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INTERPRETABILITY LOGIC IL DOES NOT HAVE FINITE SUBTREE PROPERTY

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Abstract. Usually, when a logic has finite model property (fmp), it also has a stronger, finite *sub*model property: every model can be reduced to a finite submodel. Or, at least, it has a finite *subtree* property, which is restricted to models that are trees. We prove that interpretability logic **IL** does not have finite subtree property.

1. Introduction

Interpretability logic is an extension of provability logic **GL**, and it is a modal description of the relative interpretability. The paper [7] provides the necessary definitions and detailed explanation and gives several examples of interpretations. We are only interested in interpretability logic as a system of modal logic. We introduce our notation and some basic facts, following [7].

The system **GL** is a modal propositional logic. The axioms of system **GL** are all tautologies, $\Box(A \to B) \to (\Box A \to \Box B)$, and $\Box(\Box A \to A) \to \Box A$. The inference rules of **GL** are modus ponens and necessitation $A/\Box A$.

The language of the interpretability logic contains the propositional letters p_0, p_1, ..., the logical connectives $\neg$, $\wedge$, $\vee$, $\to$ $\leftrightarrow$, the unary modal operators $\Box$ and $\Diamond$, and the binary modal operator $\rhd$. We use $\bot$ for false and $\top$ for true. We read $\rhd$ as binding stronger than binary boolean connectives, and weaker than negation and unary modal operators. The interpretability logic **IL** contains all axioms of the system **GL** and the following axioms: $\Box(A \to B) \to A \rhd B$, $(A \rhd B \wedge B \rhd C) \to A \rhd C$, $(A \rhd C \wedge B \rhd C) \to (A \vee B) \rhd C$, $A \rhd B \to (\Diamond A \to \Diamond B)$, and $\Diamond A \rhd A$. The deduction rules of **IL** are modus ponens and necessitation.

By adding the scheme $A \rhd B \to (A \wedge \Box C) \rhd (B \wedge \Box C)$ (Montagna's principle) one gets the system **ILM**. The system **ILP** is given by **IL** plus the scheme $A \rhd B \to \Box(A \rhd B)$ (principle of persistence). Further, **ILW**=**IL**+**W**, where **W** is the axiom scheme $(A \rhd B) \to (A \rhd (B \wedge \Box(\neg A)))$. Finally, the

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system $\mathbf{ILM_0}$ is given by **IL** plus the scheme $(A \rhd B) \to ((\Diamond A \wedge \Box C) \rhd (B \wedge \Box C))$.

The basic semantics for interpretability logic are Veltman models. An ordered triple $\langle W, R, (S_w : w \in W)\rangle$ is called a Veltman frame if it satisfies the following conditions: $\langle W, R\rangle$ is a **GL**–frame, i.e. W is a non–empty set, and R is a transitive and reverse well–founded relation on W; for every $w \in W$ we have $S_w \subseteq W[w] \times W[w]$, where $W[w] = \{u : wRu\}$; the relation S_w is reflexive and transitive on $W[w]$ for every $w \in W$; $wRuRv$ implies uS_wv.

An ordered quadruple $\langle W, R, (S_w : w \in W), \Vdash\rangle$ is called a Veltman model if it satisfies the following conditions: $\langle W, R, \{S_w : w \in W\}\rangle$ is a Veltman frame and $\Vdash$ is a forcing relation. We emphasize only the definition

$$w \Vdash A \rhd B \text{ if and only if } \forall u((wRu \;\&\; u \Vdash A) \Rightarrow \exists v(uS_wv \;\&\; v \Vdash B)).$$

A Veltman frame $\langle W, R, (S_w : w \in W)\rangle$ is called: an **ILM**–frame if uS_wvRz then uRz; an **ILP**–frame if uS_wv, then $uS_{w'}v$ for any w' such that wRw', $w'Ru$.; an **ILW**–frame if the converse of $R \circ S_w$ is well–founded; an $\mathbf{ILM_0}$–frame if $wRxRyS_wy'Rz$ then xRz.

We have the following completeness results: **IL** is sound and complete w.r.t. (finite) Veltman frames, **ILP** is complete w.r.t. (finite) **ILP**–frames (all in [3]), **ILW** is complete w.r.t. (finite) **ILW**–frames ([4], see also [5]), **ILM** is complete w.r.t. (finite) **ILM**–frames (in [3], also in [1]). The system $\mathbf{ILM_0}$ is complete w.r.t. $\mathbf{ILM_0}$–frames (in [5]).

2. FMP AND RELATED PROPERTIES

We say that a logic Λ has finite model property (fmp) w.r.t. some class of models $\mathcal{M}$ if $\mathcal{M} \Vdash \Lambda$ and for each formula $F \notin \Lambda$ there exists a finite model $\mathfrak{M} \in \mathcal{M}$ such that $\mathfrak{M} \nVdash F$. A logic Λ has fmp if Λ has fmp w.r.t. a class of models.

So, the systems **IL**, **ILM**, **ILP** and **ILW** have the fmp (and all the systems are decidable). A. Visser proved in [7] that interpretability logics **IL** and **ILM** do not have the fmp w.r.t. Visser models. Decidability and the fmp are two related issues that more or less seem to divide the landscape of interpretability logics into the same classes. The proof that **IL** has the fmp is relatively easy. The same can be said about **ILM**. For logics like $\mathbf{ILM_0}$ the issue seems much more involved and a proper proof of the finite model property, if one exists at all, has not been given yet.

Here we consider validity on trees. In the case of provability logic validity on trees is equivalent to validity on **GL**–frames (see [6]). In the case of interpretability logic this is not generally the case. D. de Jongh and F. Veltman in [3] proved that the formula $F \equiv \Box(p \to \neg q \wedge \Box\neg q) \wedge (p \rhd q) \to (p \rhd q \wedge \Box\bot)$ is valid on all **ILM**–models on trees, but $K_{\mathbf{ILM}} \not\models F$, where $K_{\mathbf{ILM}}$ is the class of **ILM**–frames. For **IL**, **ILW** and **ILP**, on the other hand, one can restrict oneself to tree models.

We say that a logic Λ has *finite submodel property* w.r.t. a class of models $\mathcal{M}$ if $\mathcal{M} \Vdash \Lambda$ and for every model $\mathfrak{M} \in \mathcal{M}$ and formula F, if $\mathfrak{M} \nVdash F$, then there is a finite submodel $\mathfrak{N}$ of $\mathfrak{M}$ such that $\mathfrak{N} \nVdash F$. *Finite (sub)tree property* is finite (sub)model property restricted to trees.

It is easy to see that for the logic **IL** the finite subtree property implies the fmp. Also, the ordinary "unraveling" technique (cloning every world as many times as there are paths to it from the root), which works fine on Veltman models, shows that the fmp implies finite tree property. However, as the next proposition shows, it doesn't imply the finite subtree property.

3. Main result

Proposition 3.1. *The interpretability logic* **IL** *does not have finite subtree property with respect to Veltman models.*

Proof. Consider a Veltman frame $\mathfrak{M} := \langle \mathbb{N}, R, (S_n)_{n\in\mathbb{N}} \rangle$, where ($\mathbb{N}'$ is a shortcut for $\mathbb{N} \setminus \{0\}$):

$$R := \{0\} \times \mathbb{N}' \cup \{\langle 2k-1, 2k \rangle \,:\, k \in \mathbb{N}'\}$$

$$S_0 := \{\langle x, y \rangle \in \mathbb{N}' \times \mathbb{N}' \,:\, x \leq y\}$$

$$S_{2k} := \emptyset \quad \text{and} \quad S_{2k-1} := \{\langle 2k, 2k \rangle\} \qquad \text{for } k \in \mathbb{N}'.$$

It is easily seen that $\mathfrak{M}$ really is a Veltman frame: $0 \; R \; 2k-1 \; R \; 2k$ implies $0 \; R \; 2k$ and $2k-1 \; S_0 \; 2k$; maximal length of an R–chain is 2; on respective successor sets, S_0 is a total order $\leq$, and other S–relations are universal.

It is also easily seen that $\mathfrak{M}$ is a tree: the unique maximal path from 0 to $2k$ is $0 \; R \; 2k-1 \; R \; 2k$, and other pairs of worlds are either unconnected, or connected only directly.

Consider a closed formula $\phi := (\Box\bot \rhd \Diamond\top) \to \Box\bot$. For every even R–successor n of 0 (on which $\Box\bot$ holds), there is its S_0–successor $n+1$, which is odd and therefore forces $\Diamond\bot$ (it has a successor $n+2$). So $\mathfrak{M}, 0 \Vdash \Box\bot \rhd \Diamond\top$. But $0 \; R \; 1$, so $\mathfrak{M}, 0 \nVdash \Box\bot$. That together implies $\mathfrak{M}, 0 \nVdash \phi$, so ϕ doesn't hold on $\mathfrak{M}$.

Let now $\mathfrak{N}$ be any finite submodel of $\mathfrak{M}$, and let's denote its set of worlds by S. Also let $i \in S$ be arbitrary. Assume for contradiction that $\mathfrak{N}, i \nVdash \phi$. That means two things: $\mathfrak{N}, i \Vdash \Box\bot \rhd \Diamond\top$, and $\mathfrak{N}, i \nVdash \Box\bot$. The last claim implies i having a (R–)successor, so i is 0 or odd.

If $i = 2k-1$ is odd, the only successor of i in $\mathfrak{M}$ was $i+1 = 2k$. Since in $\mathfrak{N}$ the world i still has a successor, it has to be $2k \in S$. It didn't have a successor in $\mathfrak{M}$ (because it's positive and even), so it can't have one in $\mathfrak{N}$, so $\mathfrak{N}, 2k \Vdash \Box\bot$. Since $\Box\bot \rhd \Diamond\top$ holds at i, we must have a S_i-successor of $2k$ on which $\Diamond\top$ holds—but the only possible S_i-successor or $2k$ is $2k$ itself (even in $\mathfrak{M}$, so surely in $\mathfrak{N}$), and it is even and positive, so it doesn't force $\Diamond\top$.

That leaves the case $i = 0$ to be disproved. Since $\mathfrak{N}$ is finite, S is finite, and since $0 \in S$, also S is nonempty. So it has a maximal element, $m := \max S$.

Since 0 has a R-successor in $\mathfrak{N}$, set S doesn't contain only 0, so $m > 0$. That implies $0\ R\ m$ in $\mathfrak{N}$, and m surely doesn't have a R-successor in $\mathfrak{N}$ (it could only have $m+1$ if odd, but $m+1 > m = \max S$ means $m+1 \notin S$). That means $\mathfrak{N}, m \Vdash \Box\bot$, and since $0\ R\ m$, we must have a S_0-successor n of m at which $\Diamond\top$ holds. But n would have to be in S, so $n \leq m$, and $m\ S_0\ n$ implies $m \leq n$. So the only possibility is $n = m$, which is a contradiction since $\Box\bot$ and $\Diamond\top$ cannot both hold at the same world.

We conclude that ϕ holds on every finite submodel of $\mathfrak{M}$, but it doesn't hold on $\mathfrak{M}$, so (since M is a tree) the closed fragment of **IL** doesn't have the finite subtree property. ∎

4. Conclusion

The proposition above is important since in other logics with the fmp, the proofs of the fmp usually start with a model for a formula and somehow ("unraveling", cutting width, cutting depth) produce a finite *sub*model for the same formula. We have just shown that a particular step of that procedure—namely, cutting width—cannot be universally carried out, although other steps can: $\mathfrak{M}$ is a tree (so unraveling would leave it the same), and of finite depth, equal to modal depth of ϕ (so cutting depth doesn't change it either).

Moreover, $\mathfrak{M}$ is a frame, and ϕ is closed, so cutting width doesn't even work restricted to closed fragments. Also, the counterexample uses only reasoning from **IL**, so it works in every extension of **IL** where $\mathfrak{M}$ is adequate.

It is even minimal in some way: by size, since $\mathbb{N}$ is the smallest infinite set; by depth, since on depths 0 and 1 we can't have S-relations connecting different types of worlds if we don't have propositional variables; and even by length of the formula, but that requires a precise syntax and a lot of writing to prove.

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Logika interpretabilnosti IL nema svojstvo konačnih podstabala

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Sažetak. Obično, ako logika ima svojstvo konačnih modela (*fmp*), tada također ima i jače *svojstvo konačnih podmodela*: svaki model može se reducirati na konačan podmodel. Ili, barem, ima svojstvo konačnih *podstabala*, što je svojstvo konačnih podmodela restringirano na modele koji su stabla. Ovdje dokazujemo da logika interpretabilnosti IL nema svojstvo konačnih podstabala.

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