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Computing coarse shape groups of solenoids

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Abstract. The coarse shape groups are new topological invariants which are (coarse) shape and homotopy invariants as well. Their structure is significantly richer than the structure of shape groups. They provide information (especially, about compacta) even better than the homotopy *pro*-groups. Since nontrivial coarse shape groups, even for polyhedra, are too large, it is difficult to calculate them exactly. Herein, we give an explicit formula for computing coarse shape groups of a large class of metric compacta including solenoids.

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1. Introduction and preliminaries

The coarse shape theory (founded in [4]) functorially generalizes the shape theory. A category frame for this theory is the (pointed) coarse shape category Sh^* ($Sh^*_\star$), having (pointed) topological spaces as objects and having the (pointed) shape category Sh ($Sh_\star$) as a subcategory. There exist metrizable continua having the same coarse shape type and different shape types. The coarse shape preserves many important topological and shape invariants (see [2]) as connectedness, movability, strong movability, n-movability, shape dimension, triviality of shape, stability. There are also several important algebraic coarse shape invariants. In [3], the functors $\check{\pi}^*_n : Sh^*_\star \to Grp$, $n \in \mathbb{N}$ (Grp denotes the category of groups) are introduced. The functor $\check{\pi}^*_n$ assigns to every pointed space (X, x_0) the n-th coarse shape group $\check{\pi}^*_n(X, x_0)$ having the n-th shape group $\check{\pi}_n(X, x_0)$ as its subgroup. Therefore, the coarse shape groups provide information on pointed spaces better than the shape groups. For a pointed metric compact space (pointed compactum) (X, x_0), unlike shape groups, coarse shape groups fit into a long exact sequence (see [1]). Since we know that the homotopy theory is inadequate for spaces with bad local properties, the study of coarse shape groups of such spaces can be very useful, especially when the corresponding shape groups vanish. Comparing the coarse shape groups of metric compacta with corresponding homotopy pro-groups, one can notice that coarse shape groups have many advantages. Namely, for a pointed compactum (X, x_0), pro-$\pi_k(X, x_0) \simeq \mathbf{0}$ is equivalent to $\check{\pi}^*_k(X, x_0) = 0$, for every $k \in \mathbb{N}$, but on the other hand, pro-$\pi_k(X, x_0)$ does not have an algebraic structure of a group while $\check{\pi}^*_k(X, x_0)$

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does. However, the coarse shape groups are too "massive" and it is not easy to compute them for a concrete space (except a polyhedron). In the present paper, we give an explicit formula for computing coarse shape groups of a pointed compactum whose bonding homomorphisms of its homotopy pro-groups are monomorphisms. This class of metric compactum includes many interesting spaces such as solenoids which are specially considered in the last section of this paper.

Let us recall some basic notions about the coarse shape category (see [4]). An S^*-**morphism of inverse systems** $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $\boldsymbol{Y} = (Y_\mu, q_{\mu\mu'}, M)$ in some category $\mathcal{C}$, $(f, f_\mu^n) : \boldsymbol{X} \to \boldsymbol{Y}$, consists of an index function $f : M \to \Lambda$, and of a set of $\mathcal{C}$-morphisms $f_\mu^n : X_{f(\mu)} \to Y_\mu$, $n \in \mathbb{N}$, $\mu \in M$, such that, for every related pair $\mu \leq \mu'$ in M, there exists a $\lambda \in \Lambda$, $\lambda \geqslant f(\mu), f(\mu')$, and there exists an $n \in \mathbb{N}$ so that, for every $n' \geqslant n$, $f_\mu^{n'} p_{f(\mu)\lambda} = q_{\mu\mu'} f_{\mu'}^{n'} p_{f(\mu')\lambda}$. An S^*-morphism $(f, f_\mu^n) : \boldsymbol{X} \to \boldsymbol{Y}$ of inverse systems in $\mathcal{C}$ is said to be **equivalent** to an S^*-morphism $(f', f_\mu'^n) : \boldsymbol{X} \to \boldsymbol{Y}$ denoted by $(f, f_\mu^n) \sim (f', f_\mu'^n)$, provided every $\mu \in M$ admits a $\lambda \in \Lambda$, $\lambda \geqslant f(\mu), f'(\mu)$, and an $n \in \mathbb{N}$, such that, for every $n' \geqslant n$, $f_\mu^{n'} p_{f(\mu)\lambda} = f_\mu'^{n'} p_{f'(\mu)\lambda}$. The category *pro**-$\mathcal{C}$ has as objects all inverse systems $\boldsymbol{X}$ in $\mathcal{C}$ and as morphisms all equivalence classes $\boldsymbol{f}^* = [(f, f_\mu^n)]$ of S^*-morphisms (f, f_μ^n).

Lemma 1. *Let $\boldsymbol{f}^* : (X) \to (Y_i, q_{ii+1}, \mathbb{N})$ be a morphism of pro*-$\mathcal{C}$ where (X) denotes a rudimentary system $(X_1 = X, 1_X, \{1\})$ indexed by a singleton. Then $\boldsymbol{f}^*$ admits a representative $(f_i^n) : (X) \to (Y_i, q_{ii+1}, \mathbb{N})$ such that there exists a strictly increasing sequence $\mu = (m_i, i \in \mathbb{N})$ in $\mathbb{N}$ such that $m_1 = 1$ and, for every $i > 1$ and $n \geq m_i$, $f_j^n = p_{jj'} f_{j'}^n$ holds, for every $j < j' \leq i$. The sequence μ is called a* controlling sequence *of the S^*-morphism (f_i^n).*

Proof. Let (f_i^n) be any representative of $\boldsymbol{f}^*$. For every $j < j'$, let $n_{jj'}$ denote a positive integer such that $f_j^n = p_{jj'} f_{j'}^n$, for every $n \geq n_{jj'}$. We propose to define a sequence (m_i) by induction. Put $m_1 = 1$ and let $m_2 = n_{1\,2} + 1$. For every $i > 2$, we put

$$m_i = \max\{\max\{n_{jj'} \mid j < j' \leq i\}, m_{i-1}\} + 1.$$

It is easy to check that (m_i) is the desired sequence. □

Remark 1. *The essential idea of a controlling sequence of an S^*-morphism has already appeared in [6]. In that paper, an S^*-mapping between any pair of inverse sequences, is defined. For S^*-mappings a function called commutativity radius has a similar role as a controlling sequence for S^*-morphisms.*

Let us consider any pair of categories $(\mathcal{C}, \mathcal{D})$ where $\mathcal{D}$ is a full and pro-reflective (i.e., dense) subcategory of $\mathcal{C}$ (see [5], I.2.2). Let $\boldsymbol{p} : X \to \boldsymbol{X}$ and $\boldsymbol{q} : Y \to \boldsymbol{Y}$ be $\mathcal{D}$-expansions of objects X and Y of $\mathcal{C}$, respectively (see [5], I.2.1). A coarse shape morphism $F^* : X \to Y$ is an equivalence class $\langle \boldsymbol{f}^* \rangle$ represented by a morphism $\boldsymbol{f}^* : \boldsymbol{X} \to \boldsymbol{Y}$ of *pro**-$\mathcal{D}$ (see [4]). The (**abstract**) **coarse shape category** $Sh^*_{(\mathcal{C},\mathcal{D})}$ for $(\mathcal{C}, \mathcal{D})$ has as objects all the objects of $\mathcal{C}$, and its morphisms $F^* \in Sh^*_{(\mathcal{C},\mathcal{D})}(X, Y)$ are all equivalence classes $\langle \boldsymbol{f}^* \rangle$ of morphisms $\boldsymbol{f}^* : \boldsymbol{X} \to \boldsymbol{Y}$, with respect to any choice of a pair of $\mathcal{D}$-expansions $\boldsymbol{p} : X \to \boldsymbol{X}$, $\boldsymbol{q} : Y \to \boldsymbol{Y}$.

In this paper, $\mathcal{C}$ will be the pointed homotopy category $HTop_\star$. The restriction of the class of objects to the pointed polyhedra yields the full subcategory

$HPol_\star \subseteq HTop_\star$. Recall that objects of the category $HTop_\star$ are all pointed spaces (X, x_0) and the morphisms are all homotopy classes (briefly H-maps) $[f]$ of mappings of pointed spaces, $f : (X, x_0) \to (Y, y_0)$. In this paper, the homotopy class $[f]$ of a map f, i.e., a morphism of the category $HTop_\star$ or $HTop_{\star\star}$, will be usually denoted by omitting the brackets, whenever it cannot cause misunderstandings. It is a well-known fact that $HPol_\star$ is a pro-reflective subcategory of $HTop_\star$ (see [5]). This means that for every pointed space (X, x_0) there exists an $HPol_\star$-expansion of (X, x_0) which is a morphism $\boldsymbol{p} = [(p_\lambda)] : (X, x_0) \to (\boldsymbol{X}, \boldsymbol{x}_0) = ((X_\lambda, x_\lambda), p_{\lambda\lambda'}, \Lambda)$ of *pro-*$HTop_\star$, where $(\boldsymbol{X}, \boldsymbol{x}_0)$ is an inverse system of pointed polyhedra. The construction of the **pointed coarse shape category** $Sh^*_\star$ follows now the general rule, i.e., it is the category $Sh^*_{(HTop_\star, HPol_\star)}$. Briefly, the objects of $Sh^*_\star$ are all pointed topological spaces (X, x_0), while a morphism set $Sh^*_\star((X, x_0), (Y, y_0))$ consists of all equivalence classes $F^* = \langle \boldsymbol{f}^* \rangle$ of morphisms $\boldsymbol{f}^* = \left[(f, f^n_\mu)\right] : (\boldsymbol{X}, \boldsymbol{x}_0) \to (\boldsymbol{Y}, \boldsymbol{y}_0)$ of *pro**-$HPol_\star$ ranging over the corresponding expansions. A morphism $\boldsymbol{f}^*$ is represented by an S^*-morphism $(f, f^n_\mu) : ((X_\lambda, x_\lambda), p_{\lambda\lambda'}, \Lambda) \to ((Y_\mu, y_\mu), q_{\mu\mu'}, M)$, where $f^n_\mu : \left(X_{f(\mu)}, x_{f(\mu)}\right) \to (Y_\mu, y_\mu)$ is a morphism of $HPol_\star$, for every $\mu \in M$, $n \in \mathbb{N}$.

Recall that for every pointed space (X, x_0) and for every $k \in \mathbb{N}_0$, the elements of the k-dimensional homotopy group $\pi_k(X, x_0)$ can be regarded as homotopy classes of maps $\left(S^k, s_0\right) \to (X, x_0)$, where S^k denotes the standard k-dimensional sphere. For every $k \in \mathbb{N}$, the functors $\check{\pi}^*_k : Sh^*_\star \to Grp$ and $\check{\pi}^*_0 : Sh^*_\star \to Set_\star$ ($Set_\star$ denotes the category of pointed sets and base point preserving functions) associate with every pointed space (X, x_0) the group $\check{\pi}^*_k(X, x_0)$ (for $k = 0$ pointed set) called the k-th **coarse shape group**. Its underlying set is $Sh^*_\star\left(\left(S^k, s_0\right), (X, x_0)\right)$, i.e., the elements of $\check{\pi}^*_k(X, x_0)$ are all pointed coarse shape morphisms $A^* : \left(S^k, s_0\right) \to (X, x_0)$. Notice that a representative $[(a^n_\lambda)] : \left(S^k, s_0\right) \to ((X_\lambda, x_\lambda), p_{\lambda\lambda'}, \Lambda)$ of A^* consists of $a^n_\lambda \in \pi_k(X_\lambda, x_\lambda)$, for all $\lambda \in \Lambda$, $n \in \mathbb{N}$. The functor $\check{\pi}^*_k$ associates with every coarse shape morphism $F^* : (X, x_0) \to (Y, y_0)$ a homomorphism (for $k = 0$, a base point preserving function) $\check{\pi}^*_k(F^*) : \check{\pi}^*_k(X, x_0) \to \check{\pi}^*_k(Y, y_0)$ given by the following formula:

$$\check{\pi}^*_k(F^*)(A^*) = F^* \circ A^*, \quad A^* \in \check{\pi}^*_k(X, x_0).$$

2. The main result

Let us consider a pointed metric compactum (X, x_0) and its $HPol_\star$-expansion

$$\boldsymbol{p} : (X, x_0) \to (\boldsymbol{X}, \boldsymbol{x}_0) = ((X_i, x_i), p_{ii+1}, \mathbb{N}).$$

Let $k \in \mathbb{N}$. Suppose that the homomorphism $\pi_k(p_{ii+1}) : \pi_k(X_{i+1}, x_{i+1}) \to \pi_k(X_i, x_i)$ is a monomorphism, for every $i \in \mathbb{N}$. Let us denote the group $\pi_k(X_1, x_1)$ by G^k_1 and let G^k_i denote its subgroup $\pi_k(p_{1i})(\pi_k(X_i, x_i))$, for $i > 1$. Let $\mathbb{N}^{\mathbb{N}}$ be the set of all sequences in $\mathbb{N}$ and let M denote its subset consisting of all strictly increasing sequences $\mu : \mathbb{N} \to \mathbb{N}$ such that $\mu(1) = 1$.

For every $\mu = (m_i) \in M$, consider the direct product

$$P^k(\mu) = \underbrace{G^k_1 \times \cdots \times G^k_1}_{m_2 - 1} \times \underbrace{G^k_2 \times \cdots \times G^k_2}_{m_3 - m_2} \times \underbrace{G^k_3 \times \cdots \times G^k_3}_{m_4 - m_3} \times \cdots$$

which is the subgroup of $\prod_{n\in\mathbb{N}} G_1^k = G_1^k \times G_1^k \times \cdots$.

For every $\mu = (m_i) \in M$, consider the direct sum (or an external weak direct product for $k = 1$, if G_1^1 is not abelian)

$$Q^k(\mu) = \underbrace{G_1^k \oplus \cdots \oplus G_1^k}_{m_2-1} \oplus \underbrace{G_2^k \oplus \cdots \oplus G_2^k}_{m_3-m_2} \oplus \underbrace{G_3^k \oplus \cdots \oplus G_3^k}_{m_4-m_3} \oplus \cdots ,$$

which is a subgroup of $\bigoplus_{n\in\mathbb{N}} G_1^k = G_1^k \oplus G_1^k \oplus \cdots$.

We define the union of subgroups $P^k(\mu)$ of $\prod_{n\in\mathbb{N}} G_1^k$, $\mu \in M$, by letting

$$P^k = \bigcup_{\mu\in M} P^k(\mu) .$$

Let us prove that P^k is the subgroup of $\prod_{n\in\mathbb{N}} G_1^k$. It is sufficient to prove that for every $\mu^1, \mu^2 \in M$ there exists $\mu^*_{(\mu^1,\mu^2)} \in M$ such that $P^k(\mu^1), P^k(\mu^2) \subseteq P^k(\mu^*)$. In order to find an algorithm for determining such sequence μ^* let us consider the direct product

$$\overline{P} = \prod_{n\in\mathbb{N}} \overline{P}_n$$

of the groups

$$\overline{P}_n = \begin{cases} p_n\left(P^k\left(\mu^1\right)\right); & \text{for } p_n\left(P^k\left(\mu^1\right)\right) \supseteq p_n\left(P^k\left(\mu^2\right)\right) \\ p_n\left(P^k\left(\mu^2\right)\right); & \text{for } p_n\left(P^k\left(\mu^2\right)\right) \supseteq p_n\left(P^k\left(\mu^1\right)\right) \end{cases} ,$$

where p_n denotes a natural projection, for every $n \in \mathbb{N}$. Since, for every $n \in \mathbb{N}$, $\overline{P}_n$ is one of the subgroups of the decreasing sequence $\left(G_i^k\right)$, $G_1^k \supseteq G_2^k \supseteq \cdots$, $\overline{P}_n$ is well defined. Consequently, $\overline{P}$ is the group, which contains $P^k(\mu^1)$ and $P^k(\mu^2)$, and for which there exits the sequence (d_i) in $\mathbb{N}$ such that

$$\overline{P} = \underbrace{G_1^k \times \cdots \times G_1^k}_{d_1} \times \underbrace{G_2^k \times \cdots \times G_2^k}_{d_2} \times \underbrace{G_3^k \times \cdots \times G_3^k}_{d_3} \times \cdots$$

Now, for the sequence $\mu^* = (1, d_1 + 1, d_2 + d_1 + 1, \ldots) \in M$, it is obvious that $\overline{P} = P^k(\mu^*)$. In fact, it can be proven that $\mu^* = \max\left\{\mu^1, \mu^2\right\}$.

Similarly, we define the union of subgroups $Q^k(\mu)$, $\mu \in M$, by putting

$$Q^k = \bigcup_{\mu\in M} Q^k(\mu) .$$

As in the previous case, one can prove that this union makes a group, i.e., the subgroup of $\bigoplus_{n\in\mathbb{N}} G_1^k$.

We propose to prove that, for every $k \in \mathbb{N}$,

$$\check{\pi}_k^*(X, x_0) \cong P^k/Q^k.$$

First, we define a function

$$\Phi : \check{\pi}_k^* (X, x_0) \to P^k / Q^k$$

as follows. Notice that we may fix an arbitrary $HPol_\star$-expansion $(\boldsymbol{X}, \boldsymbol{x}_0)$ of (X, x_0) with the above mentioned property. Namely, by the construction of the functor $\check{\pi}_k^*$ we may identify the group $\check{\pi}_k^* (X, x_0)$ with the group

$$\left\{ \boldsymbol{a}^* \mid \boldsymbol{a}^* \in pro^*\text{-}HPol_\star \left(\left(S^k, s_0 \right), (\boldsymbol{X}, \boldsymbol{x}_0) \right) \right\},$$

which is supplied with the standard additive operation defined in [3]. Let

$$A^* \in \check{\pi}_k^* (X, x_0) = Sh^* \left(\left(S^k, s_0 \right), (X, x_0) \right)$$

be a coarse shape morphism represented by an $\boldsymbol{a}^* : \left(S^k, s_0 \right) \to (\boldsymbol{X}, \boldsymbol{x}_0)$. Let (a_i^n) be an S^*-morphism as in Lemma 1 representing $\boldsymbol{a}^*$ and having $\mu = (m_i)$ for its controlling sequence. We define $g = (g_n) \in P^k (\mu)$, for $n \in \mathbb{N}$, by putting

$$g_n = p_{1i} a_i^n : \left(S^k, s_0 \right) \to p_{1i} \left((X_i, x_i) \right) \subseteq (X_1, x_1),$$

$g_n \in G_i^k$, where i is the unique integer such that $m_i \le n < m_{i+1}$.

In order to complete the definition of the function Φ by letting $\Phi (A^*) = [g]$ (where $[g]$ is the class of $g \in P^k$ in P^k/Q^k) it remains to verify that neither $\Phi (\langle \boldsymbol{a}^* \rangle)$ depends on the choice of representative of the morphism $\boldsymbol{a}^*$ nor it depends on its controlling sequence. Let $(a_i'^n)$ be another representative of $\boldsymbol{a}^*$ having a controlling sequence $\mu' = (m_i')$ and let $g' = (g_n') \in P^k (\mu')$,

$$g_n' = p_{1i'} a_{i'}'^n : \left(S^k, s_0 \right) \to p_{1i'} \left((X_{i'}, x_{i'}) \right) \subseteq (X_1, x_1),$$

$g_n' \in G_{i'}^k$, $n \in \mathbb{N}$, where i' is the unique integer such that $m_{i'}' \le n < m_{i'+1}'$. By using the respective properties of the controlling sequences μ and μ' one can easily check that

$$g_n = a_1^n \text{ and } g_n' = a_1'^n, \text{ for every } n \in \mathbb{N}.$$

Indeed, by the definition of $g = (g_n) \in P^k (\mu)$ and $g' = (g_n') \in P^k (\mu')$, for $n \in \mathbb{N}$, it holds $g_n = p_{1i} a_i^n$ and $g_n' = p_{1i'} a_{i'}'^n$, where i and i' are the unique integers such that $m_i \le n < m_{i+1}$ and $m_{i'}' \le n < m_{i'+1}'$, respectively. Now by Lemma 1 it follows that $a_1^n = p_{1i} a_i^n$ and $a_1'^n = p_{1i'} a_{i'}'^n$. Since $(a_i^n) \sim (a_i'^n)$ implies that there exists an $n_0 \in \mathbb{N}$ such that, for every $n' \ge n_0$, $a_1^{n'} = a_1'^{n'}$, one infers that $g_{n'} - g_{n'}' = 0$, for every $n' \ge n_0$. It follows

$$g - g' \in Q^k (\mu^*) \subseteq Q^k,$$

where the sequence $\mu^*(\mu, \mu')$ is obtained applying the previously introduced algorithm. This implies that

$$\Phi \left(\langle [(a_i^n)] \rangle \right) = [g] = [g'] = \Phi \left(\langle [(a_i'^n)] \rangle \right),$$

which means that Φ is well defined.

In order to prove that Φ is a homomorphism, suppose that A^* and $B^* \in \check{\pi}_k^*(X,x_0)$ are represented by $\boldsymbol{a}^* = [(a_i^n)]$ and $\boldsymbol{b}^* = [(b_i^n)] : (S^k, s_0) \to (\boldsymbol{X}, \boldsymbol{x}_0)$, respectively, where (a_i^n) and (b_i^n) are S^*-morphisms having controlling sequences $\mu^1 = (m_i^1)$ and $\mu^2 = (m_i^2)$, respectively. By the definition of the function Φ it follows that $\Phi(\langle \boldsymbol{a}^* \rangle)$ is represented by

$$g^1 = (g_n^1) \in P^k(\mu^1), \; g_n^1 = a_1^n = p_{1i'} a_{i'}^n : (S^k, s_0) \to G_{i'}^k, \; n \in \mathbb{N},$$

where i' is the unique integer such that $m_{i'}^1 \le n < m_{i'+1}^1$, and $\Phi(\langle \boldsymbol{b}^* \rangle)$ is represented by

$$g^2 = (g_n^2) \in P^k(\mu^2), \; g_n^2 = b_1^n = p_{1i''} b_{i''}^n : (S^k, s_0) \to G_{i''}^k, \; n \in \mathbb{N},$$

where i'' is the unique integer such that $m_{i''}^2 \le n < m_{i''+1}^2$. Thus, $[g^1] = \Phi(A^*)$ and $[g^2] = \Phi(B^*)$. Notice that $A^* + B^*$ is represented by $[(a_i^n + b_i^n)]$ (see [3]), where the sum $a_i^n + b_i^n$ denotes the H-map (the homotopy class of mapping) which is the sum in the group $\pi_k(X_i, x_i)$. It can be readily seen that an S^*-morphism $(a_i^n + b_i^n)$ has the sequence $\mu = (m_i) = \max\{\mu^1, \mu^2\}$ for its controlling sequence. Therefore $\Phi(A^* + B^*)$ is represented by

$$g = (g_n) \in P^k(\mu), \; g_n = p_{1i}(a_i^n + b_i^n) : (S^k, s_0) \to G_i^k,$$

where i is the unique integer such that $m_i \le n < m_{i+1}$. From the respective properties of the homomorphisms $\pi_k(p_{1i}) : \pi_k(X_i, x_i) \to G_i^k$, $i \in \mathbb{N}$, and the controlling sequence μ it follows

$$g_n = a_1^n + b_1^n = g_n^1 + g_n^2, \text{ for every } n \in \mathbb{N}.$$

Hence,

$$\Phi(A^* + B^*) = [g_n] = [g_n^1] + [g_n^2] = \Phi(A^*) + \Phi(B^*).$$

Next we prove that Φ is an epimorphism. Assume that $g = (g_n) \in P^k(\mu)$ is an arbitrary element of P^k. For every $i \in \mathbb{N}$, put

$$a_i^n = [\pi_k(p_{1i})]^{-1}(g_n) : (S^k, s_0) \to (X_i, x_i), \text{ for } n \ge m_i,$$

and $a_i^n = 0$ (0 is the trivial H-map), otherwise. Notice that, for $n \ge m_i$, a_i^n is defined as the unique element of $\pi_k(X_i, x_i)$ such that $g_n = p_{1i} a_i^n$. Since $\pi_k(p_{1i}) : \pi_k(X_i, x_i) \to G_i$ is an isomorphism and $g_n \in G_i$, for $n \ge m_i$, a_i^n is well defined. One can verify in a straightforward manner that

$$(a_i^n) : (S^k, s_0) \to (\boldsymbol{X}, \boldsymbol{x}_0)$$

is an S^*-morphism having μ for its controlling sequence. Thus the following equality holds

$$\Phi(\langle [(a_i^n)] \rangle) = [g].$$

Let us prove that Φ is a monomorphism. Let $A^*, B^* \in \check{\pi}_k^*(X, x_0)$ be represented by $\boldsymbol{a}^* = [(a_i^n)]$ and $\boldsymbol{b}^* = [(b_i^n)] : (S^k, s_0) \to (\boldsymbol{X}, \boldsymbol{x}_0)$, respectively, where (a_i^n) and

(b_i^n) are S^*-morphisms having controlling sequences $\mu^1 = (m_i^1)$ and $\mu^2 = (m_i^2)$, respectively. By the definition of the function Φ, it follows that $\Phi(\langle \boldsymbol{a}^* \rangle)$ is represented by

$$g^1 = (g_n^1) \in P^k(\mu^1), \; g_n^1 = p_{1i} a_i^n : (S^k, s_0) \to G_i^k,$$

where i is the unique integer such that $m_i^1 \le n < m_{i+1}^1$, and $\Phi(\langle \boldsymbol{b}^* \rangle)$ is represented by

$$g^2 = (g_n^2) \in P^k(\mu^2), \; g_n^2 = p_{1i'} b_{i'}^n : (S^k, s_0) \to G_{i'}^k,$$

where i' is the unique integer such that $m_{i'}^2 \le n < m_{i'+1}^2$. Thus, $[g^1] = \Phi(A^*)$ and $[g^2] = \Phi(B^*)$. Suppose that $\Phi(A^*) = \Phi(B^*)$. Hence,

$$g^1 - g^2 \in Q^k(\mu^*) \subseteq Q^k,$$

where $\mu^* = (m_i) = \max\{\mu^1, \mu^2\}$. Then there exists an $n \in \mathbb{N}$ such that

$$g_{n'}^1 - g_{n'}^2 = 0, \text{ for every } n' \ge n.$$

Therefore, for every $i \in \mathbb{N}$, for every $n' \ge \max\{n, m_i\}$, we have

$$0 = g_{n'}^1 - g_{n'}^2 = p_{1i} a_i^{n'} - p_{1i} b_i^{n'}.$$

Using respective properties of the monomorphism $\pi_k(p_{1i}) : \pi_k(X_i, x_i) \to G_i$ we infer that

$$0 = p_{1i}\left(a_i^{n'} - b_i^{n'}\right)$$

and finally $a_i^{n'} = b_i^{n'}$. Hence, $(a_i^n) \sim (b_i^n)$ and consequently we conclude

$$A^* = [(a_i^n)] = [(b_i^n)] = B^*.$$

This shows that Φ is a group isomorphism. Hereby we have proved the following theorem:

Theorem 1. *Let $k \in \mathbb{N}$ and let (X, x_0) be a pointed space admitting a sequential $HPol_\star$-expansion $\boldsymbol{p} : (X, x_0) \to (\boldsymbol{X}, \boldsymbol{x}_0) = ((X_i, x_i), p_{ii+1}, \mathbb{N})$ such that all bonding homomorphisms $\pi_k(p_{ii+1})$, $i \in \mathbb{N}$, of its homotopy pro-group pro-$\pi_k(\boldsymbol{X}, \boldsymbol{x}_0)$ are monomorphisms. Then*

$$\check{\pi}_k^*(X, x_0) \cong \bigcup_{\mu \in M} P^k(\mu) / \bigcup_{\mu \in M} Q^k(\mu).$$

Remark 2. *An analogous formula for $\check{\pi}_0^*(X, x_0)$ holds as well (without a group structure). Namely, one can easily check that $\check{\pi}_0^*(X, x_0) \cong \bigcup_{\mu \in M} P^k(\mu) / (\sim)$ holds, where $\sim$ denotes the equivalence relation "to be equal at all but finitely many coordinates" on the direct product of pointed sets $\prod_{n \in \mathbb{N}} G_1^k$ given by the rule: $(g_1, g_2, ...) \sim (g_1', g_2', ...)$ provided there exists an $n_0 \in \mathbb{N}$ such that $g_n = g_n'$, for every $n \geqslant n_0$.*

3. An application

An immediate consequence of Theorem 1 and Remark 2 is the following corollary which gives us a useful formula for computing the coarse shape groups of some particular pointed spaces in any dimension.

Corollary 1. *Let $k \in \mathbb{N}_0$. If a pointed space (X, x_0) satisfies the assumptions of the previous theorem, then*

$$\check{\pi}_k^* (X, x_0) \cong \left(\bigcup_{(j_i) \in \mathbb{N}^{\mathbb{N}}} \prod\nolimits_{i \in \mathbb{N}} \left(G_i^k\right)^{j_i} \right) / \sim,$$

where $\sim$ is the equivalence relation "to be equal at all but finitely many coordinates".

Proof. The corollary statement is an immediate consequence of Theorem 1 and Remark 2. It is sufficient to notice that, for every sequence $(j_i) \in \mathbb{N}^{\mathbb{N}}$, there exists a sequence

$$\mu = (1, j_1 + 1, j_2 + j_1 + 1, j_3 + j_2 + j_1 + 1, \ldots) \in M$$

such that

$$P(\mu) = \prod\nolimits_{i \in \mathbb{N}} \left(G_i^k\right)^{j_i},$$

and vice versa, for every $\mu \in M$, there exists a sequence $(j_i) \in \mathbb{N}^{\mathbb{N}}$ such that the above equality holds. □

Consider a pointed solenoid

$$\left(\Sigma_{(p_i)}, x\right) = \lim \left(\left(X_i, x_i\right), p_{ii+1}\right),$$

where $X_i = S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$, $x_i = 1$, (p_i) is a sequence in $\mathbb{N}$ and $p_{ii+1}(z) = z^{p_i}$, for every $i \in \mathbb{N}$. Notice that every bonding homomorphism

$$\pi_1 (p_{ii+1}) : \mathbb{Z} \to \mathbb{Z}$$

of *pro*-$\pi_1 \left(\Sigma_{(p_i)}, x\right)$ is the multiplication by p_i, for every $i \in \mathbb{N}$, and consequently it is a monomorphism. It follows that $\pi_1(p_{1i}) : \mathbb{Z} \to \mathbb{Z}$ is an isomorphism onto the image

$$G_i = \pi_1 (p_{1i}) (\pi_1 (X_i, x_i)) = p_1 \cdot \cdots \cdot p_{i-1} \cdot \mathbb{Z},$$

which is the subgroup of $\mathbb{Z}$ consisting of all multiples of $p_1 \cdot \cdots \cdot p_{i-1}$. Now, by Corollary 1, the coarse shape group $\check{\pi}_1^* \left(\Sigma_{(p_i)}, x\right)$ is equal, up to an isomorphism, to the group

$$\left(\bigcup_{(j_i) \in \mathbb{N}^{\mathbb{N}}} \prod\nolimits_{i \in \mathbb{N}} \left(p_1 \cdot \cdots \cdot p_{i-1} \cdot \mathbb{Z}\right)^{j_i} \right) / \sim,$$

where $\sim$ is the equivalence relation "to be equal at all but finitely many coordinates". Its elements are all equivalence classes represented by

$$g \in \mathbb{Z}^{j_1} \times (p_1\mathbb{Z})^{j_2} \times (p_1 p_2 \mathbb{Z})^{j_3} \times \cdots = \prod_{i \in \mathbb{N}} \left(p_1 \cdot \cdots \cdot p_{i-1} \cdot \mathbb{Z}\right)^{j_i}$$

for some sequence (j_i) in $\mathbb{N}$, and two such elements g and g' represent the same class if they coincide at all but finitely many coordinates. The same holds for any other base point $x \in \Sigma_{(p_i)}$ as well.

By applying the previous consideration to the pointed dyadic solenoid $(\Sigma_{\mathbf{2}}, x)$ $(p_i = 2,\ i \in \mathbb{N})$, one may conclude that the elements of $\check{\pi}_1^* (\Sigma_{\mathbf{2}}, x)$ are all sequences

$$\left(2k_1^1, ..., 2k_{j_1}^1, 4k_1^2, ..., 4k_{j_2}^2, 8k_1^3, ..., 8k_{j_3}^3, ...\right)$$

of multiples of 2, 4, 8, ...,2^i,..., in the row (where $j_i \in \mathbb{N}$, $k_l^i \in \mathbb{Z}$, $i \in \mathbb{N}$, $l = 1, ..., j_i$) and two such sequences are considered equal provided they are equal at all but finitely many coordinates.

Recall that, for every pointed solenoid $\left(\Sigma_{(p_i)}, x\right) = \lim \left(\left(X_i, x_i\right), p_{ii+1}\right)$, the corresponding 1-dimensional shape group is

$$\check{\pi}_1 \left(\Sigma_{(p_i)}, x\right) = \lim \left(\pi_1 \left(X_i, x_i\right), \pi_1(p_{ii+1})\right) = \lim_{\rightarrow} \left(\mathbb{Z} \xleftarrow{p_1} \mathbb{Z} \xleftarrow{p_2} \mathbb{Z} \cdots \right) = 0.$$

Since $\check{\pi}_1 \left(\Sigma_{(p_i)}, x\right)$ is trivial, we have lost all information about solenoid in the inverse limit process. Therefore, 1-dimensional coarse shape groups of solenoids algebraically represent their rather complicated structure much better than shape groups.

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