omega_2\omega_3$, and vector $\vec{\omega}_{\mathrm{r}}$ is the angular velocity of the body relative to the coordinate system $C\omega_1\omega_2\omega_3$. Since $(\omega_{\mathrm{e}\omega 1}, \omega_{\mathrm{e}\omega 2}, \omega_{\mathrm{e}\omega 3})^{\mathrm{T}}$ = $(-\dot{\alpha}\sin\beta,\, \dot{\beta},\, \dot{\alpha}\cos\beta)^{\mathrm{T}}$ then on the basis of (2) and (3) we obtain: $(\omega_{\mathrm{r}x}, \omega_{\mathrm{r}y}, \omega_{\mathrm{r}z})^{\mathrm{T}} = \boldsymbol{A}^{\mathrm{T}}(\dot{\alpha}\sin\beta,\, -\dot{\beta},\, \omega - \dot{\alpha}\cos\beta)^{\mathrm{T}}$.

The angular momentum vector $\vec{L}$ of the rigid body with tensor of inertia $\mathrm{diag}(A, B, C)$ (in the axes C*xyz*) has the following projections:

$$\left(L_{\omega1},\ L_{\omega2},\ L_{\omega3}\right)^{\mathrm{T}} = \boldsymbol{A}\cdot \mathrm{diag}(A,\ B,\ C)\cdot\left(\omega_x,\ \omega_y,\ \omega_z\right)^{\mathrm{T}} = \boldsymbol{\Lambda}\cdot(0,\ 0,\ \omega)^{\mathrm{T}},$$

$$\left(L_X,\ L_Y,\ L_Z\right)^{\mathrm{T}} = \boldsymbol{K}\left(L_{\omega1},\ L_{\omega2},\ L_{\omega3}\right)^{T} = \omega \boldsymbol{K}\boldsymbol{\Lambda}_1 \tag{4}$$

where

$$\boldsymbol{\Lambda} = \boldsymbol{A}\cdot \mathrm{diag}(A,\ B,\ C)\cdot \boldsymbol{A}^{\mathrm{T}},\quad \boldsymbol{\Lambda}_1 = \boldsymbol{\Lambda}\cdot(0,\ 0,\ 1)^{\mathrm{T}}.$$

Differentiating Eq. (4) with respect to time, we can write the theorem of angular momentum $\frac{\mathrm{d}}{\mathrm{d}t}\left(L_X,\ L_Y,\ L_Z\right)^{\mathrm{T}} = \left(M_X,\ M_Y,\ M_Z\right)^{\mathrm{T}}$ in the form:

$$\dot\omega\,\boldsymbol{K}\boldsymbol{\Lambda}_1 + \omega\,\dot{\boldsymbol{K}}\boldsymbol{\Lambda}_1 + \omega\,\boldsymbol{K}\dot{\boldsymbol{\Lambda}}_1 = \left(M_X,\ M_Y,\ M_Z\right)^{\mathrm{T}}. \tag{5}$$

By multiplying the Eq. (5) by the matrix $\boldsymbol{K}^{\mathrm{T}}$, we obtain:

$$(\dot\omega \boldsymbol{E} + \omega \boldsymbol{K}^{\mathrm{T}}\dot{\boldsymbol{K}})\boldsymbol{\Lambda}_1 + \omega\,\dot{\boldsymbol{\Lambda}}_1 = \left(M_{\omega1},\ M_{\omega2},\ M_{\omega3}\right)^{\mathrm{T}}. \tag{6}$$

Let us introduce into consideration matrices

$$\boldsymbol{\Omega}_\omega = \begin{pmatrix} 0 & -\omega_{\mathrm{e}\omega3} & \omega_{\mathrm{e}\omega2} \\ \omega_{\mathrm{e}\omega3} & 0 & -\omega_{\mathrm{e}\omega1} \\ -\omega_{\mathrm{e}\omega2} & \omega_{\mathrm{e}\omega1} & 0 \end{pmatrix}$$

and

$$\boldsymbol{\Omega}_x = \begin{pmatrix} 0 & -\omega_{\mathrm{r}z} & \omega_{\mathrm{r}y} \\ \omega_{\mathrm{r}z} & 0 & -\omega_{\mathrm{r}x} \\ -\omega_{\mathrm{r}y} & \omega_{\mathrm{r}x} & 0 \end{pmatrix}.$$

Then

$$\dot{\boldsymbol{K}} = \boldsymbol{K}\boldsymbol{\Omega}_\omega,\quad \dot{\boldsymbol{A}} = \boldsymbol{A}\boldsymbol{\Omega}_x.$$

Consequently,

$$\begin{aligned}\dot{\boldsymbol{\Lambda}} &= \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{A}\,\mathrm{diag}(A,\ B,\ C)\boldsymbol{A}^{\mathrm{T}}) = \\ &= \dot{\boldsymbol{A}}\,\mathrm{diag}(A,\ B,\ C)\boldsymbol{A}^{\mathrm{T}} + \boldsymbol{A}\,\mathrm{diag}(A,\ B,\ C)\dot{\boldsymbol{A}}^{\mathrm{T}} = \\ &= \boldsymbol{A}\boldsymbol{\Omega}_x\mathrm{diag}(A,\ B,\ C)\boldsymbol{A}^{\mathrm{T}} - \boldsymbol{A}\,\mathrm{diag}(A,\ B,\ C)\boldsymbol{\Omega}_x\boldsymbol{A}^{\mathrm{T}} = \\ &= \boldsymbol{A}\left[\boldsymbol{\Omega}_x\mathrm{diag}(A,\ B,\ C) - \mathrm{diag}(A,\ B,\ C)\boldsymbol{\Omega}_x\right]\boldsymbol{A}^{\mathrm{T}} = \\ &= \boldsymbol{A}\boldsymbol{\Lambda}_\Omega\boldsymbol{A}^{\mathrm{T}},\end{aligned}$$

where

$$\boldsymbol{\Lambda}_\Omega = \begin{pmatrix} 0 & (A-B)\omega_{\mathrm{r}z} & (C-A)\omega_{\mathrm{r}y} \\ (A-B)\omega_{\mathrm{r}z} & 0 & (B-C)\omega_{\mathrm{r}x} \\ (C-A)\omega_{\mathrm{r}y} & (B-C)\omega_{\mathrm{r}x} & 0 \end{pmatrix}$$

Therefore the Eq. (6) may be written in the form:

$$(\omega\boldsymbol{\Omega}_\omega + \dot\omega\boldsymbol{E})\boldsymbol{\Lambda}_1 + \omega\boldsymbol{A}\boldsymbol{\Lambda}_\Omega\boldsymbol{A}^{\mathrm{T}}(0,\ 0,\ 1)^{\mathrm{T}} = (M_{\omega1}, M_{\omega2}, M_{\omega3})^{\mathrm{T}}. \tag{7}$$

The system (7) is linear with respect to $\dot\alpha$, $\dot\beta$, $\dot\omega$ and may be resolved for these derivatives. However, before this operation we shall reduce the equations (7) to the form, which contains M_x, M_y, M_z instead of $M_{\omega1}$, $M_{\omega2}$, $M_{\omega3}$ and is more convenient for the solving of the problems where the perturbing moment $\vec{M}$ is defined by the projections on the axes, connected with the rigid body. For this purpose we can multiply (7) from the left by the matrix $\boldsymbol{A}^{\mathrm{T}}$:

$$\left[\boldsymbol{A}^{\mathrm{T}}(\omega\boldsymbol{\Omega}_\omega + \dot\omega\boldsymbol{E})\boldsymbol{\Lambda} + \omega\boldsymbol{\Lambda}_\Omega\boldsymbol{A}^{\mathrm{T}}\right](0,\ 0,\ 1)^{\mathrm{T}} = (M_x,\ M_y,\ M_z)^{\mathrm{T}}.$$

Thus, we obtained the linear system with respect to $\dot\alpha$, $\dot\beta$, $\dot\omega$. By resolving it for the derivatives we obtain:

$$\begin{aligned}\begin{pmatrix} \omega\dot\beta \\ \omega\dot\alpha\sin\beta \\ \dot\omega \end{pmatrix} &= \boldsymbol{A}\begin{pmatrix} M_x A^{-1} \\ M_y B^{-1} \\ M_z C^{-1} \end{pmatrix} + \\ &+ \frac{\omega^2}{ABC}\left[(A-B)(B-C)(A-C)A_{31}A_{32}\begin{pmatrix} A_{13} \\ A_{23} \\ A_{33} \end{pmatrix} + \right. \\ &\left. + \begin{pmatrix} AC(A-C)A_{21}A_{31} + BC(B-C)A_{22}A_{32} \\ -AC(A-C)A_{11}A_{31} - BC(B-C)A_{12}A_{32} \\ 0 \end{pmatrix}\right],\end{aligned} \tag{8}$$

where A_{ij} are the elements of the direction cosines matrix $\boldsymbol{A}$ (1).

For closing the equations (8) we consider them together with kinematic equations, which have the following form in *s*-parameters:

$$(\dot s_1,\ \dot s_2, \dot s_3)^{\mathrm{T}} = \boldsymbol{B}(\omega_{\mathrm{r}x},\ \omega_{\mathrm{r}y}, \omega_{\mathrm{r}z})^{\mathrm{T}}.$$

Here

$$\boldsymbol{B} = \frac{-1}{4}\begin{pmatrix} s_1^2 - s_2^2 - s_3^2 + 1 & 2(s_1s_2 + s_3) & 2(s_1s_3 - s_2) \\ 2(s_1s_2 - s_3) & s_2^2 - s_3^2 - s_1^2 + 1 & 2(s_2s_3 + s_1) \\ 2(s_1s_3 + s_2) & 2(s_2s_3 - s_1) & s_3^2 - s_1^2 - s_2^2 + 1 \end{pmatrix}.$$

On the basis of (3) and the easily verified equation $\boldsymbol{B}\boldsymbol{A}^{\mathrm{T}} = \frac{u_0}{16}\boldsymbol{B}^{-1} = \boldsymbol{B}^{\mathrm{T}}$ we obtain the kinematic equations in *s*-parameters:

$$\begin{pmatrix} \dot s_1 \\ \dot s_2 \\ \dot s_3 \end{pmatrix} = \boldsymbol{B}^{\mathrm{T}}\begin{pmatrix} \dot\alpha\sin\beta \\ -\dot\beta \\ \omega - \dot\alpha\cos\beta \end{pmatrix}. \tag{9}$$

Thus, the system (8), (9) represents the differential system for description of the rotational motion of rigid body. This system is convenient for computer modelling because it has the unique singular point – the pole of function, mapping the unit sphere in 4-dimensional quaternion space onto the 3-dimensional hyperplane of *s*-parameters. But the direct hit of image point into the pole of mapping function during the motion is the exception case, demanding special initial conditions. The set of such initial conditions in the space of *s*-parameters has the zero measure and so the possibility of hitting the image point into the pole of mapping function practically does not realize.

3 The configurational manifold of *s*-parameters for balanced rigid body

Let us consider the attitude motion of rigid body under the balanced external impact ($\vec{M}=0$). Such attitude motion is characterized by the constant vector of angular momentum

$$\vec{L}=(L_x,\,L_y,\,L_z)^{\mathrm T}=\mathrm{diag}(A,\,B,\,C)\cdot(\omega_x,\,\omega_y,\,\omega_z)^{\mathrm T}$$

and the constant value of kinetic energy

$$T=\frac{1}{2}\left(\omega_x,\omega_y,\omega_z\right)\begin{pmatrix}A&0&0\\0&B&0\\0&0&C\end{pmatrix}\begin{pmatrix}\omega_x\\\omega_y\\\omega_z\end{pmatrix}. \tag{10}$$

On the basis of equation

$$\begin{pmatrix}\omega_x\\\omega_y\\\omega_z\end{pmatrix}=\boldsymbol{A}^{\mathrm T}\begin{pmatrix}0\\0\\\omega\end{pmatrix}, \tag{11}$$

we can express $L_x,\,L_y,\,L_z$ through ω:

$$\vec{L}=\omega\,\mathrm{diag}(A,\,B,\,C)\,\boldsymbol{A}^{\mathrm T}\begin{pmatrix}0\\0\\1\end{pmatrix}=\omega\begin{pmatrix}AA_{31}\\BA_{32}\\CA_{33}\end{pmatrix}.$$

Consequently

$$L^2=\omega^2(A^2A_{31}^2+B^2A_{32}^2+C^2A_{33}^2). \tag{12}$$

Using (11) we can write (10) in the form:

$$2T=\omega^2(0,0,1)\,\boldsymbol{A}\mathrm{diag}(A,\,B,\,C)\boldsymbol{A}^{\mathrm T}\begin{pmatrix}0\\0\\1\end{pmatrix}=$$

$$=\omega^2(A_{31},A_{32},\,A_{33})\,\mathrm{diag}(A,\,B,\,C)\begin{pmatrix}A_{31}\\A_{32}\\A_{33}\end{pmatrix}.$$

Consequently

$$2T=\omega^2(AA_{31}^2+BA_{32}^2+CA_{33}^2). \tag{13}$$

Dividing (12) by (13) we obtain:

$$\frac{L^2}{2T}=\frac{A^2A_{31}^2+B^2A_{32}^2+C^2A_{33}^2}{AA_{31}^2+BA_{32}^2+CA_{33}^2}=D=\mathrm{const.} \tag{14}$$

We can write equation (14) in the form:

$$A_{31}^2(A^2-DA)+A_{32}^2(B^2-DB)+A_{33}^2(C^2-CD)=0. \tag{15}$$

After substituting in Eq. (15) the elements of matrix $\boldsymbol{A}$, expressed through $s_1,\,s_2,\,s_3$, we obtain

$$\begin{aligned}&\frac{16}{u_0^2}(2s_1s_3-s_2(|s|^2-1))^2(A^2-AD)+\\&+\frac{16}{u_0^2}(2s_2s_3+s_1(|s|^2-1))^2(B^2-BD)+\\&+\left(1-\frac{8}{u_0}(s_1^2+s_2^2)\right)^2(C^2-CD)=0.\end{aligned} \tag{16}$$

The equation (16) represents itself the energy integral of balanced rigid body, expressed in *s*-parameters. The fixed values of L and T determine the surface with implicit form (16) in the space of *s*-parameters. The constancy of angular momentum and kinetic energy of rotating balanced rigid body denotes that vector of *s*-parameters, defining the attitude position of the body at arbitrary moment, is on the mentioned surface. Consequently the surface (16) is the configurational manifold of balanced rigid body. It should be mentioned that the qualitative dependence of equation (16) on parameters $s_1,\,s_2,\,s_3$ is just the same as that in analogous equation obtained in [5] (but the meaning of variables s_1, s_2, s_3, is naturally another). In the paper [5] the configurational manifold of balanced rigid body was described by the equation

$$\frac{2T}{L^2}=\frac{A_{31}^2}{A}+\frac{A_{32}^2}{B}+\frac{A_{33}^2}{C}$$

or, that is the same, by the equation

$$A_{31}^2\left(\frac{1}{A}-\frac{2T}{L^2}\right)+A_{32}^2\left(\frac{1}{B}-\frac{2T}{L^2}\right)+A_{33}^2\left(\frac{1}{C}-\frac{2T}{L^2}\right)=0.$$

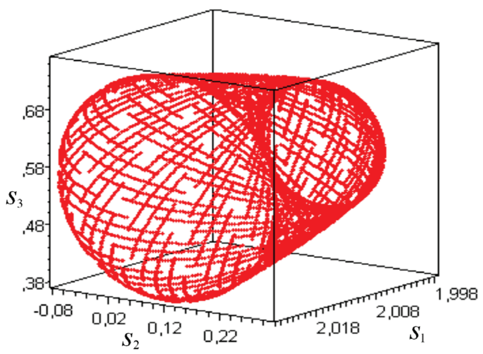

Figure 4 Configurational manifold in *s*-parameters

On the basis of (15) it follows that all conclusions made in [5] concerning the topological structure of configurational manifold and the s-vector boundedness remain in force. The example of trajectory obtained by numerical integration of equations (8), (9) at $\vec{M}=0$ is shown in Fig. 4. It is evident that the trajectory coils on the bounded toroidal surface in the space of s-parameters.

In particular case then $A=B\neq C$, the system (8) - (9) is the following

$$\begin{pmatrix} \omega\dot{\beta} \\ \omega\,\dot{\alpha}\sin\beta \\ \dot{\omega} \end{pmatrix} = \frac{\omega^2(A-C)}{A}\begin{pmatrix} A_{21}A_{31}+A_{22}A_{32} \\ -A_{11}A_{31}-A_{12}A_{32} \\ 0 \end{pmatrix}, \tag{17}$$

$$\begin{pmatrix} \dot{s}_1 \\ \dot{s}_2 \\ \dot{s}_3 \end{pmatrix} = \boldsymbol{B}^{\mathrm{T}}\begin{pmatrix} \dot{\alpha}\sin\beta \\ -\dot{\beta} \\ \omega-\dot{\alpha}\cos\beta \end{pmatrix}. \tag{18}$$

At that, as one might expect, we immediately obtain $\omega=\text{const.}$ from the third of equations (17). The energy integral (15) takes the form:

$$(A^2-DA)(A_{31}^2+A_{32}^2)+A_{33}^2(C^2-DC)=0\,.$$

In this case the configurational manifold is the regular tore, which has the following parametric representation with respect to the angles λ and γ:

$$s_1=\frac{\cos\lambda}{\sin\theta_0}(1+\cos\theta_0\cos\gamma),$$
$$s_2=\frac{\sin\lambda}{\sin\theta_0}(1+\cos\theta_0\cos\gamma),$$
$$s_3=\mathrm{ctan}\theta_0\sin\gamma.$$

Here

$\gamma\in[-\pi,\,\pi],\ \lambda\in[0,\,2\pi],$ and parameter $\theta_0\in[0,\,\pi/2]$ is defined by the equation $\sin^2 2\theta_0=\dfrac{C(D-C)}{(A-C)(A+C-D)}$.

The example of trajectory coiling on the surface of such tore is obtained by numerical integration of equations (17), (18) and is shown in Fig. 5:

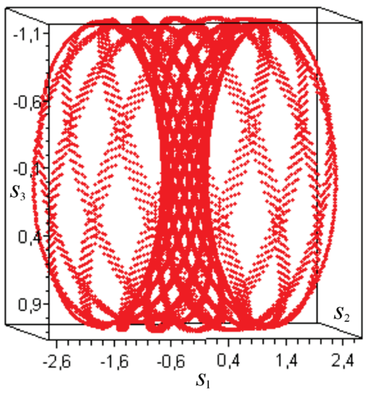

Figure 5 Configurational s-manifold for the case A=B

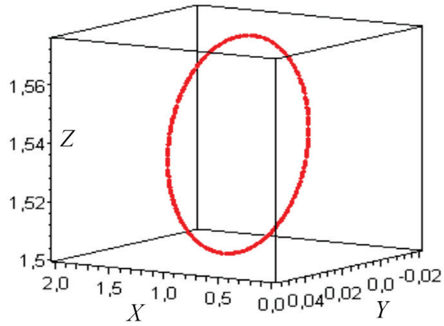

Figure 6 The trace of vector $\vec{\omega}$ on the unit sphere

The trace of vector $\vec{\omega}$ on the unit sphere, fixed in the coordinate system $CXYZ$, represents the circle (Fig. 6).

4 The example of the use of constructed equations

Let the body move in resisting medium, creating the disturbing moment $\vec{M}=-h\vec{\omega}$, proportional to the rigid body angular velocity with coefficient $h>0$. In projections on the axes $x,\,y,\,z$ we have

$$\begin{aligned} M_x &= -h\,A_{31}\omega(t), \\ M_y &= -h\,A_{32}\omega(t), \\ M_z &= -h\,A_{33}\omega(t). \end{aligned} \tag{19}$$

where A_{3i} are the elements of direction cosine matrix $\boldsymbol{A}$.

Consider the case $A=B\neq C$. Then the system (8), (9) take the form:

$$\begin{pmatrix} \omega\dot{\beta} \\ \omega\dot{\alpha}\sin\beta \\ \dot{\omega} \end{pmatrix} = \boldsymbol{A}\begin{pmatrix} M_xA^{-1} \\ M_yB^{-1} \\ M_zC^{-1} \end{pmatrix} + \frac{\omega^2(A-C)}{A}\begin{pmatrix} A_{21}A_{31}+A_{22}A_{32} \\ -A_{11}A_{31}-A_{12}A_{32} \\ 0 \end{pmatrix}, \tag{20}$$

$$\begin{pmatrix} \dot{s}_1 \\ \dot{s}_2 \\ \dot{s}_3 \end{pmatrix} = \boldsymbol{B}^{\mathrm{T}}\begin{pmatrix} \dot{\alpha}\sin\beta \\ -\dot{\beta} \\ \omega-\dot{\alpha}\cos\beta \end{pmatrix}.$$

Below there are the results of numeric integration of differential system (19) - (20) at initial conditions $s_1(0)=1,\quad s_2(0)=0,\quad s_3(0)=1,\quad \omega(0)=2,\quad \alpha(0)=0,$ $\beta(0)=\pi/2$ and the parameters values $A=B=3,\ C=2$. The trajectory of vector $\vec{\omega}$ in space of parameters $\omega,\,\alpha,\,\beta$ is shown in Fig. 7.

The trajectory of vector $\vec{\omega}$ in coordinates $x,\,y,\,z$ is shown in Fig. 8.

In Fig. 9 we can see the trajectory of s-vector in space s_1, s_2, s_3.

It is obvious that parameters $\omega,\,\alpha,\,\beta,\,s_1,\,s_2,\,s_3$ are convenient for computer modelling of rigid body motion

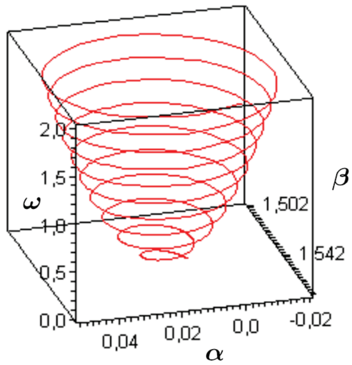

Figure 7 Trajectory of vector $\vec{\omega}$ in space ω, α, β

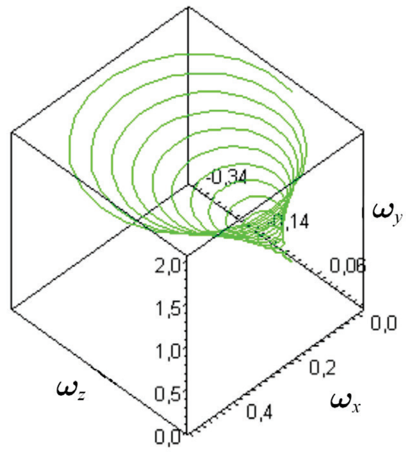

Figure 8 Trajectory of vector $\vec{\omega}$ in space x, y, z

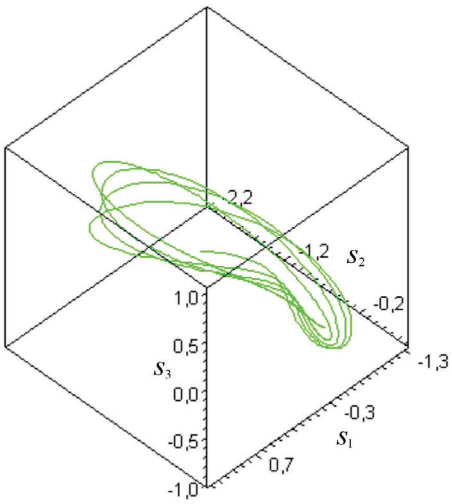

Figure 9 Trajectory of s-vector

5 Conclusions

In this paper a new form of differential equations for rigid body attitude dynamics is deduced. Three s-parameters (modified Rodrigues-Hamilton parameters) and three parameters which determine the value and direction of the body's angular velocity vector are used as unknown variables.

The new equations are reminding the earlier obtained equations from [5] since both forms are based on the use of s-parameters, but the meanings of s-parameters are quite different. Another difference between these equations is that here we use the angular velocity variables instead of the angular momentum variables. Such approach seems to be expedient as the angular velocity variables are more obvious in interpretation of the results of investigation. Built equations are particularly useful for analytical and numerical study of rotational motion of a rigid body.

The topological structure of configurational s-manifold for a balanced rigid body is investigated. The boundedness of s-vector in configurational space is shown. An example of the use of constructed equations to describe the rotational motion of a rigid body in a resisting medium is considered.

In future research, the authors will pay attention to the study of rotational motion of complex elements, such as [8, 9]. Also, in future work, the authors will be given to the study of new procedures and new effects [7, 10], that may be associated with the problems presented in this paper.

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