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Perfect Mannheim, Lipschitz and Hurwitz weight codes

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Abstract. The set of residue classes modulo an element π in the rings of Gaussian integers, Lipschitz integers and Hurwitz integers, respectively, is used as alphabets to form the words of error correcting codes. An error occurs as the addition of an element in a set $\mathcal{E}$ to the letter in one of the positions of a word. If $\mathcal{E}$ is a group of units in the original rings, then we obtain the Mannheim, Lipschitz and Hurwitz metrics, respectively. Some new perfect 1-error-correcting codes in these metrics are constructed. The existence of perfect 2-error-correcting codes is investigated by computer search.

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1. Introduction

If a code attains a bound (the sphere-packing bound) in a given metric, then it is called a perfect code. Perfect codes have always drawn the attention of coding theorists and mathematicians, since they play an important role in coding theory, both for theoretical and practical reasons. Some perfect codes with respect to the Hamming metric over finite fields are known [1, 10, 11, 15, 16]. For non-field alphabets, only trivial codes in the Hamming metric are known.

Perfect codes have been investigated not only with respect to the Hamming metric, but also to other metrics, for example the Lee metric. The Lee metric was introduced in [9]. Some perfect codes with respect to the Lee metric were discovered in [8].

Later, the Mannheim metric was introduced by Huber in [7]. It is well known that the Euclidean metric is the relevant metric for maximum-likelihood decoding. Although the Mannheim metric is a reasonable approximation to it, it is not a priori a natural choice. However, the codes proposed are very useful in coded modulation schemes based on quadrature amplitude modulation (QAM)-type constellations, for which neither the Hamming nor the Lee metric is appropriate. Two classes of codes over the Gaussian integers $\mathbb{Z}[i]$ were considered in [7], viz., the one Mannheim error-correcting codes (OMEC), and codes having the minimum Mannheim distance greater than three. The OMEC codes are perfect with respect to the Mannheim metric. Thus, some perfect codes were discovered. However, the dimension k of

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OMEC codes with parameters $[n, k, d]$ is only $n-1$. Among new perfect 1-error-correcting codes we obtain in the present study, there are perfect codes with respect to the Mannheim metric of dimension not only $n-1$, but also $n-k$, $(k>1)$.

On the other hand, the Lipschitz metric was presented, and some perfect codes over the Lipschitz integers with respect to the Lipschitz metric were introduced in [12, 13]. Let $\mathbb{Z}[i]_\pi$ denote the set of all residue classes to an element π in the ring of Gaussian integers, and similarly for $H(\mathbb{Z})_\pi$ and $\mathcal{H}_\pi$, where $H(\mathbb{Z})$ denotes the Lipschitz integers and $\mathcal{H}$ the Hurwitz integers. The main issue of this study is the construction of perfect 1-error-correcting codes with respect to the Mannheim metric in $\mathbb{Z}_\pi^n$ and the Lipschitz metric in $H(\mathbb{Z})_\pi^n$, to introduce the Hurwitz metric in $\mathcal{H}_\pi^n$ and to discuss the existence of perfect codes in this metric.

The presentation of our results is organized as follows: In the next section, we give the necessary fundamental definitions and results. The Hurwitz metric is introduced in Section 2.1. In Section 3, we discuss which algebraic properties the sets $H(\mathbb{Z})_\pi$ and $\mathcal{H}_\pi$ must have, so that our constructions works. In Section 4, we give a general construction of perfect 1-error-correcting codes in the metrics we are dealing with. These constructions are related to partitions of the set of non-zero elements in $\mathbb{Z}[i]_\pi$, $H(\mathbb{Z})_\pi$ and $\mathcal{H}_\pi$ into a kind of cosets to the groups of units in the rings $\mathbb{Z}[i]$, $H(\mathbb{Z})$ and $\mathcal{H}$. In Section 4.1, such partitions are discussed and constructed. In Section 5, we summarize which new perfect 1-error-correcting codes we have obtained thereby. In Section 6, we report on the results of computer search, where the packing condition is used to exclude the existence of perfect 2-error-correcting codes in the Mannheim, Lipschitz and Hurwitz metrics, for all but a handful of lengths less than 10 000. Finally, in Section 7, codes over $\mathcal{H}_\pi$, codes over $A_p[\rho]$, and codes over $\mathbb{Z}[i]$ are compared in terms of average energy, code rate and bandwidth occupancy.

2. Preliminaries

Fundamental in this context is

Definition 1. *A code C in a set S with a given metric is a perfect t-error-correcting code if every element, or word, $s \in S$ has the distance t or less from exactly one codeword $c \in C$.*

In Section 4, this very general definition is adjusted to the metrics we now define.

We begin with a discussion of the Mannheim metric. This metric was considered by Huber [7] and Martinez et al. [13]. Let $\mathbb{Z}[i]$ denote the set of all Gaussian integers and let $\mathbb{Z}[i]_\pi$ be the residue class of $\mathbb{Z}[i]$ modulo π. For $\beta, \gamma \in \mathbb{Z}[i]_\pi$, consider $a+bi$ in the class of $\beta-\gamma$ with $|a|+|b|$ minimum. The *Mannheim distance* d_M between β and γ is

$$d_M(\beta, \gamma) = |a| + |b|.$$

The metric induced on $\mathbb{Z}[i]_\pi$ by the Mannheim distance is in this study called the *Mannheim metric*. It is a true metric [13].[‡] More information related to the Mannheim metric and weight can be found in [7, 12, 13].

[‡]Note that the Mannheim distance defined in [7] is not a true metric.

The construction of perfect 1-error-correcting codes in the Mannheim metric that we provide later does not work, unless the norm of $\pi = a+bi$, that is $N(\pi) = \pi\pi^\star = a^2 + b^2$, is equal to an odd prime number p. It is easy to verify that in that case, the size of $\mathbb{Z}[i]_\pi$ is equal to p, and $p \equiv 1 (\mathrm{mod}\ 4)$.

The *Hamilton Quaternion Algebra* over the set of real numbers $\mathbb{R}$ denoted by $H(\mathbb{R})$, see for example [3], is the associative unital algebra given by the following representation:

i) $H(\mathbb{R})$ is the free $\mathbb{R}$ module over the symbols $1, e_1, e_2, e_3$, that is,

$$H(\mathbb{R}) = \{a_0 + a_1e_1 + a_2e_2 + a_3e_3 : \ a_0, a_1, a_2, a_3 \in \mathbb{R}\},$$

ii) 1 is the multiplicative identity,

iii) $e_1^2 = e_2^2 = e_3^2 = -1$,

iv) $e_1e_2 = -e_2e_1 = e_3,\ e_3e_1 = -e_1e_3 = e_2,\ e_2e_3 = -e_3e_2 = e_1$.

The *Lipschitz integers*, denoted here by $H(\mathbb{Z})$, is the following subset of $H(\mathbb{R})$:

$$H(\mathbb{Z}) = \{a_0 + a_1e_1 + a_2e_2 + a_3e_3 : a_0, a_1, a_2, a_3 \in \mathbb{Z}\},$$

where $\mathbb{Z}$ is the set of all integers. The set of all Lipschitz integers constitutes a ring. If $q = a_0 + a_1e_1 + a_2e_2 + a_3e_3$ is a Lipschitz integer, then its conjugate is $q^* = a_0 - (a_1e_1 + a_2e_2 + a_3e_3)$.

The *norm* $N(q)$ of q is the integer

$$N(q) = qq^* = a_0^2 + a_1^2 + a_2^2 + a_3^2.$$

It is easy to check that for any two Lipschitz integers q and q', it is true that

$$N(qq') = N(q)(q').$$

The set of units will be central in our presentation, and we denote it by $\mathcal{E}$. It follows immediately from the previous relation that

$$\mathcal{E} = \{\pm 1, \pm e_1, \pm e_2, \pm e_3\}.$$

Furthermore, if π is a Lipschitz integer such that $N(\pi)$ is equal to a prime number p, then it also follows from that relation that

$$\pi = \alpha\beta \qquad \Longrightarrow \qquad \{\alpha, \beta\} \cap \mathcal{E} \neq \emptyset.$$

Following the standard terminology, see [3], we thus say that π is a *prime Lipschitz integer* if $N(\pi)$ is a prime number. If the norm of q is an odd integer, then the Lipschitz integer q is said to be an *odd Lipschitz integer*.

Fundamental in this study is

Definition 2 ([12])**.** *Let π be a Lipschitz integer. If there exists $\lambda \in H(\mathbb{Z})$ such that $q_1 - q_2 = \lambda\pi$, then $q_1, q_2 \in H(\mathbb{Z})$ are said to be right congruent modulo π. This relation is denoted by $q_1 \equiv_r q_2$.*

The relation $q_1 \equiv_r q_2$ is an equivalence relation. The set of equivalence classes constitutes an Abelian group, which we denote by $H(\mathbb{Z})_\pi$. With another terminology, we would say that $H(\mathbb{Z})_\pi$ is the quotient group $H(\mathbb{Z})/\langle\pi\rangle$ of the additive group in the ring $H(\mathbb{Z})$, with its subgroup

$$\langle\pi\rangle = \{\lambda\pi : \lambda \in H(\mathbb{Z})\}.$$

We cannot define any multiplication in $H(\mathbb{Z})_\pi$ in a consistent way, as $\langle\pi\rangle$ is not a 2-sided ideal. This will complicate our constructions of codes. Furthermore, neither the distributive nor the associative rules are true in general.

When nothing else is stated below, we will use right congruences modulo π. Analogous results are valid for left congruences modulo π.

The next theorem was proved in [12] by Martinez et al.

Theorem 1 ([12])**.** *If π is a prime Lipschitz integer in $H(\mathbb{Z})$, then $H(\mathbb{Z})_\pi$ has $N(\pi)^2$ elements.*

The Lipschitz distance was defined in [13]. Let π be a prime Lipschitz integer. Given $\alpha, \beta \in H(\mathbb{Z})_\pi$, then the *Lipschitz distance* between α and β, denoted by $\mathrm{d}_L(\alpha, \beta)$, is equal to the integer

$$\mathrm{d}_L(\alpha, \beta) = |a_0| + |a_1| + |a_2| + |a_3|\,,$$

where $\alpha - \beta \equiv_r a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 (\mathrm{mod}\ \pi)$ with $|a_0| + |a_1| + |a_2| + |a_3|$ minimal.

The *Lipschitz weight* $\mathrm{w}_L(\gamma)$ of the element γ is defined to be equal to the integer

$$\mathrm{w}_L(\gamma) = \mathrm{d}_L(\gamma, 0).$$

Next, we define and treat the Hurwitz integers. In what follows , w denotes the element

$$w = \frac{1}{2} + \frac{1}{2}e_1 + \frac{1}{2}e_2 + \frac{1}{2}e_3,$$

and $H(\mathbb{Z} + \frac{1}{2})$ denotes the set

$$H(\mathbb{Z} + \frac{1}{2}) = w + H(\mathbb{Z}).$$

The *Hurwitz integers* $\mathcal{H}$, see e.g. [2], are the set of elements

$$\mathcal{H} = H\left(\mathbb{Z}\right) \cup H\left(\mathbb{Z} + \frac{1}{2}\right).$$

It can easily be checked that $\mathcal{H}$ is closed under quaternion multiplication and addition. Hence $\mathcal{H}$ forms a subring of the ring of all quaternions $H(\mathbb{R})$.

The next definition is also fundamental.

Definition 3. *Let π be any element in $\mathcal{H}$. If there exists $\lambda \in \mathcal{H}$ such that $q_1 - q_2 = \lambda\pi$, then $q_1, q_2 \in \mathcal{H}$ are right congruent modulo π. This relation is denoted by $q_1 \equiv_r q_2$.*

The relation $q_1 \equiv_r q_2$ is an equivalence relation, the set of equivalence classes of which constitutes an Abelian group that we denote by $\mathcal{H}_\pi$. As for $H(\mathbb{Z})_\pi$, we cannot give any well defined multiplication in $\mathcal{H}_\pi$.

We denote the set of units in $\mathcal{H}$ by $\mathcal{E}$. Let e_0 denote the element 1 in $\mathcal{H}$. It is easy to check that $\mathcal{E}$ is the union of three sets as indicated below:

$$\mathcal{E} = \{\pm e_i : i = 0, 1, 2, 3\} \cup \{\pm e_i w : i = 0, 1, 2, 3\} \cup \{\pm e_i w^\star : i = 0, 1, 2, 3\}. \quad (1)$$

We note that for any two distinct elements ϵ_1 and ϵ_2 in $\mathcal{E}$

$$N(\epsilon_1 - \epsilon_2) \in \{1, 2, 3, 4\}. \quad (2)$$

Hence, if $\pi\pi^\star$ is equal to a prime number $p \geq 5$, we may conclude that the elements in $\mathcal{E}$ represent 24 distinct elements in $\mathcal{H}_\pi$.

Observe that in case $\pi\pi^\star = 3$, the elements in $\mathcal{E}$ belong to eight distinct cosets to $\langle\pi\rangle$ in $\mathcal{H}$; the elements in set $\{\pm e_i : i = 0, 1, 2, 3\}$ can be selected to represent the elements of $\mathcal{E}$ in $\mathcal{H}_\pi$ if $N(\pi) = 3$.

Let $\mathcal{E}$ denote the set of units in the ring $\mathcal{H}$ as described in (1). We denote by $\mathcal{E}_\pi$ the image of the set $\mathcal{E}$ in $\mathcal{H}_\pi$ under the group homomorphism φ defined by

$$\varphi : \mathcal{H} \to \mathcal{H}_\pi, \qquad g \ \mapsto \ g + \langle\pi\rangle,$$

that is, $\mathcal{E}_\pi = \varphi(\mathcal{E})$. It follows from (2) that $|\mathcal{E}_\pi| = |\mathcal{E}|$ if $N(\pi) \geq 5$. Similarly, $\mathcal{E}_\pi$ is defined for the sets of units $\mathcal{E}$ in the rings $\mathbb{Z}[i]$ and $H(\mathbb{Z})_\pi$, respectively. It is easy to verify that $|\mathcal{E}_\pi| = |\mathcal{E}|$ also in both of these two cases. In order to simplify the notation at some instances we denote the elements in the set $\mathcal{E}_\pi$ by their inverse images in $\mathcal{E}$ under the map φ. So for example with $\mathcal{E} = \{\pm 1, \pm i\}$ we let

$$\mathcal{E}_\pi = \{1, -1, i, -i\} = \{1 + \langle\pi\rangle, -1 + \langle\pi\rangle, i + \langle\pi\rangle, -i + \langle\pi\rangle\}.$$

We finalize this preparatory section by proving:

Theorem 2. *If π is an odd Lipschitz prime integer, then the size of $\mathcal{H}_\pi$ is equal to $N(\pi)^2$.*

Before we prove this theorem we give an example that will also be used and considered later on.

Example 1. *Let $\pi = 2 + e_1$. Then the size of $\mathcal{H}_{2+e_1}$ is equal to 25. As furthermore $|\mathcal{E}| = 24$, we get from (2) that the elements of $\mathcal{E}$, together with the element 0, can be selected to represent the elements of $\mathcal{H}_\pi$.*

We use a simple lemma and a proposition in the proof of Theorem 2.

Lemma 1. *For any element u in $H(\mathbb{Z} + \frac{1}{2})$, the map*

$$p_u : x \mapsto x + u$$

is a bijective map from $H(\mathbb{Z})$ to $H(\mathbb{Z} + \frac{1}{2})$.

Proof. If $x+u=y+u$, then $x=y$, and consequently the map p_u is injective. As $u=b_0+b_1e_1+b_2e_2+b_3e_3$, where $b_i\in\mathbb{Z}+\frac{1}{2}$, we get that

$$u-w=(b_0-\frac{1}{2})+(b_1-\frac{1}{2})e_1+(b_2-\frac{1}{2})e_2+(b_3-\frac{1}{2})e_3\in H(\mathbb{Z}).$$

Hence, every element $z=w+x$ of $H(\mathbb{Z}+\frac{1}{2})$, where $x\in H(\mathbb{Z})$, can be expressed as the sum of an element in $H(\mathbb{Z})$ and the element u. Thus the map p_u is surjective. □

Proposition 1. *For any odd Lipschitz integer π,*

$$|\mathcal{H}_\pi|=|H(\mathbb{Z})_\pi|.$$

Proof. Assume that π is an odd Lipschitz integer. Let

$$K=\{\lambda\pi:\lambda\in H(\mathbb{Z})\}\qquad\text{and}\qquad K_{1/2}=\{\lambda\pi:\lambda\in H(\mathbb{Z}+\frac{1}{2})\}.$$

Then K is an additive subgroup of the additive group G of the ring $H(\mathbb{Z})$, in fact a left ideal. The size of $H(\mathbb{Z})_\pi$ is the number of cosets of K in G. Let $t=|H(\mathbb{Z})_\pi|$ and let h_1, h_2, ..., h_t be a set of coset representatives to K in G.

As π is an odd Lipschitz integer, we get that

$$u=w\pi\in H(\mathbb{Z}+\frac{1}{2}).$$

As every element λ in $H(\mathbb{Z}+\frac{1}{2})$ is the sum $\lambda=w+\lambda_w$ of w and an element $\lambda_w\in H(\mathbb{Z})$, we thus get that

$$K_{1/2}=\{(w+\lambda_w)\pi:\lambda_w\in H(\mathbb{Z})\}=u+K,$$

and thus a coset to G in the additive group of the ring $\mathcal{H}$.

We now observe that

$$K'=\{\lambda\pi:\lambda\in\mathcal{H}\}=K\cup K_{1/2}=K\cup(u+K),$$

and, from the definition of $\mathcal{H}_\pi$, that the size of $\mathcal{H}_\pi$ is equal to the number of cosets of the additive group K' in the additive group G' of $\mathcal{H}$.

We now apply Lemma 1. As

$$(h_1+K)\cup(h_2+K)\cup\ldots\cup(h_t+K)=H(\mathbb{Z}),$$

and as $u\in H(\mathbb{Z}+\frac{1}{2})$, by using Lemma 1 we may conclude that

$$(u+h_1+K)\cup(u+h_2+K)\cup\ldots\cup(u+h_t+K)=u+H(\mathbb{Z})=H(\mathbb{Z}+\frac{1}{2}).$$

As finally, for any $i=1,2,\ldots,t$,

$$h_i+K'=(h_i+K)\cup(u+h_i+K),$$

we can conclude that the number of cosets to K' in G' is equal to t. This proves the proposition. □

Theorem 2 now follows from Proposition 1 and Theorem 1.

2.1. The Lipschitz and the Hurwitz metric in $\mathcal{H}_\pi^n$

Consider the direct product $S = \mathcal{H}_\pi^n$ of n copies of $\mathcal{H}_\pi$, where π is an odd Lipschitz prime integer. We say that two elements, or *words*, $\bar{x}$ and $\bar{y}$ in $\mathcal{H}_\pi^n$ have distance one, $\mathrm{d}_H(\bar{x}, \bar{y}) = 1$, if there is a word $\bar{e} = (0, \ldots, 0, \epsilon, 0, \ldots, 0)$, with just one non-zero entry such that

$$\bar{y} = \bar{x} + \bar{e},$$

for a unique element ϵ in a set $\mathcal{E}_\pi$. We consider two distinct sets $\mathcal{E}$, the set of units in $H(\mathbb{Z})$ and the set of units in $\mathcal{H}$, respectively.

With terminology from graph theory, it is now easy to explain how in both cases we can define a metric in $\mathcal{H}_\pi^n$. Consider the words of S as vertices in a graph, where there is an edge between two vertices $\bar{x}$ and $\bar{y}$ if $\mathrm{d}_H(\bar{x}, \bar{y}) = 1$. The *distance* $\mathrm{d}_H(\bar{a}, \bar{b})$ between any two vertices $\bar{a}$ and $\bar{b}$ is the length of the shortest path between these two vertices. General results from graph theory give that this distance function defines a metric in S.

If $\mathcal{E}$ is defined as in (1), then the metric obtained in $\mathcal{H}_\pi^n$ is called the *Hurwitz metric*.

In case $\mathcal{E}$ consists of the elements in the set $\{\pm 1, \pm e_1, \pm e_2, \pm e_3\}$, we get a metric that we call the *Lipschitz metric in* $\mathcal{H}_\pi^n$. From Lemma 1 and Proposition 1 we get that this metric has similarities with the Lipschitz metric in $H(\mathbb{Z})_\pi$. Therefore, the Lipschitz metric in $\mathcal{H}_\pi^n$ is not treated in later sections.

Also note that if the norm of π is equal to 3, then it follows from the observation made shortly after equation (2) that the Lipschitz and Hurwitz metrics do not differ. However, note that in general the Hurwitz metric and the Lipschitz metric are not isomorphic, as shown in the next example.

Example 2. *Let $w_L(c)$ and $w_{Hz}(c)$ denote the Lipschitz weight and Hurwitz weight, respectively, of a word c. Consider $S = \mathcal{H}_\pi^1$ and the elements 2 and $q = 2w$. We note that*

$$\mathrm{w}_L(2) = 2 = \mathrm{w}_{Hz}(2),$$

while

$$\mathrm{w}_L(q) = \mathrm{w}_L(1 + e_1 + e_2 + e_3) = 4 \neq 2 = \mathrm{w}_{Hz}(q),$$

as $q = w + w$ and $w \in \mathcal{E}$.

We will sometimes refer to a perfect error-correcting code in $\mathcal{H}_\pi^n$ with the Hurwitz metric, as a *perfect Hurwitz-weight code of length n over $\mathcal{H}_\pi$*.

3. $H(\mathbb{Z})_\pi$ and $\mathcal{H}_\pi$ as modules

In later sections, we define codes and the error-correcting procedure by using a matrix $\mathbf{H}$. Thereby, we need the distributive rule to be true, more precisely

$$\mathbf{H}(\bar{a}^T + \bar{b}^T) = \mathbf{H}\bar{a}^T + \mathbf{H}\bar{b}^T \tag{3}$$

for every pair of elements $\bar{a}$ and $\bar{b}$ of $H(\mathbb{Z})_\pi^n$ and $\mathcal{H}_\pi^n$, respectively. The rule above is trivially true when we perform our calculations over $\mathbb{Z}[i]_\pi$, as it is a ring. To

solve this problem in other two instances, we let the entries of $\mathbf{H}$ be elements in the rings $H(\mathbb{Z})$ and $\mathcal{H}$, respectively, and furthermore let the groups $H(\mathbb{Z})_\pi^n$ and $\mathcal{H}_\pi^n$ be regarded as modules over these rings. This means that we define

$$h(h' + \langle\pi\rangle) = hh' + \langle\pi\rangle,$$

for $h \in \mathcal{H}$ and $(h' + \langle\pi\rangle) \in \mathcal{H}_\pi^n$. The distributive property in equation (3) then follows immediately.

We say that a selection of coset representatives $\overline{H(\mathbb{Z})_\pi}$ to $\langle\pi\rangle$ in $H(\mathbb{Z})$ is a *complete selection of coset representatives* if no two elements of $\overline{H(\mathbb{Z})_\pi}$ are congruent modulo π, and if all cosets to $\langle\pi\rangle$ are represented in $\overline{H(\mathbb{Z})_\pi}$, that is,

$$|\overline{H(\mathbb{Z})_\pi}| = |H(\mathbb{Z})_\pi|.$$

The same terminology for coset representatives to a left ideal $\langle\pi\rangle$ in $\mathcal{H}$ is used in what follows.

Let $\mathcal{E}$ denote the set of units in $H(\mathbb{Z})$ and $\mathcal{H}$, respectively, and let $\overline{H}$ be any set of mutually non-congruent elements in any of these rings. The set $\overline{H}$ is then said to be $\mathcal{E}$*-homogeneous* if

$$\bar{h}\epsilon = \bar{h}'\epsilon \qquad \Longrightarrow \qquad \bar{h} = \bar{h}'$$

for every $\epsilon \in \mathcal{E}_\pi$ and $\bar{h}, \bar{h}' \in \overline{H}$. We need to refer to the following proposition that immediately follows from the definition above.

Proposition 2. *If $\overline{H}$ is an $\mathcal{E}$-homogeneous and complete selection of coset representatives to $\langle\pi\rangle$ in $\mathcal{H}_\pi$ or $H(\mathbb{Z})_\pi$, then the equation*

$$\bar{h}\epsilon = h, \tag{4}$$

has a unique solution $\bar{h}$ in $\overline{H}$ for every pair $(\epsilon, h) \in \mathcal{E}_\pi \times \mathcal{H}_\pi$ and $(\epsilon, h) \in \mathcal{E}_\pi \times H(\mathbb{Z})_\pi$, respectively.

In $\mathbb{Z}[i]_\pi$, where the norm of π is a prime number, the conclusion of the proposition is true whenever $\overline{\mathbb{Z}[i]_\pi}$ is a complete selection of coset representatives. In $H(\mathbb{Z})_\pi$, the $\mathcal{E}$-homogeneous property is essential. For example, consider the Lipschitz prime $\pi = 1 + e_1 + 2e_2 + e_3$. The element $\bar{h} = 1 - e_1 - e_2 + 2e_3$ does not belong to the left ideal $\langle\pi\rangle$ and can hence be selected to be one of the non-zero coset representatives to $\langle\pi\rangle$ to be included in $\overline{H(\mathbb{Z})_\pi}$, or in $\overline{\mathcal{H}_\pi}$. However,

$$(1 - e_1 - e_2 + 2e_3)(e_1 + \langle\pi\rangle) = \pi + \langle\pi\rangle = 0 + \langle\pi\rangle.$$

and hence, the set $\{\bar{h}, \bar{0}\}$ is not $\mathcal{E}$-homogeneous.

4. Perfect 1-error-correcting codes and partitions

The main idea in our construction is inspired by a generalization of Herzog and Schönheim [6] of the traditional use of parity-check matrices in the construction of 1-error-correcting linear codes. They found a relation between such codes over finite fields and partitions of a vector space into subspaces just having the zero vector in common.

We now turn to the specific situation considered in this study. Here, we let H denote any of the sets $\mathbb{Z}[i]_\pi$, $H(\mathbb{Z})_\pi$, or $\mathcal{H}_\pi$ for some element π, the norm of which is equal to an odd prime number.

A *code of length* n is a subset C of the direct product H^n of n copies of H. In each of the cases we consider, H is an Abelian group, and thus the same is true for H^n. A code C is a *group code* if it is a subgroup of H^n, or equivalently as H^n is a finite group,

$$c, c' \in C \qquad \Longrightarrow \qquad c - c' \in C.$$

In the case when H is a finite field, and thus H^n is a vector space of dimension n over H, then a *linear code* is a subspace C of H^n. Here we say that a code C in H^n is an (n, k)*-code* if the size of C is equal to $|H|^k$. (The case $k = 0$ is of less interest, and thus left aside.)

From the definition of a perfect 1-error-correcting code C in these metrics, we obtain that C is a perfect code if and only if for every word x in $H^n \setminus C$ there is a unique code word c in C, such that

$$x = c + e$$

for some word e with its only non-zero entry ϵ in $\mathcal{E}_\pi$.

Theorem 3. *Let H and $\mathcal{E}_\pi$ be constituted as above, and let $\overline{H}$ be a complete selection of coset representatives to $\langle\pi\rangle$. Assume that the norm of π is an odd prime number. Let $n = (|H| - 1)/|\mathcal{E}_\pi|$.*

If $g_1 = 1$, g_2, ..., g_n are elements in $\overline{H}$, satisfying two of the following three conditions:

(i) $|g_i\mathcal{E}_\pi| = |\mathcal{E}_\pi|$, *for* $i = 2, 3, \ldots, n$,

(ii) $g_i\mathcal{E}_\pi \cap g_j\mathcal{E}_\pi = \emptyset$, *for* $i \neq j$,

(iii) $H \setminus \{0\} = \mathcal{E}_\pi \cup g_2\mathcal{E}_\pi \cup \ldots \cup g_n\mathcal{E}_\pi$,

then the null-space C of the matrix

$$\mathbf{H} = \begin{pmatrix} 1 & g_2 & \ldots & g_n \end{pmatrix}$$

is a perfect 1-error-correcting group $(n, n-1)$-code in H^n, in the metric defined by $\mathcal{E}_\pi$.

We observe that if condition *(ii)* holds, then the set $\{g_1, g_2, \ldots, g_n\}$ is $\mathcal{E}$-homogeneous.

We remind of the fact that it follows from the discussion in the previous section, that the product $g_i a$ is well defined, for every $i \in [n]$ and $a \in H$ and that the distributive rule

$$g_i(a + b) = g_i a + g_i b$$

is true for any $i \in [n]$ and any elements $a, b \in H$.

Proof. We first note that from the assumption that the norm of π is an odd prime number, it follows that n is an integer. Furthermore, a simple counting argument shows that if any two of the conditions *(i), (ii)* and *(iii)* hold, then all three of these conditions hold.

For any choice of elements c_2, c_3, ..., c_n in H, the n-tuple

$$(-(g_2c_2 + g_3c_3 + \cdots + g_nc_n), c_2, c_3, \ldots, c_n)$$

belongs to the null-space of $\mathbf{H}$. Furthermore, every element in C can be expressed in this way. This proves the statement on the size of C.

Let $\bar{x} = (x_1, x_2, \ldots, x_n)$ be any element in $H^n \setminus C$. Then, from the assumptions *(i), (ii)* and *(iii)*, it follows that

$$\mathbf{H}\bar{x}^T = g_j\epsilon, \tag{5}$$

for exactly one element $j \in [n]$ and a unique element $\epsilon \in \mathcal{E}_\pi$. It follows from the distributive property of left multiplication of elements in H with the elements in the set $\overline{H}$ that

$$\mathbf{H}(x_1, \ldots, x_j - \epsilon, \ldots, x_n)^T = g_1x_1 + \cdots + g_j(x_j - \epsilon) + \cdots + g_nx_n = \mathbf{H}\bar{x}^T - g_j\epsilon = 0.$$

This fact, together with the fact that $g_i\mathcal{E}_\pi \cap g_j\mathcal{E}_\pi = \emptyset$, for $i \neq j$, proves that C is a perfect 1-error-correcting code.

From the distributive rule, when multiplying elements of H from the left with elements of $\overline{H}$, it follows that the null-space of the matrix $\mathbf{H}$ is a group code. □

It must be remarked that in the case $H = \mathbb{Z}[i]_\pi$, the null space C of the matrix $\mathbf{H}$ also has the property

$$c \in C,\ \lambda \in H \qquad \Longrightarrow \qquad \lambda c \in C,$$

and thus that C is a linear code.

Furthermore, the proof shows how to correct an error in a received word $\bar{x}$, which has the *syndrome* given by (5).

It must also be remarked that condition *(i)* of the theorem above is not always true. Consider for example $H(\mathbb{Z})_\pi$, where $\pi = 2 - e_1 + e_2 + e_3$. The element $a = 3 + e_1 + 2e_2$ can be verified not to belong to $\langle\pi\rangle$, and hence a represents a non-zero element in $H(\mathbb{Z})_\pi$. However,

$$(3 + e_1 + 2e_2) - (3 + e_1 + 2e_2)e_1 = 2\pi \in \langle\pi\rangle,$$

that is, ae_1 and $a1$ belong to the same coset to $\langle\pi\rangle$ in $H(\mathbb{Z})$. It follows that

$$|(3 + e_1 + 2e_2)\mathcal{E}_\pi| \leq |\mathcal{E}_\pi| - 1.$$

Given any partition of H satisfyingconditions *(i)*, *(ii)* and *(iii)* of Theorem 3, under certain conditions we can derive a partition of H^k for any non-negative integer k. The next theorem can be proved in exactly the same way as Theorem 3.

We assume that H and $\mathcal{E}_\pi$ are constituted as in Theorem 3.

Theorem 4. *Let $\overline{H}$ be an $\mathcal{E}$-homogeneous and complete selection of coset representatives to $\langle\pi\rangle$, where the norm of π is an odd prime number. Let $m = (|H|^k - 1)/|\mathcal{E}_\pi|$ and $n = (|H| - 1)/|\mathcal{E}_\pi|$.*

If $g_1 = 1$, g_2, ..., g_n are coset representatives to $\langle\pi\rangle$ in H satisfying two of the following three conditions:

(i) $|g_i\mathcal{E}_\pi| = |\mathcal{E}_\pi|$, *for* $i = 2, 3, \ldots, n$,

(ii) $g_i\mathcal{E}_\pi \cap g_j\mathcal{E}_\pi = \emptyset$, *for* $i \neq j$,

(iii) $H \setminus \{0\} = \mathcal{E}_\pi \cup g_2\mathcal{E}_\pi \cup \ldots \cup g_n\mathcal{E}_\pi$,

then the null-space of the $k \times m$-matrix, the columns of which are

$$\begin{pmatrix} 0 & \cdots & 0 & g_i & h_\nu & \cdots & h_k \end{pmatrix}^T$$

for $i \in [n]$, $\nu \in \{2, \ldots, k\}$, and where $(h_\nu, \ldots, h_k) \in \overline{H}^{k+1-\nu}$, is a perfect 1-error-correcting group $(m, m-k)$-code in H^m.

Proof. By Proposition 2, the columns of **H** multiplied to the right by the elements in the set $\mathcal{E}_\pi$ induce a partition of H^k. The remaining part of the proof uses similar arguments as those used in the proof of Theorem 3. □

4.1. Some constructions of partitions

Lemma 2. *Let G be any subgroup of the set of units in a ring R. Then the following relation is valid:*

$$aG \cap bG \neq \emptyset \qquad \Longrightarrow \qquad aG = bG. \tag{6}$$

For the sake of completeness, we include a trivial proof of the lemma.

Proof. Assume $aG \cap bG \neq \emptyset$, that is, $ag_a = bg_b$ for two elements g_a and g_b of G. Then, for every $g \in G$,

$$ag = ((bg_b)g_a^{-1})g \in bG.$$

Thus $aG \subseteq bG$. Similarly, we can deduce that $bG \subseteq aG$. □

Hence, in any finite ring R, and with any subgroup G of the group of units in R, it is easy to partition the non-zero elements of R into left cosets to G by successively forming cosets r_0G, r_1G, ..., r_nG, such that for $i = 1, 2, \ldots, n-1$,

$$r_{i+1} \notin r_0G \cup r_1G \cup \ldots \cup r_iG.$$

As $\mathbb{Z}[i]_\pi$ is a ring for every $\pi \in \mathbb{Z}[i]$ and the set $G = \{\pm 1, \pm i\}$ is a subgroup of the set of units in $\mathbb{Z}[i]_\pi$, we easily get partitions of every such $\mathbb{Z}[i]_\pi$ into cosets to G.

Example 3. *Let $\pi = 3 + 2i$. Following the "naive" recipe described above we get the following partition*

$$\mathbb{Z}[i]_\pi \setminus \{0\} = 1\{\pm 1, \pm i\} \cup 2\{\pm 1, \pm i\} \cup (1+i)\{\pm 1, \pm i\}.$$

Observe that this naive method of forming partitions of the set of non-zero elements into cosets of a group of units has in general no success in $H(\mathbb{Z})_\pi$ or $\mathcal{H}_\pi$. Consider for example the Lipschitz prime $\pi = 3 + 2e_1 + 2e_2$. Let us start with the coset $1\mathcal{E}_\pi$. Then, for example $a = -2 + 3e_1 + e_2 - 2e_3$ does not belong to this coset, while it is easy to check that

$$a(-e_1) \in a\mathcal{E}_\pi \cap 1\mathcal{E}_\pi,$$

which makes the naive method fail. Evidently, relation (6) is not always true either in $H(\mathbb{Z})_\pi$ or in $\mathcal{H}_\pi$.

Our constructions of perfect 1-error-correcting codes also require that the number of elements in each coset $g_i\mathcal{E}_\pi$ is equal to the number of elements in $\mathcal{E}_\pi$, as otherwise the error-correcting process fails. Let for example $\pi = 2 + 2i$ in the preceding example. Then the set $2\{\pm 1, \pm i\}$ consists of exactly two distinct elements in the ring $\mathbb{Z}[i]_\pi$. However, as remarked in Section 2, if $N(\pi)$ is equal to a prime number $p > 5$, then $\mathbb{Z}[i]_\pi$ is a commutative ring with p elements, and thus a finite field. Then the set $\mathcal{E}_\pi = \{\pm 1, \pm i\}$ is a subgroup of the multiplicative group of that field. This proves the following theorem:

Theorem 5. *Assume that $\pi \in \mathbb{Z}[i]$ has a norm equal to an odd prime $p \geq 5$. Then the set of non-zero elements of $\mathbb{Z}[i]_\pi$ admits exactly one partition*

$$\mathbb{Z}[i]_\pi \setminus \{0\} = \mathcal{E}_\pi \cup g_2\mathcal{E}_\pi \cup \ldots \cup g_t\mathcal{E}_\pi$$

into $t = (p-1)/4$ mutually disjoint cosets of $\mathcal{E}_\pi$, all of the same size.

Despite the failure in general of the construction above of partitions in $H(\mathbb{Z})_\pi$, we can provide partitions $H(\mathbb{Z})_\pi \setminus \{0\}$ into left cosets of $\mathcal{E}_\pi$. The coset representatives to the left ideal $\langle\pi\rangle$ must however be chosen with great care. In this context, the choice of coset representatives to $\langle\pi\rangle$ given in the next proposition turns out to be fruitful.

Proposition 3. *Let $\pi = a_0 + a_1e_1 + a_2e_2 + a_3e_3$ be a Lipschitz prime with $p = \pi\pi^\star$. Then, for any two distinct elements e_i and e_j in $\{e_0 = 1, e_1, e_2, e_3\}$ such that p does not divide $a_i^2 + a_j^2$, it is true that*

$$C_{i,j} = \{x_ie_i + x_je_j : x_i, x_j \in \mathbb{Z}_p\}$$

is a complete selection of non-congruent coset representatives to $\langle\pi\rangle$ in $H(\mathbb{Z})$. Furthermore, $C_{i,j}$ is $\mathcal{E}$-homogeneous.

Proof. We show that no two elements in a set $C_{i,j}$ are congruent modulo π. The first part of the proposition then follows, as by Theorem 1 the size of such a set is equal to the size of $H(\mathbb{Z})_\pi$.

We consider the case $i = 0$ and $j = 1$, and assume that $a_0^2 + a_1^2 \not\equiv 0 \pmod{p}$. The other cases are treated in the same way. Assume that

$$x_0e_0 + x_1e_1 - (x_0'e_0 + x_1'e_1) = (t_0 + t_1e_1 + t_2e_2 + t_3e_3)\pi,$$

for some integers t_0, t_1, t_2, t_3. Then,

$$\begin{pmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & a_3 & -a_2 \\ a_2 & -a_3 & a_0 & a_1 \\ a_3 & a_2 & -a_1 & a_0 \end{pmatrix} \begin{pmatrix} t_0 \\ t_1 \\ t_2 \\ t_3 \end{pmatrix} = \begin{pmatrix} x_0 - x_0' \\ x_1 - x_1' \\ 0 \\ 0 \end{pmatrix}. \tag{7}$$

We multiply by the transpose of the 4×4-matrix above and get

$$\begin{pmatrix} pt_0 \\ pt_1 \\ pt_2 \\ pt_3 \end{pmatrix} = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ -a_1 & a_0 & -a_3 & a_2 \\ -a_2 & a_3 & a_0 & -a_1 \\ -a_3 & -a_2 & a_1 & a_0 \end{pmatrix} \begin{pmatrix} x_0 - x_0' \\ x_1 - x_1' \\ 0 \\ 0 \end{pmatrix}. \tag{8}$$

Thus we obtain the following system of equations

$$\begin{cases} a_0(x_0 - x_0') + a_1(x_1 - x_1') \equiv 0 \pmod{p}, \\ -a_1(x_0 - x_0') + a_0(x_1 - x_1') \equiv 0 \pmod{p}. \end{cases}$$

The assumption that p is a prime number that does not divide the determinant $a_0^2 + a_1^2$ of the system above implies that p divides both $x_0 - x_0'$ and $x_1 - x_1'$. This means that x_0 and x_0' represent the same element in the ring $\mathbb{Z}_p$, and similarly for x_1 and x_1'.[§]

Finally, p belongs to $\langle \pi \rangle$, and so does pe_i, for $i = 1, 2, 3$. This implies that we can add elements in $C_{0,1}$ in the same way as we add elements in $\mathbb{Z}_p \times \mathbb{Z}_p$.

Now let us turn to the question of $\mathcal{E}$-homogeneity. We note that

$$(x_0 + x_1 e_1)e_1 = -x_1 + x_0 e_0.$$

Hence, if $\bar{h}$ and $\bar{h}'$ are two distinct elements in $C_{0,1}$, then $\bar{h}e_1$ and $\bar{h}'e_1$ are distinct, and similarly for multiplication to the right with $-e_1$, (and -1). Multiplying a coset representative in $C_{0,1}$ by $\pm e_2$ and $\pm e_3$, gives a coset representative in $C_{2,3}$. As $C_{2,3}$ is also complete, this means that distinct elements of $C_{0,1}$ by the latter multiplication by each of the units above, give distinct elements in $C_{2,3}$. Hence $C_{0,1}$ is $\mathcal{E}$-homogeneous. □

Theorem 6. *For any Lipschitz prime $\pi = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3$ with $p = \pi\pi^\star$ and such that $p \equiv 3 (\mathrm{mod}\ 4)$, we can find a set of coset representatives to $\langle \pi \rangle$ forming the set $H(\mathbb{Z})_\pi$ and admitting a partition*

$$H(\mathbb{Z})_\pi \setminus \{0\} = g_1 \mathcal{E}_\pi \cup g_2 \mathcal{E}_\pi \cup \ldots \cup g_{(p^2-1)/8} \mathcal{E}_\pi,$$

where $g_i \mathcal{E}_\pi \cap g_j \mathcal{E}_\pi = \emptyset$, for $i \neq j$.

[§]It follows that the size of $H(\mathbb{Z})_\pi$ equals p^2, that is, that Theorem 1 is valid. That $C_{0,1}$ contains coset representatives from p^2 distinct cosets to $\langle \pi \rangle$ is proved above. Let

$$\mathbf{A} = \begin{pmatrix} a_0 & -a_1 \\ a_1 & a_0 \end{pmatrix} \qquad \text{and} \qquad \mathbf{B} = \begin{pmatrix} a_2 & -a_3 \\ a_3 & a_2 \end{pmatrix}.$$

Let $(x' \ \ y')^T = (a_2^2 + a_3^2)^{-1} \mathbf{A}\mathbf{B}^T (x \ \ y)^T$ for any two arbitrary elements x, y. Then

$$-xe_2 - ye_3 + x' + y'e_1 \in \langle \pi \rangle.$$

It follows that every coset to $\langle \pi \rangle$ contains an element in $C_{0,1}$. (For details, see the proof of Theorem 6, in particular equation (9)).

Note that it follows from the theorem that $|g_i\mathcal{E}_\pi| = |\mathcal{E}_\pi|$, for $i \in [(p^2-1)/8]$.

Proof. Assume that $a_0^2 + a_1^2 \not\equiv (\bmod\ p)$. The other cases are treated similarly. We choose the elements g_i from the selection $C_{0,1}$ of coset representatives defined in the previous proposition. For such an element $g_i = x + ye_1$,

$$(x + ye_1)(\pm e_i) \in C_{2,3},$$

if $i = 2, 3$. Because of this fact, we start by searching for the relation between coset representatives to cosets of $\langle\pi\rangle$ in $C_{2,3}$ and $C_{0,1}$. The fact is that every element $xe_2 + ye_3$ in $C_{2,3}$ is congruent modulo π to a unique element

$$x' + y'e_1 = \mathbf{j}(xe_2 + ye_3)$$

in $C_{0,1}$.

There are integers t_0 and t_1 such that

$$\begin{cases} t_0a_2 - t_1a_3 \equiv x\ (\bmod\ p), \\ t_0a_3 + t_1a_2 \equiv y\ (\bmod\ p), \end{cases}$$

that is, $x = t_0a_2 - t_1a_3 + b_2p$ and $y = t_1a_2 + t_0a_3 + b_3p$ for some integers b_2 and b_3. Such integers t_0 and t_1 can be found by using the formula

$$\begin{pmatrix} t_0 \\ t_1 \end{pmatrix} = (a_2^2 + a_3^2)^{-1} \begin{pmatrix} a_2 & a_3 \\ -a_3 & a_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

where $(a_2^2 + a_3^2)^{-1}$ denotes the unique integer c in the interval $0 < c < p$ with the property

$$c(a_2^2 + a_3^2) \equiv 1(\bmod\ p).$$

Let

$$\begin{cases} x' = (-t_0a_0 + t_1a_1)\ (\bmod\ p), \\ y' = (-t_0a_1 - t_1a_0)\ (\bmod\ p), \end{cases}$$

and thus, $x' = -t_0a_0 + t_1a_1 + b_0p$ and $y' = -t_0a_1 - t_1a_0 + b_1p$ for some integers b_0 and b_1. Then we get the equality

$$xe_2 + ye_3 - x' - y'e_1 = (t_0 + t_1e_1)\pi - (b_0 + b_1e_1 - b_2e_2 - b_3e_3)\pi^\star\pi.$$

The element on the right-hand side belongs to $\langle\pi\rangle$. Hence, $xe_2 + ye_3$ and $x' + y'e_1$ are congruent modulo π.

Our calculations above thus show that the following relation in $\mathbb{Z}_p \times \mathbb{Z}_p$ between the coefficients (x, y) of an element $xe_2 + ye_3$ in $C_{2,3}$ and the coefficients (x', y') of its congruent element $x' + y'e_1$ in $C_{0,1}$, is true:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = (a_2^2 + a_3^2)^{-1} \begin{pmatrix} -a_0 & a_1 \\ -a_1 & -a_0 \end{pmatrix} \begin{pmatrix} a_2 & a_3 \\ -a_3 & a_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{9}$$

We use the notation: if (x, y) and (x', y') are related as in the equation above, then

$$(x', y') = \mathbf{j}((x, y)).$$

We define the *p-norm* of an element (x, y) in $\mathbb{Z}_p \times \mathbb{Z}_p$ to be the following element in $\mathbb{Z}_p$:

$$N_p(x, y) = (x^2 + y^2)(\text{mod } p).$$

The relation between the p-norm of an element $\alpha \in \mathbb{Z}_p \times \mathbb{Z}_p$ and the p-norm of the element $\mathbf{j}(\alpha)$ is essential in our proof. By using the relation in (9), we get

$$(x' \;\; y') \begin{pmatrix} x' \\ y' \end{pmatrix} = (a_0^2 + a_1^2)(a_2^2 + a_3^2)^{-1}(x \;\; y) \begin{pmatrix} x \\ y \end{pmatrix}.$$

As $(a_0^2 + a_1^2) + (a_2^2 + a_3^2) \equiv 0(\text{mod } p)$, we get that

$$x'^2 + y'^2 \equiv -(x^2 + y^2)(\text{mod } p).$$

Thus

$$N_p(\mathbf{j}(\alpha)) = -N_p(\alpha). \tag{10}$$

Let

$$\mathcal{E}_0 = \{\pm 1, \pm e_1\}, \qquad \mathcal{E}_1 = \{\pm e_2, \pm e_3\},$$

and let $L(x, y)$ denote the set

$$L(x, y) = (x + ye_1)\mathcal{E}_0 = \{x + ye_1, -x - ye_1, -y + xe_1, y - xe_1\}.$$

It follows from (9) that the relation between the coefficients in $xe_2 + ye_3$, with its congruent element $x' + y'e_1$, is of the form

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

for some elements a and b in $\mathbb{Z}_p$. By using the relation above, it is easy to verify that

$$x' + y'e_1 = \mathbf{j}(xe_2 + ye_3) \qquad \Longrightarrow \qquad \mathbf{j}(L(x, y)) = L(x', y'), \tag{11}$$

and hence, as

$$(x + ye_1)\mathcal{E}_1 = \{xe_2 + ye_3, -xe_2 - ye_3, -ye_2 + xe_3, ye_2 - xe_3\},$$

we can conclude that

$$(x + ye_1)\mathcal{E}_1 = L(x', y').$$

It follows that for each $a = x + ye_1$ in $C_{0,1}$, the set $a\mathcal{E}_\pi$ is the union of two sets $L(x, y)$ and $L(x', y')$. These two sets are mutually disjoint, as the p-norm of elements in the set $L(x', y')$ differs from the p-norm of the elements in $L(x, y)$.

Let (x_0, y_0) be any element in $\mathbb{Z}_p \times \mathbb{Z}_p$ such that $x_0^2 + y_0^2 \not\equiv 0(\text{mod } 4)$, and let (x_i, y_i) denote the element

$$(x_i, y_i) = \mathbf{j}^i(x_0, y_0) = \mathbf{j} \circ \cdots \circ \mathbf{j}(x_0, y_0).$$

Let $N(x_0, y_0)$ denote the smallest positive integer N such that

$$\mathbf{j}^N(L(x_0, y_0)) = L(x_0, y_0).$$

We observe that

$$N_p(x_i, y_i) = (-1)^i N_p(x_0, y_0).$$

As p is an odd integer, we get that $a \neq -a$ in $\mathbb{Z}_p$ if $a \neq 0$. We may thus conclude that $N(x_0, y_0)$ must be an even integer.

We may now adjust the "naive" method in order to produce a partition of $H(\mathbb{Z})_\pi$ into cosets of $\mathcal{E}_\pi$. We start with the elements contained in the cycle of sets $L(x_i, y_i)$ found above and form the following cosets to $\mathcal{E}_\pi$

$$\begin{aligned}
(x_0 + y_0 e_1)\mathcal{E} &= (x_0 + y_0 e_1)\mathcal{E}_0 \cup (x_0 + y_0 e_1)\mathcal{E}_1 = L(x_0, y_0) \cup L(x_1, y_1) \\
(x_2 + y_2 e_1)\mathcal{E} &= (x_2 + y_2 e_1)\mathcal{E}_0 \cup (x_2 + y_2 e_1)\mathcal{E}_1 = L(x_2, y_2) \cup L(x_3, y_3) \\
&\vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots \\
(x_s + y_s e_1)\mathcal{E} &= (x_s + y_s e_1)\mathcal{E}_0 \cup (x_s + y_s e_1)\mathcal{E}_1 = L(x_s, y_s) \cup L(x_{s'}, y_{s'}),
\end{aligned}$$

where $s' = s + 1 = N(x_0, y_0) - 1$. Then we continue with a new cycle, starting with an element $z_0 + u_0 e_1$ not belonging to any of the sets in the enumeration above and such that $z_0^2 + u_0^2 \not\equiv 0 (\mathrm{mod}\ p)$.

The element -1 is a non-square in $\mathbb{Z}_p$, if $p \not\equiv 1 (\mathrm{mod}\ 4)$. Hence in that case

$$x + ye_1 \in C_{0,1} \qquad \Longrightarrow \qquad x^2 + y^2 \not\equiv 0 (\mathrm{mod}\ p),$$

for every pair of elements $(x, y) \in \mathbb{Z}_p \times \mathbb{Z}_p \setminus \{(0, 0)\}$. We may thus derive a partition of all non-zero elements in $H(\mathbb{Z})_\pi$ into cosets of $\mathcal{E}_\pi$. □

We give an example of a partition of $H(\mathbb{Z})_\pi$ in the case the Lipschitz prime π has a norm p, that is congruent to 1 modulo 4.

Example 4. *Let $\pi = 2 + e_1$ and let $\mathcal{E} = \{\pm 1, \pm e_1, \pm e_2, \pm e_3\}$. Then*

$$H(\mathbb{Z})_\pi = 1\mathcal{E}_\pi \cup (1 + e_2)\mathcal{E}_\pi \cup (1 + e_3)\mathcal{E}_\pi.$$

We have not yet found any general method for finding partitions of $\mathcal{H}_\pi$ into cosets of the set of units in $\mathcal{H}$. Anyway, let us give an example, verifying the existence of partitions in one particular case.

Example 5. *Let $\pi = 2 + e_1 + e_2 + e_3$, and let $\mathcal{E}$ denote the set of units in $\mathcal{H}$, that is,*

$$\mathcal{E} = \{\pm 1, \pm e_1 \pm e_2, \pm e_3\} \cup w\{\pm 1, \pm e_1 \pm e_2, \pm e_3\} \cup w^\star\{\pm 1, \pm e_1 \pm e_2, \pm e_3\}.$$

Then we have the following partition of $\mathcal{H}_\pi \setminus \{0\}$ into cosets of $\mathcal{E}_\pi$:

$$\mathcal{H}_{2+e_1+e_2+e_3} \setminus \{0\} = \mathcal{E}_\pi \cup (1 + e_1)\mathcal{E}_\pi.$$

5. Some new perfect codes

We are now able to combine the results of Section 4 to construct some new perfect 1-error-correcting codes.

5.1. Perfect 1-error-correcting codes in the Mannheim metric

We consider the ring $H = \mathbb{Z}[i]_\pi$, where $\pi = a + bi$, and $a^2 + b^2$ is equal to an odd prime number p. Then $p = 4n + 1$ for some integer n. By Theorem 5, there exists a partition of the set $\mathbb{Z}[i]_\pi \setminus \{0\}$ into $n = (p-1)/4$ cosets of $\mathcal{E}_\pi = \{\pm 1, \pm i\}$. By Theorems 3 and 4, we can thus find perfect 1-error-correcting (m, k)-linear codes in the Mannheim metric, with $m = (p^l - 1)/4$ and $k = (p^l - 1)/4 - l$, for any non-negative integer l.

If C is any perfect 1-error-correcting code in $\mathbb{Z}[i]_\pi^m$ with the Mannheim metric, then

$$(4m + 1)|C| = p^m.$$

As p is a prime number, we can deduce that $m = (p^l - 1)/4$ and $\log_p |C| = (p^l - 1)/4 - l$ for some non-negative integer l. This shows that every perfect 1-error-correcting code in the Mannheim metric must have the same length and size as some of those we have found.

Also note that by the well-known Christmas theorem of Fermat, first proved by Euler, it follows that to every prime number p congruent to 1 modulo 4, there are two unique non-negative integers a and b such that $p = a^2 + b^2$. Thus, we can construct perfect 1-error-correcting Mannheim weight codes for every prime p with $p \equiv 1 (\mathrm{mod}\ 4)$.

5.2. Perfect 1-error-correcting codes in $H(\mathbb{Z})_\pi^n$

Let $\pi = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3$ be a Lipschitz prime of a norm p such that $p \equiv 3 (\mathrm{mod}\ 4)$. Then, by combining Theorem 3, Theorem 4, and Theorem 6, we can find perfect 1-error-correcting Lipschitz weight group (n, k)-codes over $H(\mathbb{Z})_\pi$. The parameters of these codes are $n = (p^{2l} - 1)/8$ and $k = (p^{2l} - 1)/8 - l$, where l can be any non-negative integer.

By a theorem of Lagrange, every non-negative integer is a sum of four squares of integers. Thus the construction above works for every prime number p such that $p \equiv 3 (\mathrm{mod}\ 4)$.

5.3. Perfect 1-error-correcting codes in $\mathcal{H}_\pi^n$

By combining Theorem 3 with Example 5, we get perfect 1-error-correcting Hurwitz weight group $(2, 1)$-codes over $\mathcal{H}_\pi$ for $\pi = 2 + e_1 + e_2 + e_3$.

5.4. Another example of a perfect code

Let π be any prime Lipschitz integer of norm 5, for example $\pi = 2 + e_1$. Let C' be any perfect 1-error-correcting code in a direct product $S' = \mathrm{GF}(25)^n$ of n copies of the finite field with 25 elements. For example, let C' be the Hamming code of length

$$n = 25^s + 25^{s-1} + \cdots + 25 + 1$$

for some integer s. Let φ be any bijective map from GF(25) to $\mathcal{H}_\pi$, such that $\varphi(0) = 0$. We extend φ to a bijective map, also denoted by φ, from S' to $S = \mathcal{H}_\pi^n$. As

$N(\pi) = 5$, we know by Example 1 that the non-zero elements of $\mathcal{H}_\pi$ can be identified with the elements of $\mathcal{E}$. Consequently, with $\mathrm{d}_H(\bar{x}, \bar{y})$ denoting the Hamming distance in S' and d_{Hz} the Hurwitz distance in S, we then get that

$$\mathrm{d}_H(\bar{x}, \bar{y}) = \mathrm{d}_{Hz}(\varphi(\bar{x}), \varphi(\bar{y})).$$

In other words, the map φ is an isometry from S' to S. It follows that the code

$$C = \varphi(C')$$

is a perfect 1-error-correcting Hurwitz weight code in S.

6. Perfect 2-error-correcting codes in the Mannheim, Lipschitz and Hurwitz metric

We have not yet found any multiple-error-correcting codes in the metrics we are studying. Computer search has made it possible to exclude the existence of perfect 2-error-correcting codes for a large number of lengths.

While one error during transmission of a word occurs in exactly one coordinate position, two errors either occur in one coordinate position as a sequence of two single errors made in that coordinate position, or in two distinct coordinate positions. In the latter case, the number of possibilities to make two errors in a word of length n is equal to

$$\binom{n}{2}|\mathcal{E}_\pi|^2,$$

where $|\mathcal{E}_\pi|$ is equal to 4 in the Mannheim metric, 8 in the Lipschitz metric and 24 in the Hurwitz metric.

We need to identify the error vectors $\bar{e} = (0, \ldots, 0, \epsilon_i, 0, \ldots, 0)$, with all but one coordinate position equal to zero, and such that the distance between $\bar{e}$ and all zero word is equal to 2. If that is the case, then the non-zero element ϵ_i belongs to the following set:

$$\mathcal{E}_2 = (\mathcal{E}_\pi + \mathcal{E}_\pi) \setminus (\mathcal{E}_\pi \cup \{0\}), \tag{12}$$

where $\mathcal{E}$ is the set of units related to the metric under consideration. The size of $\mathcal{E}_2$ depends in general on π. For instance, in the case of the Hurwitz metric in $\mathcal{H}_\pi$, where $\pi = 2 + e_1 + e_2 + e_3$, we get that

$$\mathcal{E}_2 = \mathcal{E}_2(\pi) = \{\pm(1 \pm e_i) : i \in [3]\} \cup \{\pm(e_i \pm e_j) : i, j \in [3], i \neq j\},$$

and hence that $|\mathcal{E}_2(2 + e_1 + e_2 + e_3)| = 24$.

Let H_π denote either the sets $\mathbb{Z}[i]_\pi$, $H(\mathbb{Z})_\pi$ or $\mathcal{H}_\pi$. In general, the following well-known necessary condition for the existence of a perfect 2-error-correcting code in H_π^n is true:

Lemma 3. *If a perfect 2-error-correcting code C exists in the direct product H_π^n where the norm of π is equal to an odd prime number p, then*

$$1 + n|\mathcal{E}_\pi| + \binom{n}{2}|\mathcal{E}_\pi|^2 + n|\mathcal{E}_2(\pi)| = |H_\pi|^n p^{-k},$$

where k denotes the integer $k = \log_p(|C|)$.

Note that the size of H_π is equal to p if $H_\pi = \mathbb{Z}[i]_\pi$ and p^2 in the other cases.

The lemma above is called the *sphere packing condition*. It is the main tool used in our computer search that gives the non-existence results reported in the subsections that follow.

The number of known non-trivial multiple-error-correcting codes is very limited. The two Golay codes found in 1948 [4] may be the only ones. Thus it is of greatest interest to investigate the existence of multiple-error-correcting codes in these new metrics.¶

6.1. Perfect 2-error-correcting codes in the Mannheim metric

We assume that the element $\pi = a + bi$ of $\mathbb{Z}[i]$ has the norm p, where p is a prime number. In the ring $\mathbb{Z}[i]_\pi$ we get that

$$\mathcal{E}_2(\pi) = \{\pm 2, \pm 2i\} \cup \{\pm(1 \pm i)\}.$$

Thus, if a perfect 2-error-correcting Mannheim weight code C of length n exists, then, by Lemma 3

$$p^t = \begin{cases} 8n^2 - 4n + 1 \text{ if } p = 5, \\ 8n^2 + 4n + 1 \text{ if } p \geq 13, \end{cases} \tag{13}$$

where $t = n - k$.

Computer search for solutions of equation (13) in the case when p is a prime number and $(n+1, t) \in [9999] \times [26]$ gave just one solution:

$$(p, n, t) = (29, 10, 2).$$

Consequently, we cannot use the packing condition to exclude the existence of a perfect 2-error correcting code C in $\mathbb{Z}[i]_\pi^{10}$, if $\pi = \pm 2 \pm 5i$ or $\pi = \pm 5 \pm 2i$. The size of C would be $|C| = 29^8$.

6.2. Perfect 2-error correcting codes in the Lipschitz metric

Let π be any Lipschitz prime of norm p. In $H(\mathbb{Z})_\pi$, we get that $\mathcal{E}_2(\pi)$ is a union of the following four sets.

$$G_2 = \{\pm(1 \pm e_1), \pm(e_2 \pm e_3)\}, \; G_3 = \{\pm(1 \pm e_2), \pm(e_1 \pm e_3)\},$$
$$G_4 = \{\pm(1 \pm e_3), \pm(e_1 \pm e_2)\}, \; G_5 = \{\pm 2, \pm 2e_1, \pm 2e_2, \pm 2e_3\}.$$

Hence, by using the relation in (12), applying Lemma 3, and using the fact that in this case $|\mathcal{E}_\pi| = 8$, we can derive the following necessary conditions for the existence of a perfect 2-error-correcting Lipschitz weight (n, k)-code over $H(\mathbb{Z})_\pi$:

$$p^t = \begin{cases} 32n^2 - 8n + 1 \text{ if } p = 5, \\ 32n^2 + 1 \quad\quad\; \text{if } p = 7, 11, \\ 32n^2 + 8n + 1 \text{ if } p \geq 13, \end{cases} \tag{14}$$

¶Furthermore, as one of the reviewers pointed out to us, the existence of 2-perfect codes in the Mannheim and Lipschitz metrics implies the existence of tilings of $\mathbb{Z}^{2n}$ or $\mathbb{Z}^{4n}$ with radius-2 Lee balls, which are periodic, with period p or p^2 in all basic directions. That is, we will get 2-perfect Lee codes in $\mathbb{Z}_p^N$ or $\mathbb{Z}_{p^2}^N$. This is a partial case of the well-known Golomb-Welch conjecture that such codes do not exist [5].

where $t = 2(n-k)$. Computer search for solutions to the equations above, in the cases when p is a prime number, showed that the only solutions in the intervals $(n+1,t) \in [9999] \times [24]$ occur when the 3-tuple (p,n,t) is equal to one of the following two:

$$(29,5,2), \qquad (33461,5915,2).$$

The sizes of codes with these parameters would be 29^8 and 33461^{11828}, respectively.

6.3. Perfect 2-error-correcting codes in the Hurwitz metric

In this subsection, we presume that π is an odd Lipschitz prime integer with norm $N(\pi) = p > 3$. Let $e_0 = 1$. By applying the relation in (12), some tedious but trivial calculations modulo π in $\mathcal{H}_\pi$ give the following:

Proposition 4. *There are six possible sizes of the sets $\mathcal{E}_2(\pi)$, where π is a Lipschitz prime integer with $N(\pi) \geq 5$. These sizes are*

$$|\mathcal{E}_2(\pi)| = \begin{cases} 0 & \text{if } N(\pi) = 5, \\ 24 & \text{if } N(\pi) = 7, \\ 72 & \text{if } N(\pi) = 11, \\ 88 & \text{if } N(\pi) = 13 \text{ and } \pi \in \{\pm e_{i_0} + (\pm 2)e_{i_1} + (\pm 2)e_{i_2} + (\pm 2)e_{i_3}\}, \\ 112 & \text{if } N(\pi) = 13 \text{ and } \pi \in \{\pm 3e_{i_0} \pm 2e_{i_1}\}, \\ 112 & \text{if } N(\pi) > 13. \end{cases}$$

As $|\mathcal{E}_1| = 24$, from Lemma 3 we get that if a perfect 2-error-correcting Hurwitz weight (n,k)-code exists in $\mathcal{H}_\pi^n$, then the following equality must be satisfied:

$$p^t = \begin{cases} 288n^2 - 264n + 1 \text{ if } p = 5, \\ 288n^2 - 240n + 1 \text{ if } p = 7, \\ 288n^2 - 184n + 1 \text{ if } p = 11, \\ 288n^2 - 176n + 1 \text{ if } p = 13 \text{ with } \pi \in \{\pm e_{i_0} + (\pm 2)e_{i_1} + (\pm 2)e_{i_2} + (\pm 2)e_{i_3}\}, \\ 288n^2 - 152n + 1 \text{ if } p = 13 \text{ with } \pi \in \{\pm 3e_{i_0} \pm 2e_{i_1}\}, \\ 288n^2 - 152n + 1 \text{ if } p > 13, \end{cases}$$

where $t = 2(n-k)$.

Our computer search gave that if p is an odd prime number, then none of these equations has a solution for $(n+1,t) \in [9999] \times [24]$.

7. Comparison between codes over $\mathcal{H}$, codes over $\mathbb{Z}[\rho]$ and codes over $\mathbb{Z}[i]$

In this section, we compare codes over $\mathcal{H}$, codes over $\mathbb{Z}[\rho]$ defined in [14], and codes over $\mathbb{Z}[i]$ in terms of average energy, code rate and bandwidth occupancy, where $\rho = \frac{1+i\sqrt{3}}{2}$. We first compare the average energy of codes over $\mathcal{H}_\pi$ with the average energy of codes over $\mathbb{Z}[\rho]$. Let $\pi = 2 + e_1 + e_2 + e_3$ and $\alpha = 5 + 3\rho$. Then the elements of the set $H_{2+e_1+e_2+e_3} - \{0\}$ correspond to the vertices of the 48−cell polytope, see Figure 1A, and the elements of the set $\mathbb{Z}_\alpha[\rho]$ form a lattice, see Figure 1B. Considering constellations with the same cardinality, we show that the average

energy for the transmitted signal is smaller in the case of $\mathcal{H}$ than in the case of $\mathbb{Z}[\rho]$, see Table I.

Note that the average energy is calculated as:

$$E = \frac{1}{M} \sum_{s=0}^{M-1} |q_s|^2,$$

where q_s belongs to the signal space and has a magnitude (distance from the origin) of $|q_s| = \sqrt{q_{s,0}^2 + q_{s,1}^2 + q_{s,2}^2 + q_{s,3}^2}$, and where M denotes the cardinality of the signal set.

Alphabet	Base ring	Average energy
$GF(49)$	$\mathcal{H}$	1.47
$GF(49)$	$\mathbb{Z}[\rho]$	7.22

Table I: *Comparison between codes over $\mathcal{H}_\pi$ and $\mathbb{Z}_\alpha[\rho]$.*

We now compare the average energy of codes over $\mathcal{H}$ with the average energy of codes over $\mathbb{Z}[i]$. Let $\pi = 2 + e_1$ and $\alpha = 4 + 3i$. Then the elements of the set $H_{2+e_1} - \{0\}$ correspond to the vertices of the 24−cell polytope, see Figure 1C, and the elements of the set $\mathbb{Z}[i]_\alpha$ form a lattice, see Figure 1D. Considering constellations with the same cardinality, we show that the average energy for the transmitted signal is smaller in the case of $\mathcal{H}_\pi$ than in the case of $\mathbb{Z}[i]_\alpha$, see Table II.

Alphabet	Base ring	Average energy
$GF(25)$	$\mathcal{H}$	0.96
$GF(25)$	$\mathbb{Z}[i]$	4.16

Table II: *Comparison between codes over $\mathcal{H}_\pi$ and $\mathbb{Z}[i]_\alpha$.*

Then we compare the code rate with the bandwidth occupancy of the codes over $\mathcal{H}_\pi$ and $\mathbb{Z}_\alpha[\rho]$, $\mathbb{Z}[i]_\alpha$, when the alphabets considered have the same cardinality. The codes given in Section 5.3 and the OMEC codes presented in [7] can be generalized to the lengths $n = \frac{p^2-1}{24}$ and $n = \frac{p^2-1}{4}$, respectively. Let $p \equiv 1 \pmod{12}$. We now have $p \equiv 1 \pmod 6$ and $p \equiv 1 \pmod 4$. In this case, a code C_1 over $\mathcal{H}_\pi$ has the length $n_1 = \frac{p^2-1}{24}$, a code C_2 over $\mathbb{Z}_\alpha[\rho]$ has the length $n_2 = \frac{p^2-1}{6}$ and a code C_3 over $\mathbb{Z}[i]_\alpha$ has the length $n_3 = \frac{p^2-1}{4}$. Hence, if the dimensions k_1, k_2 and k_3 of the codes C_1,C_2 and C_3 equal k, then the rate R_1 of C_1 is greater than the rate R_2 of C_2 and the rate R_3 of C_3, since $R_1 = \frac{k_1}{n_1} = \frac{24k}{p^2-1}$, $R_2 = \frac{k_2}{n_2} = \frac{6k}{p^2-1}$ and $R_3 = \frac{k_3}{n_3} = \frac{4k}{p^2-1}$. For example, let $P = 13$ and $k = 1$. Then we get $R_1 = \frac{1}{7}$, $R_2 = \frac{1}{28}$ and $R_3 = \frac{1}{42}$. It is shown that the bandwidth occupancy of the code C_1 is smaller than the bandwidth occupancy of the code C_2 and the bandwidth occupancy of the code C_3.

8. Remarks

We have not yet found any general constructions of partitions of either $H(\mathbb{Z})_\pi$ or $\mathcal{H}_\pi \setminus \{0\}$ into cosets $a_i\mathcal{E}$ of the appropriate set of units $\mathcal{E}$. We have found examples

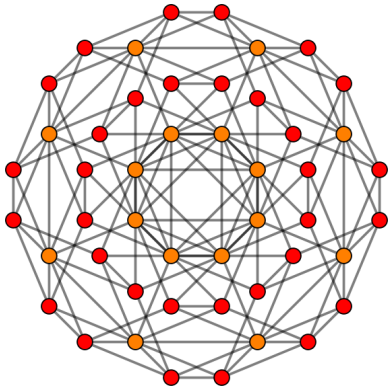

Figure 1A: *The set* $\mathcal{H}_{2+e_1+e_2+e_3} - \{0\}$ *is displayed as the vertices of the* $48-$ *cell polytope*

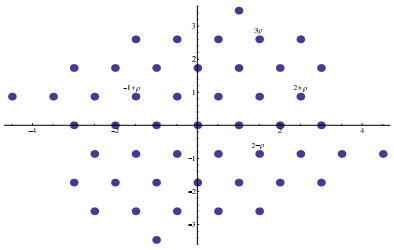

Figure 1B: *The set* $\mathbb{Z}_\alpha[\rho]$ *is displayed as the points in the complex plane, where* $\alpha = 5 + 3\rho$

that indicate that there actually exist such partitions whenever π is any Lipschitz prime. Nevertheless, we think that such partitions deserve a study of their own.

There are no perfect multiple-error-correcting codes in the Hamming metric over alphabets of a size equal to a power of a prime number, see [15] or [16]. Hence, from Section 5.4 it follows that there are no perfect multiple-error-correcting Hurwitz weight codes in $\mathcal{H}_\pi$ if $N(\pi) = 5$.

Just in some instances, perfect codes with given parameters (q, n, e), where q denotes the size of the alphabet, n the length and e the number of errors that can be corrected, are unique up to equivalence. As our construction is dependent on the selection of elements to be placed as entries in the parity-check matrices $\mathbf{H}$, we are convinced that there will exist some non-equivalent 1-error-correcting perfect codes in the metrics and spaces we have treated in the present study. By using the fact that the partition of $\mathbb{Z}[i]_\pi$ described in Theorem 5 is unique, it can be proved that all perfect 1-error-correcting Mannheim weight linear codes in $\mathbb{Z}[i]_\pi^n$, for given π and n, are unique up to equivalence. Hence, all such codes are now classified and characterized.

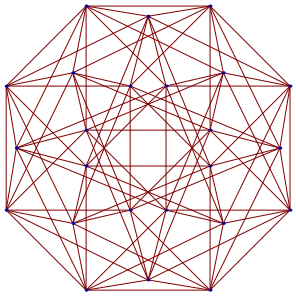

Figure 1C: *The set* $\mathcal{H}_{2+e_1} - \{0\}$ *is displayed as the vertices of the* 24*− cell polytope*

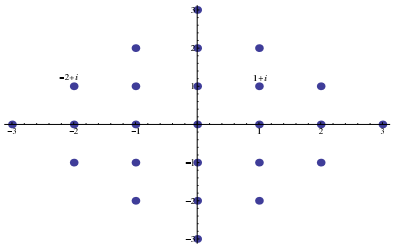

Figure 1D: *The set* $\mathbb{Z}_{4+i3}$ *is displayed as the points in the complex plane*

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