

CURRICULAR APPROACHES TO CONNECTING SUBTRACTION TO ADDITION AND FOSTERING FLUENCY WITH BASIC DIFFERENCES IN GRADE 1

Arthur J. Baroody

Six widely used US Grade 1 curricula do not adequately address the following three developmental prerequisites identified by a proposed learning trajectory for the meaningful learning of the subtraction-as-addition strategy (e.g., for $13 - 8$ think “what $+ 8 = 13$?”): (a) reverse operations (adding 8 is undone by subtracting 8); (b) common part-whole relations ($5 + 8$ and $13 - 8$ share the same whole 13 and parts 5 and 8); and (c) the complement principle in terms of part-whole relations (if parts 5 and 8 make the whole 13, then subtracting one part from the whole leaves the other part).

Keywords: Fact fluency; Learning trajectories; Part-whole relations; Primary-grade mathematics curricula; Subtraction-as-addition strategy

Aproximaciones curriculares para conectar la sustracción con la adición y promover la fluidez con las diferencias básicas en primer curso de educación primaria

Seis currículos ampliamente usados en Estados Unidos para primero de primaria no tratan adecuadamente tres prerrequisitos identificados por una trayectoria de aprendizaje propuesta para el aprendizaje significativo de la estrategia de sustracción como adición (ejemplo, para $13 - 8$ piensa “¿qué $+ 8 = 13$?”): (a) operaciones inversas (sumar 8 se deshace restando 8); (b) relaciones parte-todo comunes ($5 + 8$ y $13 - 8$ comparten el mismo todo 13 y las partes 5 y 8) y (c) principio de complemento en relaciones parte-todo (si 5 y 8 dan el todo 13, al restar una parte al todo se obtiene la otra parte).

Términos clave: Currículos de matemáticas de primero de educación primaria; Estrategia de la sustracción como suma; Fluidez informativa; Relaciones parte-todo; Trayectorias de aprendizaje

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Two key primary-level goals of the “Common Core State Standards” (Council of Chief State School Officers [CCSSO], 2010) are: (a) connecting subtraction to addition, and (b) achieving fluency with basic subtraction combinations. A hypothetical learning trajectory (HLT) based on current theory and empirical evidence is proposed for achieving these goals. The proposed HLT raises issues regarding the make-up of primary-level mathematics curricula and the timing of addition and subtraction instruction.

KEY PRIMARY GOALS

The Common Core State Standards (CCSSO, 2010) specify as Goal 1.OA.4 (Grade 1, operations and algebraic thinking domain, Goal 4): “Understand subtraction as an unknown-addend problem. For example, subtract $10 - 8$ by finding the number that makes 10 when added to 8” (p. 15). The process of using the known sum of an addition combination to deduce the unknown difference of a related subtraction combination will hereafter be called the *subtraction-as-addition strategy*. Goal 1.OA.6 stipulates that “using the relationship between addition and subtraction (e.g., knowing that $8 + 4 = 12$, one knows $12 - 8 = 4$)” (p. 15). Common Core Goals 1.OA.4 and 1.OA.6 can facilitate Goal 2.OA.2: “Fluently add and subtract within 20 using mental strategies” (p. 19) [such as the subtraction-as-addition strategy].

Fostering relational learning and the learning of reasoning strategies are generally viewed as important to helping students achieve fluency with basic combinations. The meaningful memorization of a basic combination or a family of combinations typically involves three overlapping phases (Baroody, 1985; Steinberg, 1985; Verschaffel, Greer, & De Corte, 2007). In Phase 1, children use counting strategies to determine an answer. In Phase 2, they use the patterns and relations discovered in Phase 1 to invent reasoning strategies, which they apply in a deliberate manner—consciously and relatively slowly. In Phase 3, children achieve fluent retrieval—that is, they can efficiently, appropriately, and adaptively produce sums and differences from a memory network via automatic reasoning processes or fact recall. Phase 2 can serve as a bridge between the relatively inefficient counting strategies of Phase 1 and the fluent retrieval of Phase 3 (CCSSO, 2010; National Council of Teachers of Mathematics, 2000; National Mathematics Advisory Panel, 2008; National Research Council, 2001; Rathmell, 1978; Sarama & Clements, 2009; Thompson, 1997; Thornton, 1990; Thornton & Toohey, 1985).

HLT FOR THE SUBTRACTION-AS-ADDITION STRATEGY

In this section, a 10-step HLT for fostering Phases 2 and 3 with the subtraction-as-addition strategy is described. The premise of the HLT is that fostering the meaningful learning of the strategy (Step 9) and its fluency (Step 10) depends on constructing a deep (richly interconnected) understanding of the relations between addition and subtraction—what Piaget (1965) called *additive composition*. The HLT, delineated below and summarized in Figure 1, is based on a rational task analysis (theory, research, and logic).

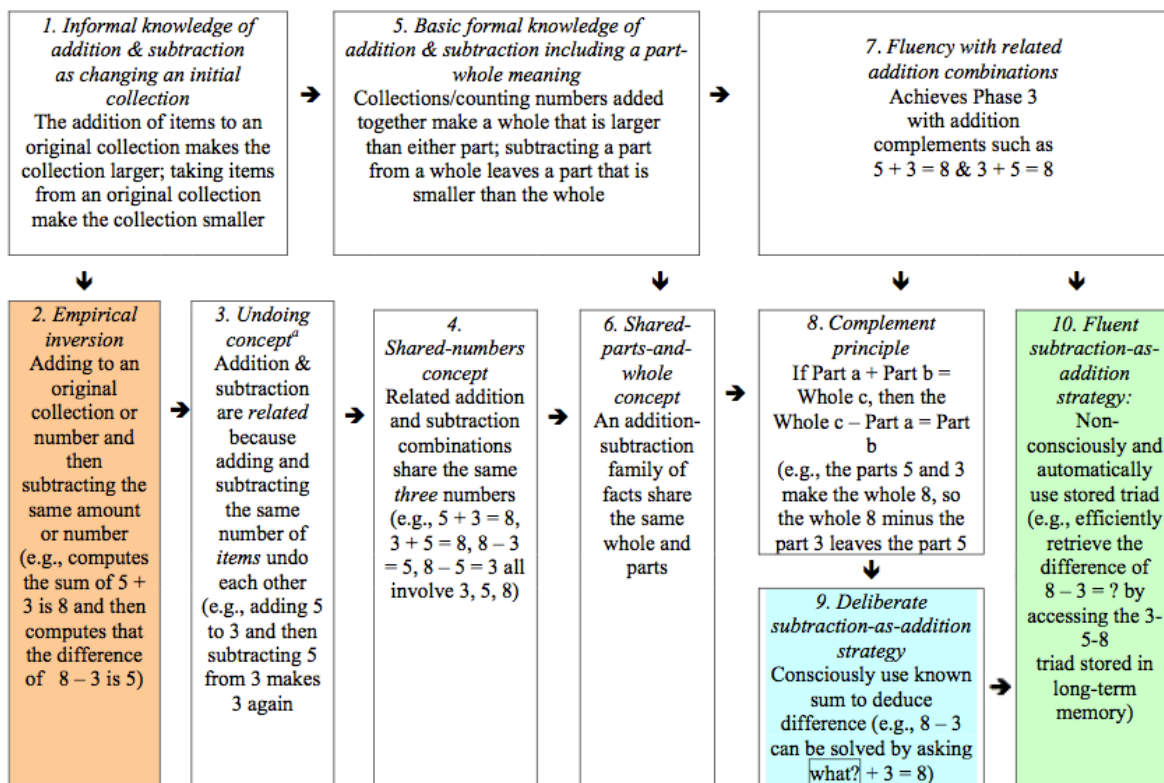


Figure 1. HLT for the meaningful development of the subtraction-as-addition reasoning (subtraction) strategy¹

¹ All un-shaded cells are conceptual knowledge. The orange-shaded Cell 2 is an experience that can lead to a concept. The blue-shaded cell 9 is conscious procedural knowledge. The green-shaded cell 10 is compiled integrated conceptual procedural knowledge of the complement principle and subtraction strategy.

^a A general undoing concept can evolve into formal and explicit knowledge of the inverse principle, which includes the ability to summarize the principle algebraically as $a + b - b = a$ or $a - b + b = a$.

Step 1: Informal Knowledge of Addition and Subtraction as Changing an Initial Collection

Children informally recognize that addition of items makes an initial collection larger and taking away makes an initial collection smaller, even before they develop counting-based strategies (Huttenlocher, Jordan, & Levine, 1994; Jordan, Huttenlocher, & Levine, 1992, 1994). Although children's initial informal *change* concepts of addition and subtraction involve viewing these operations as *unrelated*, they provide a direct basis for Steps 2 and 5 and an indirect basis for other succeeding steps.

Step 2: Empirical Inversion

Empirical inversion involves adding items to an original collection and then subtracting the same amount from the result to restore the original collection—without expecting that the final outcome will be the original collection. For example, empirical inversion would entail adding 2 item to the initial collection of 3 to get 5 items and then—because the final outcome cannot be foreseen—taking away 2 items from the 5 to arrive again at 3 items. Some children may first experiment with empirical inversion with formal symbols instead of with objects. For $3 + 2 - 2$, a child might add $3 + 2$ first to get the sum 5, then—unable to anticipate the outcome—compute the difference of $5 - 2$. Alternatively, children may use a computational shortcut for symbolic *inversion problem* such as the *associative strategy*: Operating from right to left and successively applying subtractive identity (as any number minus itself is zero, $2 - 2 = 0$) and then additive-identity (as any number plus zero is itself, $3 + 0 = 3$) (Baroody & Lai, 2007; Klein & Bisanz, 2000). Although children engage in empirical inversion because they view addition and subtraction as unrelated operations, restoring an original amount or number via this process, especially with small subitizable collections or small and highly familiar numbers, can serve as the first step toward viewing these operations as interdependent (i.e., Step 3).

Step 3: Undoing Concept

By reflecting on experiences with empirical inversion (either with concrete objects, mental images, or symbolic numbers), children may induce or discover an informal *undoing concept*—the idea that adding and then subtracting the same (small) amount or the same (familiar) number (or vice versa) undo each other and thus restores the starting amount/number. This concept enables children to immediately predict the outcome (without actual calculations) of adding a few objects (a familiar number) to an initial collection (number) and then subtracting the same amount (number) or vice versa. The *informal* undoing concept may only be implicit in nature and applied in a limited fashion—initially to adding and then subtracting one item and then to

somewhat larger numbers. In time, the concept is generalized to larger collections and symbolic numbers—a development that permits the logical recognition that adding and then subtracting the same symbolic number (or vice versa) cancel each other (e.g., immediately recognizing that $5 + 3 - 3$ is 5 without actual calculations). A generalized undoing concept is sometimes called *inversion* or the *inverse principle*. As a result of formal instruction, children learn that the inverse principle can be algebraically represented as $a + b - b = a$ or $a - b + b = a$. More germane to the present topic, the undoing concept is hypothesized to directly support Step 4 and indirectly support Steps 6 and 8.

Step 4: Shared-Numbers Concept

Further reflection on experiences with empirical inversion, involving small collections or familiar numbers especially, may lead to an expansion of the undoing concept. Specifically, whereas the undoing concept is primarily concerned with two elements (the starting amount/number and the amount/number involved in the doing/undoing process), the shared number concept entails the explicit insight that all elements in the undoing or subtraction process (not merely the starting and end amount) have corresponding elements in the doing or addition process. For example, for $5 + 3 = 8$ and $8 - 3 = 5$, the amount added and taken away are both 3, the starting point of adding and the end point of subtracting are both 5, and the outcome of adding $5 + 3$ corresponds to the starting point for the undoing process (8). This expansion of the undoing concept to include all three numbers in related sums and differences and additive commutativity (the order of adding numbers does not affect the outcome) results in the recognition of that there are families of number combinations—a shared numbers concept (e.g., $5 + 3 = 8$, $3 + 5 = 8$, $8 - 3 = 5$, and $8 - 5 = 3$ are members of the same addition-subtraction family because these combinations share the same triad of numbers: 3, 5, and 8). A number of curricula underscore the shared number concept with the visual analogy of the *fact triangle* (see Figure 2). The elaboration of the shared numbers concept (Step 4) with an understanding of part-whole relations (Step 5)—described next—results in a shared-parts-whole concept (Step 6).

Fact Triangles

The fact triangle shown below were downloaded from http://instruction.aaps.k12.mi.us/EM_parent_hdbk/activities.html, and illustrate two different types of fact triangles—complete and incomplete triangles—used in *Everyday Mathematics* (University of Chicago School Mathematics Project, 2005; see, e.g., p. 503). For both types, the whole is starred, and a plus and minus appear in the middle of the triangle.

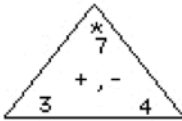


Figure 2A

Complete triangles. As shown in the Figure 2A, complete triangles represent the whole at the apex both parts such at a base angle. One exercise with complete triangles asks a pupil to construct related addition and subtraction number sentence—that is, elements of an addition-subtraction family (e.g., $3 + 4$, $4 + 3$, $7 - 3 = 4$, and $7 - 4 = 3$).

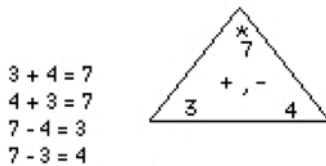


Figure 2B

As shown in Figure 2B, a family of complementary sums and differences may accompany the complete fact triangle to illustrate an “addition-subtraction family” (shared numbers concept or Step 4 of the HLT). Other potential uses of complete triangles are used to underscore empirical inversion/undoing concept (Steps 2 & 3), common parts and wholes (Step 6), and the subtraction-as-addition strategy (Step 9).



Figure 2C

Incomplete triangles. Incomplete triangles either represent both parts but not the whole to practice sums or (as shown in Figure 2C) the whole and only one part to practice subtraction. Incomplete triangles are used to practice basic sums and difference and promote fluency (Steps 7 to 10 in the HLT).

Fact Rectangles

Note that in the Fact Rectangle below is may be a more concrete representation of part-whole relations than the relatively abstract fact triangle in two ways: (1) The representation of 7 is literally the same length (and area) as 3 and 4 combined and is, thus, size proportional. (2) The Fact Rectangle representation builds directly on children’s counting-based (discrete quantity) view of numbers as collections.

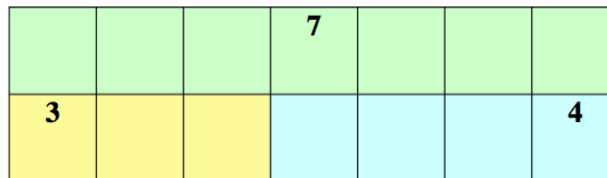


Figure 2. Fact triangles and fact rectangles

Step 5: Basic Part-Whole Relations

Children gradually supplement their informal and active view of addition and subtraction (i.e., a physical action that changes an initial collection by making it larger or smaller) with the formal (passive) meaning of these operations in terms of parts and whole. For example, instead of informally viewing $5 + 3$ as five and three more and as different from $3 + 5$ (three and five more) and expecting different outcomes for $5 + 3$ and $3 + 5$ (Baroody, Wilkins, & Tiilikainen, 2003), they learn that adding the parts 5 and 3 or parts 3 and 5 result in the same whole 8, which is larger than either part. With subtraction, children may realize that the whole 8 take away the part 3 leaves the other part 5, which must be smaller than the whole. A more formal understanding of addition and subtraction in terms of part-whole relations is an important conceptual leap that some scholars have hypothesized permits understanding missing-addend situations (Briars & Larkin, 1984; Canobi, 2005; Piaget, 1965; Riley, Greeno, & Heller, 1983; Sophian & Vong, 1995; Sophian & McCorgray, 1994)—without an understanding of the subtraction-as-addition strategy is not possible. Step 5 is hypothesized to be critical for transforming Step 4 into Step 6.

Step 6: Shared-Parts-and-Whole Concept

The integration of part-whole knowledge (Step 5) and shared-numbers concept (Step 4) creates the explicit and even more detailed knowledge of the shared-number concept, namely the *shared parts-and-whole concept* (Step 6). Students realize, for example, $5 + 3 = 8$, $3 + 5 = 8$, $8 - 3 = 5$, and $8 - 5 = 3$ all share the *same parts* 3 and 5 and the *same whole* 8. Step 6 and next step pave the way for inducing Step 8.

Step 7: Fluency with Basic Sums

Fluency with basic sums (Step 7) may serve to foster the learning of the addition-as-subtraction strategy (Step 9) and fluency with basic subtraction combinations (Step 10) in two ways. One is that fluency with basic sums makes it more likely children will discover the complement principle (Step 8). Research indicates that children appear to discover and use the complement principle (use the subtraction-as-addition shortcut) first in cases where sums are well known such as the doubles (Baroody, Ginsburg, & Waxman, 1983). Discovery of the complement relation between addition and subtraction without such prerequisite knowledge would seem less likely. A second reason is that, logically, fluency with basic sums also serves to foster the fluency with the addition-as-subtraction strategy, which involves retrieving the related addition combination (Step 10; Baroody, 1999; Campbell, 2008; Peters, De Smedt, Torbeyns, Ghesquire, & Verschaffel, 2010; Thornton, 1990). Research has consistently shown that fluency with developmental prerequisites is necessary to achieve fluency with a reasoning strategy, including the subtraction-as-addition strategy

(Baroody, Eiland, Purpura, & Reid, 2012, 2013; Baroody, Purpura, Eiland, & Reid, 2014; Baroody, Purpura, Eiland, Reid, & Paliwal, in press; Eiland, 2014).

Step 8: Complement Principle

A logical consequence of integrating the undoing concept (Step 3) and shared-parts-whole concept (Step 6) is the *complement principle*: If Part $a +$ Part $b =$ Whole c , then the Whole $c -$ Part $a =$ Part b (or the Whole $c -$ Part $b =$ Part a). For example, if the parts 5 and 3 create the whole 8, then logically taking away the part 3 from the whole 8 should result in leaving the other part 5 (or taking away the part 5 from the whole 8 should result in leaving the other part 3). Step 8 (the complement principle) serves as the conceptual rationale for the addition-as-subtraction strategy (Step 9).

Step 9: Deliberately Executed Subtraction-as-Addition Strategy

The complement principle provides an explanation for why the subtraction-as-addition strategy works. As $8 - 3 = \boxed{?}$ can be viewed as the whole 8 take away the part 3 and $\boxed{?}$ can be thought of as the unknown remaining part, then solving for the unknown is simply a matter of determining what part must be added to the part 3 to make the whole 8. As with other reasoning strategies, children initially must consciously or deliberately use the subtraction-as-addition strategy (Step 9). Step 9—Phase 2 in the meaningful learning of subtraction combinations—provides children with a more efficient means of determining differences than Phase 1 counting-based strategies, which can be difficult to execute (see, e.g., Baroody, 1984, or Baroody et al., in press). Moreover, Step 9 serves as a basis for Step 10—Phase 3 in the meaningful learning of subtraction combinations: the automatic or *fluent retrieval* from a memory network composed of facts, relations, and reasoning processes.

Step 10: Automatically Executed Subtraction

Reasoning strategies can aid in the meaningful memorization of basic subtraction combinations and thus fluent retrieval in two ways. One is that, as children achieve fluency with basic sums (the developmental prerequisite for fluency with subtraction-as-addition strategy) and practice implementing the subtraction-as-addition strategy itself, the strategy becomes automatic and can be applied efficiently without conscious oversight (Baroody et al., 2014; Baroody & Varma, 2006; Eiland, 2014; cf. Fayol & Thevenot, 2012). A second way is that they can provide an organizing framework for learning and storing both practiced and unpracticed combinations (Canobi, Reeve, & Patterson, 1998; Dowker, 2009; Rathmell, 1978; Sarama & Clements, 2009). Specifically, meaningful instruction, such as that outlined by the proposed HLT, may prompt a reorganization of retrieval system in which sums and differences are represented as an integrated *mental fact triangle* (Baroody et al., in press).

ISSUES RAISED BY THE HLT REGARDING THE MAKEUP OF PRIMARY MATHEMATICS CURRICULA

If valid, the proposed HLT has clear implications regarding the instructional elements needed for the meaningful learning of the subtraction-as-addition strategy.

Does Research Support Complement Problem Mediation?

Theoretically, practice with complementary sums (Step 7), applying the complement principle (Step 8), and deliberately using the subtraction-as-addition strategy (Step 9) should produce the automatic use of the strategy (Step 10) and *complement problem mediation*—greater long-term improvement in the fluency of related, but unpracticed, subtraction combinations (Baroody, 1999; Campbell, 2008; Peters, De Smedt, Torbeyns, Ghesquire, & Verschaffel, 2010). Such transfer is indicative of a general and meaningful strategy. Although a number of intervention studies have failed to produce transfer from practiced sums to related, but unpracticed, differences (Baroody, 1999; Walker, Bajic, Kwak, & Rickard, 2014; Walker, Mickes, Bajic, Nailon, & Rickard, 2013), these efforts may not have been particularly meaningful (Baroody et al., in press). Specifically, Baroody's (1999) intervention simply entailed practicing related sums and reiterating the subtraction-as-addition strategy without its conceptual rationale (involved Steps 7 and 9 in the HLT exclusively). One training condition in the Walker et al. (2013) simply involved drill of related sums (HLT Step 7) only, and the second condition only involved translating complete fact triangles into addition and subtraction equations (i.e., creating addition and subtraction family; HLT Step 4). Instruction that focuses narrowly on one or few steps of the HLT is not likely to be effective. Recent research revealed that a computer-assisted intervention involving all the elements of the HLT resulted in the transfer of fluency to unpracticed subtraction combinations (Baroody et al., in press).

Which Step or Combination of Steps is Most Pedagogically Useful?

Although previous intervention with all the elements of the HLT achieved complement mediation (Baroody et al., in press), it is unclear whether all were needed and, if not, which step or combination of steps is the most efficacious in promoting the learning of the subtraction-as-addition strategy and achieving fluency with basic differences, including transfer to unpracticed subtraction combinations.

Theoretically, empirical inversion and the undoing concept or the inversion principle would seem particularly important to include in a primary-level curriculum. Baroody, Torbeyns, and Verschaffel's (2009) conjectured that empirical inversion underlies the development of complement and inversion principles. Specifically, empirical inversion (Step 2) should be a helpful, if not a necessary, element of a cur-

riculum, because it directly supports the discovery of the undoing concept (Step 3) and the shared number concepts (Step 4)—both of which, at least indirectly, support the construction of a meaningful complement principle (Step 8). The localized undoing concept, but not the shared number concept, is also hypothesized to be the basis for an explicit and general inverse principle.

Parenthetically, it is unclear whether a local and implicit undoing concept or a general and explicit undoing concept (inverse principle) is needed to facilitate or construct an understanding of the complement principle or the learning the subtraction-as-addition strategy in a meaningful fashion (Step 9). Although the local undoing concept, which includes the key ideas that addition and subtraction are related operation and that one can undo (at least with small or familiar quantities/numbers), would seem sufficient as a developmental prerequisite for Steps 4, 6, 8, and 9, a more general and explicit inverse principle may be helpful or even needed. The issue would be moot if research shows that the general inverse principle develops before Steps 4, 6, 8, and 9.

Some evidence indicates that the inverse principle does develop before the complement principle (Step 8) or the subtraction-as-addition strategy (Step 9). Canobi (2004) found children who understood the inverse principle typically also knew the complement principle. Nunes, Bryant, Hallett, Bell, and Evans (2009) found direct causal evidence that inverse knowledge can facilitate learning of the complement principle—that the inverse principle is the developmental prerequisite or a necessary condition for the complement principle and the addition-as-subtraction strategy. Specifically, an intervention designed to foster the inverse relation resulted in improving participants' performance on complement problems also.

However, Canobi (2004) and Nunes et al.'s (2009) evidence does not clearly indicate whether the more robust knowledge of the inverse principle (Step 3A) should be added to the HLT. Canobi's evidence does not address whether inverse knowledge develops before Steps 4 and 6 or the direction of causation. Moreover, the helpful aspect of Nunes et al.'s training may have been the opportunity to engage in empirical inversion and constructing an undoing concept. This learning, in turn, may have benefitted enough participants to produce significant improvement with the complement principle/subtraction-as-addition strategy and the inverse principle. Clearly, further research is needed to examine the links among empirical inversion, the informal undoing concept, the more formal and general inverse principle, the complement principle, and the subtraction-as-addition strategy.

Is an Analysis of all the Developmental Separate Steps Needed or Useful?

Van den Heuvel-Panhuizen and Treffers (2009) implicitly questioned the need to distinguish between the complement principle (which they represent as

$a - b = ? \rightarrow b + ? = a$ principle) and the inversion principle. Specifically, they argued that such a distinction is meaningless from a mathematical, psychological, and/or educational point of view. However, empirical inversion ($a + b \Rightarrow c$ and $c - b \Rightarrow a$; where \Rightarrow indicates where the outcome cannot be predicted but must be determined by computing), the inverse principle ($a + b - b = a$ or $a - b + b = a$), the complement principle (if $a + b = c$, then $c - a = b$ or $c - b = a$), and the subtraction-as-addition strategy ($a - b = ? \rightarrow b + ? = a$) each have a different algebraic representation. More importantly, there are four reasons why it may be psychological or educational useful to view these constructs—including the undoing concept or inverse principle and the complement principle—as separate steps, especially when designing early childhood mathematics curricula.

- ◆ For both research and pedagogical purposes, it can be useful to distinguish between procedural knowledge and conceptual knowledge (the motivation or rationale for a procedure). In the present analysis, re-representing $a - b = ?$ as $b + ? = a$ is treated as procedural knowledge (the subtraction-as-addition strategy or Step 9), not as conceptual knowledge as in previous analyses by Van den Heuvel-Panhuizen and Treffers' (2009) and others (e.g., Baroody, Torbeyens, et al., 2009). Put differently, the subtraction-as-addition strategy, which can be learned either by rote or meaningfully, is differentiated from its distal conceptual rationale (e.g., the undoing principle, the shared parts and whole concept) and its immediate conceptual rationale (the complement principle).
- ◆ Children appear to learn a rudimentary form of the undoing concept well before formal instruction begins, the general inverse principle later, and the relatively advanced complement principle even later (Baroody et al., 1983; Baroody & Lai, 2007; Baroody, Lai, Li, & Baroody, 2009). For example, using the *magic task* Starkey and Gelman (1982) found that 3-year olds could differentiate between changes in appearances (the re-arrangement of a small collection) and numerical transformation (e.g., the addition or subtraction of an item from a small collection) and could indicate how to undo numerical transformation of plus or minus one but not those involving larger numbers. Although empirical inversion cannot be discounted, these results are consistent with children constructing an initial (albeit a highly local) understanding of the undoing concept well before formal schooling. As noted previously, Canobi (2004) and Nunes et al.'s (2009) evidence is consistent that the development of the inverse principle prior to that of the complement principle. In contrast, research indicates that the complement relation and the subtraction-as-addition strategy are not salient to young children, even those in the primary grades (Canobi, 2009; Putnam, deBettencourt, & Leinhardt, 1990).

For example, Baroody et al. (1983) found that Grade 1 to 3 pupils had but a local knowledge of the complement principle involving only the well-learned doubles.

- ◆ Whereas empirical inversion and the complement principle (if Part a + Part b = the Whole c , then the Whole c – Part a = Part b or the Whole c – Part b = Part a) involve three numbers (a , b , and c) and support or are supported by the shared numbers and shared parts and whole concepts, the same is not true of the formal inverse principle ($a + b - b = a$ or $a - b + b = a$), which does not involve the third element c . So the inverse principle may not be psychologically equivalent to the shared numbers concept, the shared parts and whole concepts, and the complement principle.
- ◆ Explicitly distinguishing among empirical inversion, the inverse principle, the complement principle, and the subtraction-as-addition strategy may better ensure that all of these steps in the HLT are included school curricula and done so in the appropriate developmental order—something that, as discussed in the following subsections, is currently not done.

What Components of the HLT do US Curricula Have?

One supplemental and six mainstream textbooks were analyzed for their consistency with HLT (see Table 1).

Table 1
Approaches to Teaching Subtraction by Grade 1 Curriculum

| Component | Relevant Aspect(s) of the HLT (Step in Figure 1) | Curriculum | | | | | | |
|--|---|------------|----|----|----|----|----|----|
| | | T1 | T2 | T3 | T4 | T5 | T6 | T7 |
| 1. Reverse operations (e.g., the effects of adding 8 can be reversed by subtracting 8 or vice versa) | Empirical inversion (Step 2) and undoing/inversion concept (Step 3) | ★? | – | ★? | ✓ | ✓ | ✓ | ✓ |
| 2. Fact families (e.g., $5 + 8$, $8 + 5$, $13 - 8$, and $13 - 5$ all share the same three numbers: 5, 8, and 13) | Shared-numbers concept (Step 4) | – | ✗ | ★? | ✗ | ✗ | ★? | ✗ |

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Approaches to Teaching Subtraction by Grade 1 Curriculum

| Component | Relevant Aspect(s) of the HLT (Step in Figure 1) | Curriculum | | | | | | |
|--|--|------------|----|----|----|----|----|----|
| | | T1 | T2 | T3 | T4 | T5 | T6 | T7 |
| 3. Common part-whole relations (e.g., fact triangle with 13 at the apex explicitly identified as the whole and 5 and 8 at vertices of the base clearly identified as the parts) | Shared-parts-and-whole concept (Step 6) | – | ✓ | ✓ | ✓ | ✓ | – | ✓ |
| 4. Rationale for the subtraction as-addition strategy in terms of shared part-whole relations (e.g., If adding the parts 5 and 8 make the whole 13, then taking the part 8 from the whole 13 leaves the other part 5) | Complement principle (Step 8) | – | – | ✗ | ✗ | – | – | ✗ |
| 5. An unknown subtraction combination can be solved by using a related known addition combination (e.g., Find $13 - 8 = \text{what?}$ by thinking what $+ 8 = 13$?) | Deliberate (Phase 2) subtraction-as-addition strategy (Step 9) | – | ✗ | ✗ | ✗ | ✓ | ✓ | ✓ |

Note. T1=Bridges in mathematics (Math Learning Center, 2009); T2= Everyday Mathematics (UCSMP, 2005); T3=Go Math! (Houghton Mifflin Harcourt, 2015); T4=Math Connects (Macmillan/Mc-Graw Hill, 2009); T5= Math Expressions (Fuson, 2006), T6=Saxon Math (Larson, 2008), T7=Scott Foresman Mathematics (2005). *Bridges in Mathematics* is a supplemental program; the other six curricula listed are mainstream or complete programs. A (★)=clearly a substantive (prevalent, explicit, and systematic) characteristic that builds on prior components in the trajectory; a cross check (✗)=substantive treatment *that does not capitalize on prior components in the trajectory*; a single check (✓)=an infrequent, implicit, or non-systematic characteristic; and a dash (–)=not characteristic; a question mark (?)=indicates a rating with qualifications or reservations.

Although Step 1 can be assumed to develop before Grade 1, essentially all Grade 1 curricula provide at least a cursory review of an informal change view of addition and subtraction. This foundational step is not discussed further. The curricula were

analyzed in terms of five components that include one step or two closely related steps delineated in the HLT.

Component 1 (Reverse Operations): Step 2 (Empirical Inversion) and Step 3 (Undoing/Inverse Concept)

Only two of the seven curricula analyzed use reverse operations as basis for highlighting that addition and subtraction are related operations on a direct and sustained basis and earn a star in Table 1. The supplemental *Bridges in Mathematics* (Math Learning Center, 2009) program introduces both operations as *hopping along a number line*. An addition item such as $5 + 3 = 8$, which is represented as hopping to the right 5 and then 3; is followed by the its related subtraction items such as $8 - 3 = 5$, which is represented as hopping from 8 in the opposite direction (to the left) three times. Two qualifications apply to the star for *Bridges in Mathematics*. One is that empirical inversion and the undoing concept are introduced exclusively using the relatively abstract model of the number line and symbolic arithmetic (instead of initially with more concrete models) to help children connect their informal knowledge of inversion (the undoing principle) with symbolic addition and subtraction (Ernest, 1985; Fuson, 2009; National Research Council, 2001; Saxe et al., 2010). Another is that the instruction is relatively implicit, which future research may or may not reveal is sufficient for children to construct the undoing concept or connect to formal arithmetic.

Similarly, the mainstream curriculum *Go Math!* (Houghton Mifflin Harcourt, 2015) entails a concerted effort to involve children, if only implicitly, with empirical inversion, (re-)discovering the undoing concept, and connecting the concept to symbolic representations. For example, on p. 257E (Professional Development: Teaching for Depth), the teacher's edition includes the note that children can conceptualize the relation between addition and subtraction by using differently colored connecting cubes. Teachers are encouraged to have children describe their connecting-cube models (e.g., "I made a train with 5 red cubes and 4 blue cubes to show $5 + 4 = 9$. I break off 4 blue cubes to show $9 - 4 = 5$ "). Children are given ample opportunities to (implicitly) notice the undoing concept concretely via modeling the addition of cubes to a starting number of cubes and then the subtraction the same of cubes from the total, semi-concretely via pictures of such models, and symbolically by solving sequentially presented addition items and related subtraction problems. For example, such activities are introduced in Lesson 5.2 by asking children to model, solve, and symbolically represent problems involving $7 + 1$ and $8 - 1$ and *working through the model and pictures* of the model and symbolic representations to show how knowing one addition fact, such as $4 + 5 = 9$, can help find the related facts $9 - 5 = 4$, $5 + 4 = 9$ (via additive commutativity), and $9 - 4 = 5$ (p. 261). How-

ever, the undoing concept is not explicitly highlighted by noting, for instance, that adding 4 to 5 can be undone or reversed by subtracting 4 from the sum of 5 and 4. Moreover, inconsistent with the undoing concept, sometimes the elements of the related addition and subtraction equations cancelling each other are in different positions. For example, on p. 266 and elsewhere, an addition fact such as $4 + 7 = 11$, not $7 + 4 = 11$, is related to $11 - 4 = 7$. Future research needs to gauge whether consistent pairing of inverse pairs ($4 + 7 = 11$ with $11 - 7 = 4$ and $7 + 4 = 11$ with $11 - 4 = 7$) is more effective (at least initially) than mixing such pairs (e.g., relating $4 + 7 = 11$ with $11 - 7 = 4$ or $11 - 4 = 7$ and $7 + 4 = 11$ with $11 - 4 = 7$ or $11 - 7 = 4$).

Four other mainstream curricula briefly touch on empirical inversion or the undoing concept. Exercises in *Math Connects* (Macmillan/McGraw Hill, 2009) include showing two collections (e.g., 6 bees in a circle and 3 bees in a line) and asking a child to write one addition sentence ($6 + 3 = 9$ or $3 + 6 = 9$) and one subtraction sentence ($9 - 3$ or $9 - 6 = 9$). This might implicitly involve empirical inversion *if* a child thought: “6 and 3 more is 9, and 9 take away the 3 is, oh, 6 again.” For Unit 2 in *Math Expressions* (Fuson, 2006), a teacher is encouraged to show that $8 - 3 = 5$ and an iconic representation (○○○|○○○○) are the “reverse story of $5 + 3 = 8$.” However, this relation is not shown in the textbook or a (seatwork or homework) worksheet, with one exception (an exercise for students “on [grade] level” in the “Extending the Lesson—Differentiated Instruction/Activities for Individualizing” section). *Saxon Math* (Larson, 2006) only briefly engages pupils in empirical inversion. One enrichment worksheet in *Scott Foresman Mathematics* (2005) illustrate, for example, 1 green interlocking cube added to 4 white cubes with equation $4 + \boxed{1} = 5$ and the statement, “Add the green cube. 4 and 1 more is 5.” Immediately below this is the equation $5 - \boxed{1} = 4$ and the statement, “Take away the green cube. 5 take away 1 more is 4.” Unfortunately, as an enrichment activity, it may not be used in most classes.

Component 2 (Fact Families): Step 4 (Shared Numbers Concept)

The idea that addition and subtraction complements or addition-subtraction fact families share common numbers—is characteristic of all six mainstream curricula surveyed. For example, *Math Connects* (Macmillan/McGraw Hill, 2009) defines related addition and subtraction facts as having the same numbers (for example, $1 + 6 = 7$ and $7 - 6 = 1$ or $3 + 7 = 10$ and $10 - 3 = 7$). Unit 6.3 of *Everyday Mathematics* (University of Chicago School Mathematics Project, 2005) goes further by introducing addition-subtraction families as combinations with same number triad (for example, $3 + 5 = 8$, $5 + 3 = 8$, $8 - 3 = 5$, and $8 - 5 = 3$ all share the same three numbers), finding both sums and differences using dominoes (fact triples: 3, 5, 8), and looking for the equivalent names for sums and differences using the same Addition/Subtraction

Fact Table. Unit 6.4 does the same sums using fact triangles (see Figure 2). Unit 6.5 uses Addition/Subtraction Fact Table to determine differences and fact triangles to generate addition-subtraction fact families.

Reverse operations (and additive commutativity) provide a mechanism for generating fact families that share the same three numbers. Two curricula earn a star in Table 1 because they build on reverse operations (empirical inversion and the undoing concept) to introduce the shared numbers concept. Page 257E (Professional Development: Teaching for Depth) of the teacher's edition in *Go Math!* (Houghton Mifflin Harcourt, 2015) includes the note that "related addition and subtraction facts show the inverse relationship of addition and subtraction for a group of numbers" and examples of fact families, such as, $6 + 7 = 13$, $13 - 6 = 7$, $7 + 6 = 13$, and $13 - 7 = 6$. The work page for re-teaching Lesson 5.2 involves showing the numbers 6, 4, and 10; pictorial models of $6 + 4 = 10$, $10 - 4 = 6$, $4 + 6 = 10$, and $10 - 6 = 4$, and the advice "THINK: Each number is in all four facts" (p. 261). The exercise involves four sets of three of numbers. In Lesson 5.3, students are encouraged to at least implicitly use empirical inversion/the undoing concept to construct (partial) addition-subtraction families by asking pupils to use a picture of 3 yellow leaves and 9 green leaves and 3 yellow leaves and 9 green leaves crossed out to write two facts ($3 + 9 = 12$ and $12 - 9 = 3$). The reminder on page 268 is: "These are related facts. If you know one of these facts, you also know the other fact." Lesson 5.3 also includes an exercise that involves determining the unknown of pairs of symbolic addition and subtraction equations and circling related facts. For example, $6 + 4 = 10$ and $10 - 4 = 6$ would be circled but $6 + 3 = 9$ and $12 - 3 = 9$ would not be (though a strong argument could be made that $6 + 3$ and $12 - 3$ are both other names for "9" and, thus, are related). Lesson 5.3 further includes an interesting open-ended problem in which students are asked to use the numbers 4 to 9 and 1 to 14 to write related addition and subtraction sentences. A number of exercises ask a student to decide if two facts are related (e.g., $13 - 8 = 5$ and $5 + 8 = 13$, yes; $5 + 7$ and $7 - 5$, no).

Saxon Math's (Larson, 2006) star in Table 1 was awarded with reservations, because the link between reverse operations and the shared numbers concept is minimal and inconsistent. For instance, in Lesson 132, blue and red linking cubes are used to model $4 + 1 = 5$ and then $5 - 1 = 4$, $1 + 4 = 5$ and then $5 - 4 = 1$ (examples of empirical inversion) to introduce the concept of "addition and subtraction families" and as a method for learning $9 - 4$, $9 - 5$, $9 - 3$, and $9 - 6$. However, although $7 - 3$, $7 - 4$, $8 - 3$, and $8 - 5$ are used in Lesson 134 to introduce addition and subtraction families, these families are not related to reverse operations.

Component 3 (Common Part-Whole Relations): Step 5 (Part-Whole Relations) and Step 6 (Shared Parts and Whole Concept)

Common part-whole relations serve to deepen further the understanding of addition-subtraction fact families by underscoring *why* family members have the same three numbers (e.g., $3 + 5 = 8$ and $8 - 5 = 3$ share the same parts 5 and 3 and the same whole 8). This component was not utilized at all in the supplemental curriculum and one mainstream curriculum and only inconsistently, implicitly, incompletely, or non-systematically in the other five mainstream Grade 1 curricula surveyed. For instance, for Unit 3, *Everyday Mathematics* (UCSMP, 2005) teachers are instructed: “Point out that the domino has a part with 3 dots and a part with 5 dots and that the whole domino has 8 dots” (p. 234), and children practice translating various dominoes into part-part-whole diagrams. However, this is not continued later (e.g., Unit 6.2) when the focus is on addition facts and never related to subtraction facts. Indeed, there is no mention of “parts” and “wholes” in Unit 6.2. In brief, the models used in this unit, such as fact triangles only *implicitly* represent part-part-whole relations of related addition and subtraction combinations.

Curiously, in *Go Math!* (Houghton Mifflin Harcourt, 2015), part-whole relations are noted only once in Lesson 5.4 to answer the (important) question: “Why can you use addition to check subtraction?” (p. 274). The answer is: “You subtract one part from the whole. The difference is the other part. When you add the parts, you get the same whole” (p. 274). Lesson 5.6 introduces fact triangles in the form of three interconnected squares with whole inscribed in the top square and the parts inscribed in the two bottom squares. However, no systematic or explicit effort is made to relate fact families to part-whole relations (e.g., explicitly teach the shared part-whole concept) or the if-then connection between a known sum and an unknown difference (explicitly teach the complement principle in terms of part-whole relations).

Component 4 (Rationale for the Subtraction-as-Addition Strategy): Step 8 (Complement Principle)

As Table 1 shows, the addition-subtraction complement principle is not a component in the Grade 1 supplemental curriculum and three of the mainstream curricula surveyed. Although the remaining three mainstream curricula surveyed included substantial treatment of the complement principle, none earned a star, because they failed to capitalize on common part-whole relations. For example, *Math Connects* (Macmillan/McGraw Hill, 2009) includes exercises that suggests, for instance, “Think $5 + 9 = \square$, so $14 - 9 = \square$ ” and circle the addition fact that will help you subtract $12 - 9 = \square$). Similarly, *Scott Foresman Mathematics* (2005) notes: “You can use addition to help you subtract” and “think if $5 + 8 = 13$, then $13 - 8 = \text{what?}$ ” (p. 439). Unfortunately, such propositions—barren of part-whole meaning—may not

be transparent to many students, particularly those struggling with mathematics. That is, statements of the complement principle without reference to common part-whole relations do not illuminate the logical connection between $5 + 8 = 13$ and $13 - 8 = ?$ In brief, none of the surveyed curricula provide an explicit and clearly meaningful basis for conceptually understanding the complement principle and why thinking of an addition combination can help determine the difference of a related subtraction combination.

Component 5 (Exploiting Known Missing-Addend Addition): Steps 9 and 10 (Deliberate and Fluent Subtraction-as-Addition Strategy)

As Table 1 shows, the supplemental curriculum did not have this component and three of the mainstream curricula had but brief or indirect treatments of the strategy. For example, *Saxon Math* (Larson, 2006) inconsistently related subtraction to addition. On one hand, for example, although adding with 1 is practiced immediately before subtracting by 1 is introduced in Lesson 44, the two are not related. Similarly, the addition facts with sums to 10 are introduced in Lessons 94 and 95. Although Lessons 101 and 102 begin with practicing these sums, they are not related to the focus of the lesson—subtracting a number from 10. Although Lesson 121, which introduces the “Differences of 1 Subtraction Facts,” relates $10 - 9 = 1$ to using addition to check a difference ($1 + 9 = 10$), add-with-1 combinations are not recommended as vehicle for determining the differences of such subtraction combinations. In Lesson 68 adding 2 to an odd or an even number with linking cubes is reviewed immediately before subtracting 2 from odd and even numbers is introduced with cubes. The former is summarized by the rule that “adding 2 is like saying the next add even or odd number” and the later is summarized as “subtracting 2 is like saying the even or odd numbers backwards by 2’s.” Although these rules may be useful, addition again is not used as a shortcut for determining differences. On the other hand, in Lesson 129, unknown subtraction combinations are linked to known addition combinations. For “subtracting half of a double” such as $14 - 7$, $12 - 6$, and $8 - 4$, children are asked: “What do you notice about each these problems?” [they are the doubles going the other way]...How can we remember these answers? We will call these problems the ‘subtracting half of a double facts.’” Although the text of *Scott Foresman Mathematics* (2005) specifies that “the sum of an addition fact is the first number in [a related] subtraction fact” (p. 138), it does not explain that the difference (missing part) in a subtraction equation is an addend (known part) in a related addition equation.

Three other mainstream Grade 1 curricula surveyed provide explicit instruction on the subtraction-as-addition strategy but fail to take advantage of other components, such as common part-whole relations, to teach the strategy in a deeply mean-

ingful manner. *Everyday Mathematics* (UCSMP, 2005) explicitly introduces the subtraction-as-addition strategy in Lesson 6.5 (Using the Addition/Subtraction Facts Table to Solve Subtraction Problems): “To find the answer to $15 - 9$, ask yourself: ‘9 plus what number is 15?’” (p. 510). However, an analogous strategy is not explicitly recommended when working with other models (fact triples such as 3, 5, 8 represented by dominoes or fact triangles). A teacher note in Unit 6, specifies (p. 500): “For many first graders, it is helpful to think about $8 - 5 = ?$ as $5 + \text{what number?} = 8$. This approach encourages ‘adding up’ to subtract, a strategy that also works well with multidigit numbers.” Encouraging thinking of subtraction as addition and using a counting-up strategy to solve for the missing addends can be a meaningful and useful for step toward, but is not the same as recommending, using known sums as a vehicle for shortcutting subtraction computation—for deducing differences.

Go Math! (Houghton Mifflin Harcourt, 2015) notes on work page 286: “You can use an addition fact to find a related subtraction fact.” The solution process is illustrated with finding the difference of $10 - 3$ using a 10-3-7 fact triangle and the hint: “I know that $3 + _ = 10$, so $10 - 3 = _$.” Simplified fact rectangles were used sparingly to represent and solve missing-addend problems (e.g., 4 rabbits in the garden, some more came, now there are 12 rabbits) and missing-subtrahend subtraction (e.g., 16 turtles, some swam away, 9 turtles were left on the beach). A simplified fact rectangle involves a rectangle divided into two parts the lengths of which were proportional to the magnitude of parts (e.g., for $6 + 4$, the 6 part of the rectangle is 50% longer than the 4 part) with a line running the length of the bottom of the rectangle and labeled 10.) Unfortunately, this representation was not used to illustrate and solve missing-difference problems or to relate such problems to an addition complement (i.e., to teach the complement principle or subtraction-as-addition strategy). Nevertheless, numerous and various reminders, activities, and exercises are used throughout Lesson 5 to underscore the subtraction-as-addition strategy. In addition to the page 286 reference in the previous footnote, on p. 310, students are asked model and draw a subtraction word problem that is accompanied by: “What is $10 - 4$? $4 + \square = 10$. So $10 - 4 = _$ ” and the reminder: “THINK. I can use a related addition fact to solve $10 - 4$.” Add to Subtract Bingo, for example, involves finding an addition fact that helps with a subtraction fact.

However, as Table 1 illustrates, none of the curricula earned a star by introducing the subtraction-as-addition strategy meaningful in terms of part-whole relations. For example, no curriculum explains: “Solving ‘the *whole* 13 take away the *part* 8 leaves what *part*’ can be found by ‘thinking what *part* and the *part* 8 makes the *whole* 13?’” Furthermore, *Go Math!* (Houghton Mifflin Harcourt, 2015) misses opportunities to build on the undoing principle. Work page 280 of Lesson 5.5 instructs: “Use what you know about related facts to find the unknown.” The missing-addend

addition equation $8 + \square = 11$ is illustrated with a train of 8 red interlocking cubes and 3 blue cubes. The missing-difference subtraction problem $11 - 8 = \square$ is illustrated with the “starting amount” 8 red interlocking cubes broken off from the remaining train of 3 blue cubes. Building clearly on the undoing concept would entail using $\square + 8 = 11$ to solve $11 - 8 = \square$, where 8 is the amount added to the (unknown) starting amount and the amount subtracted from their total. In effect, by not building (effectively) on Components 1 to 4, current curricula do not provide a clear explanation of *why* a known sum can be used to determine an unknown difference.

Research and Pedagogical Issues Raised by the Curriculum Analysis

Hypothesis 1: If the proposed HLT about how a deep and interconnected understanding of how addition and subtraction develops is correct, then curricula *that include ALL components* should be more efficacious than less complete programs of instruction. None of the six mainstream curricula surveyed substantially—directly, explicitly, consistently, and systematically—involved all five components hypothesized for the meaningful learning of the subtraction-as-addition strategy. If Hypothesis 1 is corroborated, then five of the six and all six mainstream curricula surveyed would need to upgrade their treatment of reverse operations and common part-whole relations, respectively.

Hypothesis 2: If the proposed HLT about how a deep and interconnected understanding of how addition and subtraction develops is correct, then curricula that capitalize effectively on earlier aspects of the HLT should be more efficacious than less well-connected programs of instruction. In particular, should Component 2 (fact families) and Component 3 (common part-whole relations) build on Component 1 (reverse operations)? Should Components 4 (rationale for the subtraction-as-addition strategy) and 5 (the strategy itself) build on Components 1 (reverse operations) and 3 (common part-whole relations)? If Hypothesis 2 is supported, then all six mainstream curricula surveyed would need to be revised to capitalize effectively on the reverse operations (Component 1) and part-whole relations (Component 3) in order to explain the logic of fact families, the complement principle, and subtraction-as-addition strategy in clear and meaningful manner.

Instructional Timing Issues

Formal subtraction instruction can either follow that of addition or be done simultaneously. A number of unresolved questions remain, including those regarding how the implementation of the HLT might impact order and vice versa.

Why is Timing Even an Issue?

A case can be made for both the *sequential/addition-first* and *simultaneous/integrated* approach and some (indirect) evidence supports each approach.

Traditionally, instruction has been based on a sequential/addition-first approach. The conventional wisdom—as embodied by the HLT in Figure 1 and five of the six mainstream curricula surveyed—is that fluency with addition combinations should be fostered first and that this fluency could serve as a basis for promoting fluency with subtraction. James (1958) observed that meaningful and secure memorization of new information can be achieved by relating it to what a child already knows (cf. Piaget's, 1964, concept of assimilation). Baroody, Eiland, and Thompson (2009) attempted to teach addition and subtraction of 1 simultaneously to at-risk preschoolers by relating these operations first to nonverbal tasks, then to word problems, and finally to symbolic expressions. Subtraction instruction was suspended, though, because switching between the operations confused the preschool participants. Moreover, Baroody et al. (1983) found that primary-grade pupils were far more likely to use the subtraction strategy with subtraction complements of the doubles than with non-doubles and concluded that this may have been due to the fact they could readily relate, for instance, $12 - 6$ to the already memorized double $6 + 6 = 12$ but failed to see the connection between $8 - 5$ and their non-fluent knowledge of $3 + 5 = 8$ or $5 + 3 = 8$. Indeed, Steinberg (1985) concluded that, even for second graders, using known facts to deduce unknown basic combinations was difficult, and using the relations between addition and subtraction to reason out differences was particularly difficult to understand. Thornton (1990) similarly concluded: “Thus, it seems that it might be quite important to separate addition and subtraction units to allow more time for children to consolidate newly-learned addition facts before trying to apply them to obtain solutions for subtraction” (p. 245). In brief, some (indirect) evidence suggests that encouraging a general use of the subtraction strategy might better be put off until the time children are fluent with the range of basic sums.

Five reasons support a case for the simultaneous/integrated approach.

- ◆ Some kindergartners and first-graders overgeneralize the add-1 rule and need additional counterexamples of the rule (Baroody, 1989, 1992; Baroody et al., 2012, 2013; Baroody, Eiland, & Thompson, 2009; Baroody, Purpura, Eiland, & Reid, 2015; Dowker, 2003). Subtraction of 1 provides a clear contrast with adding 1.
- ◆ The use of contrasting examples or non-examples has been recommended to help children learn a variety of concepts and skills, including the meaningful learning of the first few number words *one* and *two* or *two and three* (Baroody, Lai, & Mix, 2006; Bloom & Wynn, 1997; Durkin & Rittle-Johnson, 2012; Frye et al., 2013; Mix, 2009; Palmer & Baroody, 2011; Rittle-Johnson & Star, 2011; Sarnecka & Carey, 2008), recognition of numerals with a similar appearance such as 2 and 5 or 6 and 9 (Baroody, 1988; Baroody & Kaufman, 1993), or learning number after and before (Dyson,

Jordan, & Glutting, 2013). Hattikudur and Alibali (2010) found that teaching a relational meaning of equals in conjunction with non-examples—other relational symbols such as the inequality, more than, and less than symbols was significantly more effective than in promoting third- and fourth-grade students conceptual understanding of equals, equation encoding, and problem solving than teaching the equals sign alone or regular classroom instruction. Rittle-Johnson and Star (2007) found that simultaneously contrasting algebraic solution procedures enhanced posttest performance and transfer more than did sequential instruction of the two methods. 3.

- ◆ Jordan (Dyson et al., 2013; Dyson, Jordan, & Hassinger-Das, in press; Jordan, Glutting, Dyson, Hassinger-Das, & Irwin, 2012) has recently advocated a compare and contrast approach teaching $n + 1$ and $n - 1$ as a component of their successful number sense intervention with at risk kindergartners.
- ◆ “Everyday Mathematics”, which uses a simultaneous/integrated approach, is more successful in promoting mathematics achievement than traditional curricula, which typically use a sequential/addition-first approach (Carroll & Isacs, 2003; What Works Clearinghouse [WWC], 2006).
- ◆ Practicing the subtraction-as-addition strategy can improve the fluency of related sums not practiced during the training (Baroody et al., 2014).

What Issues Need to be Resolved?

The HLT depicted in Figure 1 is based on the assumption of at least a partially sequential approach. Specifically, Step 7 (fluency with sums) is represented as a prerequisite for Steps 8 and 9 (discovering the complement principle and the subtraction-as-addition strategy) and achieving Step 10 (fluency with the subtraction-as-addition strategy). However, if instruction is meaningful (i.e., involves an empirically verified learning trajectory), might children discover the complement principle without necessarily achieving fluency with sums first and might such instruction be more effective than addition-first training in helping children construct the mental triads needed for fluent use of the subtraction-as-addition strategy? Put differently, might prior instruction that facilitates Step 3 (undoing concept), Step 4 (the shared numbers concept), and Step 6 (the shared parts and whole concept) provide a basis for simultaneously learning the complement principle (Step 8) and subtraction-as-addition strategy (Step 9) and achieving fluency with basic differences (Step 10) and sums (Step 7)? If so, the proposed HLT will require amendment.

Unfortunately, a sequential and a simultaneous approach to timing have not been directly compared. Research, then, is needed to determine whether addition and subtraction should be taught successively or simultaneously in order to have the greater impact learning the connections between these operations and fluency with

basic subtraction combinations. If instruction is meaningful (i.e., involved an empirically supported learning trajectory), are the two approaches to timing equally effective? If not, is a sequential/addition-first approach or simultaneous/integrated instruction more efficacious? If both meaningful approaches to timing prove to be equally effective, it would undercut the universal constraint to instructional/ curriculum planning based on the assumption that order does matter. If the sequential/addition-first approach should prove to be more efficacious than the simultaneous/integrated approach, the widely used “Everyday Mathematics” (UCSMP, 2005), which uses a simultaneous approach, would need a major revision and reorganization to be consistent with best practices. The implications for the HLT is that efforts to promote Step 10 (fluency with the subtraction-as-addition strategy) would need to be postponed until children achieve fluency with its developmental prerequisite, namely fluency with sums. Indeed, efforts to promote Steps 8 (the complement principle) and 9 (deliberate use of the subtraction-as-addition strategy) might also benefit from such a postponement. If the simultaneous/integrated approach should prove to be more efficacious than the sequential approach, Steps 8 to 10 would need to be done concurrently. This would mean nearly all US Grade 1 mathematics curricula, which are based on the assumption that addition should be introduced before subtraction, would need to be revised and re-organized to meet the standard of best practices.

CONCLUSIONS

Currently, there is no direct empirical evidence on which component or combination of components is the more/most effective in helping Grade 1 pupils achieve the two crucial Common Core goals of (a) connecting subtraction to addition, including learning the subtraction-as-addition strategy, and (b) promoting fluency with the subtraction strategy. Currently, it also remains unclear whether addition and subtraction should be taught sequentially or simultaneously in Grade 1 to best achieve these goals. Research, then, is needed to determine what component or combination of components and what instructional order most/more successfully promotes the learning of the subtraction-as-addition strategy and achieving fluency with basic differences. Such research might indicate the need for major changes in existing US mathematics primary-grade curricula.

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REFERENCES

- Baroody, A. J. (1984). Children's difficulties in subtraction: Some causes and questions. *Journal for Research in Mathematics Education*, *15*, 203-213. doi:10.2307/748349
- Baroody, A. J. (1988). A cognitive approach to writing instruction for children classified as mentally handicapped. *Arithmetic Teacher*, *36*(2), 7-11.
- Baroody, A. J. (1989). Kindergartners' mental addition with single-digit combinations. *Journal for Research in Mathematics Education*, *20*, 159-172. doi: 10.2307/749280
- Baroody, A. J. (1992). *The development of kindergartners' mental-addition strategies*. *Learning and Individual Differences*, *4*, 215-235. doi: 10.1016/1041-6080(92)90003-W
- Baroody, A. J. (1999). Children's relational knowledge of addition and subtraction. *Cognition and Instruction*, *17*, 137-175. doi: 10.1207/S1532690XCI170201
- Baroody, A. J., Eiland, M. D., Purpura, D. J., & Reid, E. E. (2012). Fostering kindergarten children's number sense. *Cognition and Instruction*, *30*, 435-470. doi: 10.1080/07370008.2012.720152
- Baroody, A. J., Eiland, M. D., Purpura, D. J., & Reid, E. E. (2013). Can computer-assisted discovery learning foster first graders' fluency with the most basic addition combinations? *American Educational Research Journal*, *50*, 533-573. doi: 10.3102/0002831212473349
- Baroody, A. J., Eiland, M. D., & Thompson, B. (2009). Fostering at-risk preschoolers' number sense. *Early Education and Development*, *20*, 80-120. doi: 10.1080/10409280802206619
- Baroody, A. J., Ginsburg, H. P., & Waxman, B. (1983). Children's use of mathematical structure. *Journal for Research in Mathematics Education*, *14*, 156-168. doi: 10.2307/748379
- Baroody, A. J., & Kaufman, L. C. (1993). The case of Lee: Assessing and remedying a numerical-writing difficulty. *Teaching Exceptional Children*, *25*(3), 14-16.
- Baroody, A. J., & Lai, M. L. (2007). Preschoolers' understanding of the addition-subtraction inverse principle: A Taiwanese sample. *Mathematics Thinking and Learning*, *9*, 131-171. doi: 10.1080/10986060709336813

- Baroody, A. J., Lai, M. L., Li, X., & Baroody, A. E. (2009). Preschoolers' understanding of subtraction-related principles. *Mathematics Thinking and Learning, 11*, 41-60. doi: 10.1080/10986060802583956.
- Baroody, A. J., Lai, M.-L., & Mix, K. S. (2006). The development of young children's number and operation sense and its implications for early childhood education. In B. Spodek & O. Saracho (Eds.), *Handbook of research on the education of young children* (pp. 187-221). Mahwah, NJ: Erlbaum.
- Baroody, A. J., Purpura, D. J., Eiland, M. D., & Reid, E. E. (2014). Fostering first-graders' fluency with basic subtraction and larger addition combinations via computer-assisted instruction. *Cognition & Instruction, 32*, 159-197. doi: 10.1080/07370008.2014.887084
- Baroody, A. J., Purpura, D. J., Eiland, M. D., & Reid, E. E. (2015). The impact of highly and minimally guided discovery instruction on promoting the learning of reasoning strategies for basic add-1 and doubles combinations. *Early Childhood Research Quarterly, 30*, 93-105. doi: 10.1016/j.ecresq.2014.09.003
- Baroody, A. J., Purpura, D. J., Eiland, M. D., Reid, E. E., & Paliwal, V. (in press). Does fostering reasoning strategies for relatively difficult basic combinations promote transfer by K-3 students? *Journal of Educational Psychology*.
- Baroody, A. J., Torbeyns, J., & Verschaffel, L. (2009). Young children's understanding and application of subtraction-related principles: Introduction. *Mathematics Thinking and Learning, 11*, 2-9. doi: 10.1080/10986060802583873
- Baroody, A. J., & Varma, S. (2006, November). *The active construction view of basic number fact knowledge: New directions for cognitive neuroscience*. Invited presentation at the Neural Basis of Mathematical Cognition Conference, Nashville, TN.
- Baroody, A. J., Wilkins, J. L. M., & Tiilikainen, S. (2003). The development of children's understanding of additive commutativity: From protoquantitative concept to general concept? In A. J. Baroody & A. Dowker (Eds.), *The development of arithmetic concepts and skills: Constructing adaptive expertise* (pp. 127-160). Mahwah, NJ: Erlbaum.
- Bloom, P., & Wynn, K. (1997). Linguistic cues in the acquisition of number words. *Journal of Child Language, 24*, 511-533. doi: 10.1017/S0305000997003188
- Briars, D. J., & Larkin, J. G. (1984). An integrated model of skills in solving elementary word problems. *Cognition and Instruction, 1*, 245-296. doi: 10.1207/s1532690xci0103_1
- Campbell, J. I. D. (2008). Subtraction by addition. *Memory & Cognition, 36*, 1094-1102. doi: 10.3758/MC.36.6.1094
- Canobi, K. H. (2004). Individual differences in children's addition and subtraction knowledge. *Cognitive Development, 19*, 81-93.

- Canobi, K. H. (2005). Children's profiles of addition and subtraction understanding. *Journal of Experimental Child Psychology, 92*, 220-246.
- Canobi, K. H. (2009). Conceptual-procedural interactions in children's addition and subtraction. *Journal of Experimental Child Psychology, 102*, 131-149. doi: 10.1016/j.jecp.2008.07.008
- Canobi, K. H., Reeve, R. A., & Patterson, P. E. (1998). The role of conceptual understanding in children's problem solving. *Developmental Psychology, 34*, 882-891. doi: 10.1037/0012-1649.34.5.882
- Carroll, W. M., & Issacs, A. (2003). Achievement of students using the University of Chicago School Mathematics Project's Everyday Mathematics. In S. L. Senk & D. R. Thompson (Eds.), *Standards-based school mathematics curricula: What are they? What do students learn?* (pp. 79-108). Mahwah, NJ: Erlbaum
- Council of Chief State School Officers (2010). *Common core state standards: Preparing America's students for college and career*. Retrieved from <http://www.corestandards.org/>.
- Dowker, A. D. (2003). Young children's estimates for addition: The zone of partial knowledge and understanding. In A. J. Baroody & A. Dowker (Eds.), *The development of arithmetic concepts and skills: Constructing adaptive expertise* (pp. 35-74). Mahwah, NJ: Erlbaum.
- Dowker, A. (2009). Use of derived fact strategies by children with mathematical difficulties. *Cognitive Development, 24*, 401-410. doi: 10.1016/j.cogdev.2009.09.005
- Durkin, K., & Rittle-Johnson, B. (2012). The effectiveness of using incorrect examples to support learning about decimal magnitude. *Learning and Instruction, 22*, 206-214. doi: 10.1016/j.learninstruc.2011.11.001
- Dyson, N., Jordan, N., & Glutting, J. (2013). A number sense intervention for low-income kindergartners at risk for mathematics difficulties. *Journal of Learning Disabilities, 46*, 166-181. doi: 10.1177/0022219411410233
- Dyson, N., Jordan, N. C., & Hassinger-Das, B. (in press). Kindergarten number operations: Connecting the dots. *Teaching Children Mathematics*.
- Eiland, M. D. (2014). *Fostering fluency with basic addition and subtraction facts using computer-aided instruction* (Unpublished doctoral dissertation). https://www.ideals.illinois.edu/bitstream/handle/2142/50759/Michael_Eiland.pdf?sequence=1
- Ernest, P. (1985). The number line as a teaching aid. *Educational Studies in Mathematics, 16*, 411-424.
- Fayol, M., & Thevenot, C. (2012). The use of procedural knowledge in simple addition and subtraction. *Cognition, 123*, 392-403. doi:10.1016/j.cognition.2012.02.008

- Frye, D., Baroody, A. J., Burchinal, M. R., Carver, S., Jordan, N. C., & McDowell, J. (2013). *Teaching math to young children: A practice guide*. Washington, DC: National Center for Education Evaluation and Regional Assistance (NCEE), Institute of Education Sciences, US Department of Education.
- Fuson, K. C. (2006). *Math expressions*. Boston, MA: Houghton Mifflin.
- Fuson, K. C. (2009). Avoiding misinterpretations of Piaget and Vygotsky: Mathematical teaching without learning, learning without teaching, or helpful learning-path teaching. *Cognitive Development*, 24, 343-361. doi: 10.1016/j.cogdev.2009.09.009
- Hattikudur, S., & Alibali, M. W. (2010). Learning about the equal sign: Does comparing with inequality symbols help? *Journal of Experimental Child Psychology*, 107, 15-30. doi: 10.1016/j.jecp.2010.03.004
- Houghton Mifflin Harcourt (2015). *Go math!* Boston, MA: Author.
- Huttenlocher, J., Jordan, N. C., & Levine, S. C. (1994). A mental model for early arithmetic. *Journal of Experimental Psychology: General*, 123, 284-296.
- James, W. (1958). *Talks to teachers on psychology: And to students on some of life's ideals*. New York, NY: Norton & Company.
- Jordan, N. C., Huttenlocher, J., & Levine, S. C. (1992). Differential calculation abilities in young children from middle- and low-income families. *Developmental Psychology*, 28, 644-653. doi: 10.1037/0012-1649.28.4.644
- Jordan, N. C., Huttenlocher, J., & Levine, S. C. (1994). Assessing early arithmetic abilities: Effects of verbal and nonverbal response types on the calculation performance of middle- and low-income children. *Learning and Individual Differences*, 6, 413-432. doi: 10.1016/1041-6080(94)90003-5
- Jordan, N. C., Glutting, J., Dyson, N., Hassinger-Das, B., & Irwin, C. (2012, April). *Building kindergartners' number sense: A randomized controlled study*. Paper presented at the annual meeting of the National Council of Teachers of Mathematics, Philadelphia.
- Klein, J. S., & Bisanz, J. (2000). Preschoolers doing arithmetic: The concepts are willing but the working memory is weak. *Canadian Journal of Experimental Psychology*, 54, 105-114. doi: 10.1037/h0087333
- Larson, N. (2008). *Saxon math 1*. Austin, TX: Harcourt Achieve. Macmillan/McGraw Hill.
- Macmillan/McGraw Hill (2009). *Math Connects*. New York, NY: Author
- Mix, K. S. (2009). How Spencer made number: First uses of the number words. *Journal of Experimental Child Psychology*, 102, 427-444. doi: 10.1016/j.jecp.2008.11.003
- National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics: Standards 2000*. Reston, VA: Author.

- National Mathematics Advisory Panel (2008). *Foundations for success: The final report of the National Mathematics Advisory Panel*. Washington, DC: US Department of Education.
- National Research Council (2001). *Adding it up: Helping children learn mathematics*. Washington, DC: National Academy Press.
- Nunes, T., Bryant, P., Hallett, D., Bell, D., & Evans, D. (2009). Teaching children about the inverse relation between addition and subtraction. *Mathematical Thinking & Learning, 11*, 61-78.
- Palmer, A., & Baroody, A. J. (2011). Blake's development of the number words "one," "two," and "three." *Cognition and Instruction, 29*, 265-296. doi: 10.1080/07370008.2011.583370
- Peters, G., De Smedt, B., Torbeyns, J., Ghesquire, P., & Verschaffel, L. (2010). Adults' use of subtraction by addition. *Acta Psychologica, 135*, 323-329. doi: 10.1016/j.actpsy.2010.08.007
- Piaget, J. (1964). Development and learning. In R. E. Ripple & V. N. Rockcastle (Eds.), *Piaget rediscovered* (pp. 7-20). Ithaca, NY: Cornell University.
- Piaget, J. (1965). *The child's conception of number*. New York, NY: Norton.
- Putnam, R. T., deBettencourt, L. U., & Leinhardt, G. (1990). Understanding of derived fact strategies in addition and subtraction. *Cognition and Instruction, 7*, 245-285. doi: 10.1207/s1532690xci0703_3
- Rathmell, E. C. (1978). Using thinking strategies to teach basic facts. In M. N. Suydam & R. E. Reys (Eds.), *Developing computational skills* (pp. 13-50). Reston, VA: National Council of Teachers of Mathematics.
- Riley, M. S., Greeno, J. G., & Heller, J. I. (1983). Development of children's problem solving ability in arithmetic. In H. P. Ginsburg (Ed.), *The Development of mathematical thinking* (pp. 153-196). New York, NY: Academic Press.
- Rittle-Johnson, B., & Star, J. R. (2007). Does comparing solution methods facilitate conceptual and procedural knowledge? An experimental study on learning to solve equations. *Journal of Educational Psychology, 99*, 561-574. doi: 10.1037/0022-0663.99.3.561
- Rittle-Johnson, B., & Star, J. R. (2011). The power of comparison in learning and instruction: Learning outcomes supported by different types of comparisons. In J. P. Mestre & B. H. Ross (Eds.), *The psychology of learning and motivation: Cognition in education* (Vol. 55, pp. 199-225). Oxford, United Kingdom: Academic Press.
- Sarama, J., & Clements, D. H. (2009). *Early childhood mathematics education research: Learning trajectories for young children*. New York, NY: Routledge.

- Sarnecka, B. W., & Carey, S. (2008). How counting represents number: What children must learn and when they learn it. *Cognition*, *108*(3), 662-674. doi: 10.1016/j.cognition.2008.05.007
- Saxe, G. B., Earnest, D., Sitabkhan, Y., Haldar, L. C., Lewis, K. E., & Zheng, Y. (2010). Supporting generative thinking about integers on number lines in elementary mathematics, *Cognition & Instruction*, *28*, 433-474. doi: 10.1080/07370008.2010.511569
- Scott Foresman-Addison Wesley (2005). *Mathematics (first grade)*. Pearson Education. Upper Saddle River, NJ: Author.
- Sophian, C., & McCorgay, P. (1994). Part-whole knowledge and early arithmetic problem-solving. *Cognition and Instruction*, *12*, 3-33. doi: 10.1207/s1532690xcil201_1
- Sophian, C., & Vong, K. I. (1995). The parts and wholes of arithmetic story problems: Developing knowledge in the preschool years. *Cognition and Instruction*, *13*, 469-477. doi: 10.1207/s1532690xcil303_5
- Starkey, P., & Gelman, R. (1982). The development of addition and subtraction abilities prior to formal schooling in arithmetic. In T. P. Carpenter, J. M. Moser, & T. A. Romberg (Eds.), *Addition and subtraction: A cognitive perspective* (pp. 99-116). Hillsdale, NJ: Erlbaum.
- Steinberg, R. M. (1985). Instruction on derived facts strategies in addition and subtraction. *Journal for Research in Mathematics Education*, *16*, 337-355. doi: 10.2307/749356
- Thompson, I. (1997). The role of counting in derived fact strategies. In I. Thompson (Ed.), *Teaching and learning early number* (pp. 52-61). Buckingham, United Kingdom: Open University Press.
- Thornton, C. A. (1990). Solution strategies: Subtraction number facts. *Educational Studies in Mathematics*, *21*, 241-263. doi: 10.1007/BF00305092
- Thornton, C., & Toohey, M. (1985). Basic math facts: Guidelines for teaching and learning. *Learning Disabilities Focus*, *1*, 44-57. doi: 10.2307/748382
- University of Chicago School Mathematics Project (UCSMP) (2005). *Everyday mathematics teacher's lesson guide* (Volume 1). Columbus, OH: McGraw-Hill
- Van den Heuvel-Panhuizen, M., & Treffers, A. (2009). Mathe-didactical reflections on young children's understanding and application of subtraction-related principles. *Mathematical Thinking and Learning*, *11*, 102-112. doi: 10.1080/10986060802584046
- Verschaffel, L., Greer, B., & De Corte, E. (2007). Whole number concepts and operations. In F. K. Lester Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 557-628). Charlotte, NC: NCTM & Information Age Publishing.

- Walker, D., Bajic, D., Kwak, J., & Rickard, T. C. (2014). Specificity of children's arithmetic learning. *Journal of Experimental Child Psychology*, *122*, 62-74. doi: 10.1016/j.jecp.2013.11.018
- Walker, D., Mickes, L., Bajic, D., Nailon, C. R., & Rickard, T. C. (2013). A test of two methods of arithmetic fluency training and implications for educational practice. *Journal of Applied Research in Memory and Cognition*, *2*, 25-32. doi: 10.1016/j.jarmac.2013.0

Arthur J. Baroody
University of Illinois at Urbana-
Champaign and the University of Denver
baroody@illinois.edu