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## ***Khipu* UR19: Inca Measurements of the Moon's Diameter and its Distance from the Earth**

### ***Quipu* UR19: Mediciones realizadas por los Incas del diámetro de la Luna y de la distancia que nos separa de la Luna**

Alberto Saez-Rodriguez<sup>1</sup>

#### **Abstract**

This study explores the functions of *tukapu* as possibly important elements that are directly involved in the *khipu* construction process, i.e., when used by *khipukamayusqs* as a useful rudimentary outline for organising ideas, points, and details when constructing their forthcoming *khipu*. My re-study of the *khipu* sample UR19, which was previously investigated by Urton (2003) for other purposes, involved using the additional pieces of information that were identified in the graphical representations of the organisation of three different knot types that were tied into pendant and subsidiary strings. Proceeding one step further, we can sketch out a structure for the various meanings that are contained in these devices, which takes us much closer to developing an approach that could help us attempt to decipher the *khipu*. The irrational number  $\pi$  (first used by Archimedes and Liu Hui in the third century B.C.), the diameter of the Moon, and its distance from Earth (first measured by Aristarchus in the third century B.C.) could account for the observed results.

**Keywords:** Inca *khipu*; *Tukapu*; *Khipu* construction; Subsidiary strings; Pi value; Moon's diameter.

#### **Resumen**

En este estudio se explora el papel desempeñado por el *tukapu* como un elemento importante que haya posiblemente estado directamente implicado en la fase preliminar del proceso de construcción del *quipu*, es decir, que haya sido utilizado por los *khipukamayusqs* como un croquis primitivo o en calidad de plan de trabajo para organizar las ideas que pudieran servir de base para la confección del futuro *quipu*. El presente estudio de la muestra UR19, que ha sido previamente investigada por Urton (2003) por diferentes razones, comprende el uso de la representación gráfica en la organización de los tres tipos de nudos tejidos en los cordeles del *quipu*, ya sean colgantes o subsidiarios. De tal manera, avanzando un paso más en el estudio del *quipu*, ello nos permitirá estructurar los posibles significados encontrados en la muestra de *quipu* para poderlos aprovechar en futuros intentos de interpretación o desciframiento. El valor del número irracional Pi (usado por primera vez por Arquímedes y Liu Hui en el siglo III a.C.), así como la medida del diámetro de la Luna y de la distancia entre la Tierra y la Luna (calculado por primera vez por Aristarco de Samos en el siglo III a.C.) se encuentran entre los resultados más importantes de este trabajo de investigación.

**Palabras clave:** *Quipu* inca; *Tukapu*; Construcción del *quipu*; Cuerdas subsidiarias; Valor de Pi; Diámetro de la Luna.

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## RELATED LITERATURE AND CONCEPTUAL FRAMEWORK

According to Spanish chronicles and documents that were recorded throughout the first few decades of the establishment of Spanish rule in Peru (1532), *kipu* (also spelled *quipu* or *quipo*, which is a term in Quechua, the native Inca language) records were kept of censuses, tributes assessed and performed, goods stored in the Inca storehouses (*Collcacapac*, in Quechua), astronomical periodicities and calendrical calculations, royal genealogies, and historical events (see M. Asher, 1986, 1991; Murra, 1982; Salomon, 2004; Urton 2006, 2003, 2001). Blas Valera argued that even in his own time, the colonisers had remained essentially ignorant of how *kipu* (knot; to knot) functioned in record keeping (for a study of Blas Valera's ideas about and commentary on *kipu*, see Laurencich Minelli, 1999; Cantù, 2001; Hyland, 2002; Laurencich-Minelli and Magli, 2009). To construct a *kipu*, the *kipu* maker was concerned mainly with how the information would be recorded on future *kipus*. Although calculations were performed using a *yupana* (Quechua for "counting tool"), the *kipukamayuc* faced the challenge of struggling with the decoding of the information, which is comparable to the challenge of designing and implementing an outline or plan. To appreciate certain characteristics of *kipu* construction, however, we have had to wait for the results of careful scientific study of museum samples. As Urton (2003) emphasised, the best way to approach studying *kipu* is through the description and careful study of variations and patterns in its construction. Although many people were working with weaving materials and skills (e.g., spinning, plying, and knotting threads) in every Inca community, (presumably) few specialists were using similar techniques to manipulate threads to produce knotted-string statistical records. Conklin (2002) first indicated the existence of knot directionality variations in the *kipu*. The spelling of Quechua words in this article usually follow the orthographic conventions that are used in Antonio Cusihuamán's *Diccionario Quechua Cuzco-Collao* (1976). Rather than the Hispanicised forms *quipu* and *quipucamayoq*, I use the phonologically more accurate forms: *kipu* and *kipukamayuc* (see Urton, 2003).

In this paper, we examine the *kipu* analyses as non-graphic and two-dimensional (instead, the *kipu* was three-dimensional and tactile) provided that this approach can also help us to identify a connection between *kipu* and *tukapu* (but not absolutely), a long-standing

objective of Inca studies. According to Brokaw (2003; my emphasis), the Inca might have imposed specific features of their ethnic dress for a reason other than facilitating ethnic identification; instead, Urton (2003, p. 3) raised the highly speculative question as to whether they could possibly provide “cues” to aid the Inca administrator, who made each specific sample in such a way as to recall a specific body of memorised information.

**Tukapu: A rough sketch with no numerical values.**

*Tukapu* (see Figure 9) are elaborately woven tapestry designs that contain geometric figures that are enclosed by rectangles or squares. The small squares contain many heraldic-like geometric designs, which constitute the whole surface pattern of the highest status Inca tunics. Mary Frame (2001, p.135) makes the following observations and suggestions:

Just as quipu cords can combine even or odd numbers at each plying stage, the t'oqapu contain number attributes range up to nine, and some multiples beyond, through spatial divisions. Inversions in color, value, and direction suggest that meaning in t'oqapu is often couched in binary oppositions, a basic principle in fabric structure imagery.

The basic argument that I present here began to take shape in a research study that was entitled “An Ethnomathematics Exercise for Analysing a *Khipu* Sample from Pachacamac” (Saez-Rodriguez, 2012). To state that the Inca did not produce a graphic script does not imply that they could not produce geometrical two-dimensional representations and other images, and they could do so rather sketchily, as observed in their woven geometrical *tukapu* designs (Arellano, 1999; Rowe, 1997; Micelli & Crespo, 2011). What use would producing two-dimensional representations have served? What form might these conventionalised recording units (the *tukapu*) have taken? If the *tukapu* was an empty physical schema onto which ideas were projected, why would the *khipukamayuyq* have needed or wanted to construct such objects and have stored them away for later referral into *khipu*? Such a practice appears to be reasonable because the record keepers could not have accomplished the same ends with a purely mental image, recovering lost or forgotten information from it. Although the colours were considered within an overall geometrical framework, a significant modification and expansion of the coding units was required to

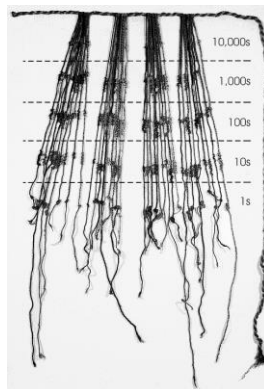
symbolise these components (geometrical two-dimensional representations) of the *tukapu* information system.

**The Inca concept of zero: A symbol of nothing and a real number**

In their description of Inca numbers (i.e., using Hindu-Arabic numerals), the Spanish chroniclers do not mention the Inca zero (*ch'usah* or *mana imapas*, which can be translated as empty or nothing). One mathematical concern that could potentially hold great interest and importance for numeracy studies is whether the Incas would have developed traditions of viewing numerical signs (e.g., 0, 1, 2, 3...) as having iconic or symbolic values.



**Figure 1.** *Khipukamayuuq* with *khipu* and *yupana*, drawn by Felipe Guaman Poma de Ayala. Photograph courtesy of the Royal Library, Copenhagen (Nueva crónica y buen gobierno, fol. 360).



**Figure 2.** Decimal hierarchical organization of knots (Museum für Völkerkunde, Berlin, VA 47083). (Source: Urton, 2003)

Although the Incas used a base ten counting system, when we closely analyse ideas about natural numbers in Quechua, we find an overall organisation in these numbers that is similar to that in Western numbering systems, i.e., “one, two, three...” or [in Quechua]

“*huk, iskay, kinsa, ... = 1, 2, 3...*”. Although knot makers/keepers used the concept of zero to denote an empty place, i.e., the absence of a knot in an appropriate position (as in the schematic *khipu* diagram shown in Figure 7 and Tables 1 and 2, the Inca record-keeping system does not include a visualised symbol for zero. However, records on such matters (i.e., to describe the position of an object in the sky, see especially Saez-Rodriguez, 2012) show that Inca astronomers appeared to conceive of ways of representing zero in positional astronomy and time. In this specific sample, when a representation of nothing stands by itself on a pendant string (e.g., the absence of a long knot in the appropriate position), the absence of a knot on a string is transcribed as 0. Nonetheless, in the case of the Subsidiary String no. Z-6s1 (an empty string), there is a missing factor in the multiplication. To clarify the information that is presented in Figure 7, the column on the right-hand side of the figure, under the heading “1/10”, contains values that represent “factors” or “multiplicands”, as shown in the right-hand column, under the heading “× Subsidiary”. However, it is precisely this last question that leads me to a major point that I want to make with respect to Subsidiary no. 6s1, which is an unknotted pendant string, which represents a partial inscription (see Urton, 2003, p. 42). More precisely, Subsidiary no. 6s1 is not equal to zero as a whole number. Urton (2003, p. 103) examined the question of whether the binary reading of *khipu* knots is incompatible with interpreting anomalous *khipu* or other analyses that argue for patterning of information using repeated number sequences (e.g., M. Ascher 1991, 1986; Ascher and Ascher, 1997; Pereyra, 1996), fraction or ratio representations (Ascher and Ascher, 1997, pp. 143–151), or numbers that are arranged in calendrically significant patterns (e.g., Nordenskiöld, 1925a, 1925b; Urton, 2001; Zuidema, 1989).

We should now ask the following: Have we conducted our analysis and theorising of *khipu* signs as far as we presently can? I want to introduce and explore *tukapu* as a preliminary phase of *khipu* construction. However, Spanish writers differ on the Spanish equivalents for Incan units of measure; thus, it is uncertain how precise the Incan units truly were. Rowe (1946, p. 323) stated that the units of length were based on the human body, e.g., a fathom (64 inches). As Figure 4 shows, the Incan astrologer (see Guaman Poma, 1980 [1615], p. 829 [883]) is depicted as a barefoot man holding a measuring stick (*cqta-k'aspi*, in

Quechua) in his right hand and a *kipu* in his left hand. Originally based on the distance between a man's outstretched arms, this length (i.e., a fathom) was kept as a legal check (González Holguín, 1608, pp. 117, 127, 315, 326, and 373; see also Urton, 1984, p. 37).

## RESEARCH QUESTIONS AND HYPOTHESES

According to my main hypothesis, the *kipukamayus* (see Figure 1) might have used drawings, diagrams or jottings (notes) as complementary techniques (see Figures 7 and 8 and Table 3) to effectively handle large data sets (e.g., accounting, census information and astronomical calculations). I hypothesise that handling this tremendous amount of data and putting this information into knotted strings should have demanded new tools and strategies in Inca times, before beginning *kipu* construction and even before creating the first knot. We need a textile diagram-like theory because, given our present lack of understanding of how full messages beyond numerical values (as magnitudes) were encoded and decoded in the *kipu*, we can have no sensible idea of how best to advance our studies if we do not have a general theory of the previous phase, i.e., before beginning *kipu* construction, to inform and direct our future research.

In a second hypothesis, I postulate that the *kipukamayus* could have developed an early *kipu* construction phase before material selection procedures (i.e., spinning and plying of fibers, attachment of strings to primary cords, and tying of knots into pendant strings). Such materials comprised textile diagrams that were worn by *kipukamayus* similar to *tukapus* and that contained fully coded information about their forthcoming *kipu* (see Figures 7, 8 and 10), e.g., components of a sum that could have held administrative, calendrical, or purely mathematical value for state record keepers.

These hypotheses concern procedures that went into the early *kipu* construction phases, even before creating the first knot. The process would have begun with textile diagrams before selecting the construction material. To test these hypotheses, I focus on analysing the numerical data that is registered on the *kipu* sample and examining the information that is contained on both sides of the vertical axis to formulate testable hypotheses. I then utilise conventional arithmetical tools. In this study, I examine three relevant research questions, as follows:

Q1. Where do we locate a model for conceptualising the organisation and meaning of knots in the *khipu* sample? This model must be consistent with the principle of binary organisation that is employed in constructing this device.

Q2. How did the Incas calculate the ratio of the circumference of a circle to its diameter?

Q3. How did the Inca astronomers calculate the distance between the Moon and the Earth?

The first research question investigates using models to visualise the three-dimensional and tactile system of *khipu* on a planar representation. I hypothesise that the *tukapu* was an empty physical schema onto which memories were projected when the *khipukamayuc* needed or wanted to construct such objects in the first place for his forthcoming *khipu*. The second research question explicitly considers the ways in which the Incas could divide the circumference (the distance around a circle) by the diameter and always get exactly the same number. The third question is more novel because it raises fundamental questions that pertain to the purpose and manner of using *khipu*. I address whether the numerical *khipu* indeed constituted the basis of recording astronomical calculations in the statistical (quantitative) *khipu*.

## **MATERIALS AND METHODS**

### **Khipu Sample Description**

The *khipu* sample UR19 (Museum Centro Mallqui, Leymebamba, Peru) has the Centro Mallqui catalog number CMA480/LCI-109.3. This sample is one of seven Chachapoya *khipus* that were found tied together and deposited in the burial chambers at Laguna de los Cóndores, northern Peru (Guillén, 1999; Urton, 2001; von Hagen, 2000). The sample comprises a main, or primary, cord, to which 13 pendant coloured strings are attached (See Table 1. The actual observed and recorded colours in the sample are provided in the colour column in the tabular description of this same *khipu* in Table 1. Moving down the body of the string from the point where the *khipu* pendant string attaches to the primary cord (see Tables 1, 2 and 4), one encounters single knots that are tied on successively lower levels, which indicates decreasing powers of ten; one then encounters a long knot, which signifies



units 2 to 9. This *kipu* has two single figure-eight knots on the lower parts of pendants nos. 4 and 7. Pendant strings have attached five secondary, or subsidiary, strings to them. This sample has an unusual (although not unique) vertical axis: to one side of the *kipu* sample (e.g., Pendants nos. 1 to 7), all of the knots are tied rightward (Z/S), except for the single figure-eight knot that is tied on Pendant no. 4, whereas all of the knots are tied leftward (S/Z) on the other side of the *kipu* (e.g., pendants nos. 7s1 to 13). Table 1 reproduces Urton's diagram but has added elements (e.g., material, Z-Spin/S-Ply, and decimal values). What does the *kipu* sample look like, i.e., how is it organised? How might we understand its significance and meaning, especially considering the organisation of the information? The guiding principle behind the proposal that the organising knots use the numeration decimal system is the same as in our own (decimal) system: any place can hold up to nine units (e.g., 99); when a value is higher, one moves to the next higher place value ( $99 + 1 = 100$ ). Using Urton's terminology, i.e., *recto/verso*, it is immediately apparent that in the *recto* and *verso* designations for any given string attachment, the *verso* is the "back" side and the *recto* the "front" side of the attachment.

Sample UR19 is a *kipu* from a collection of thirty-two *kipus* that were recently discovered in Chachapoyas, which is in northern Peru (Urton, 2001). According to Urton (2001), one often finds admixtures of woolen strings in the predominately cotton *kipus* that are found in Chachapoyas, with an overwhelming preference for the combination Z-spun/S-plied and a significant degree of variation in how the *kipu* pendant strings are attached to primary cords. Most of the *kipus* that have been studied in the collection from Chachapoyas have a "mixed *recto/verso*" pendant attachment pattern (Urton, 2001, p. 72).

Table 1 (adapted from Urton, 2003, p. 145) is based on a standardised recording format and coding symbol set for registering systematic observations of certain features in *kipu* samples. This information includes the following:

- (1) the *kipu* study number assignment (e.g., for Urton, this number bears the prefix UR19);
- (2) the name and location of the museum in which the sample is located and its catalog number;
- (3) the colour of the main (or primary) cord;
- (4) a table that shows the primary cord to each pendant or top string; a sequential numbering of the pendants and subsidiaries;
- (5) observations on each 'cord' (i.e., pendant, subsidiary,

or top string), which includes the (a) cord number; (b) the number of knots in each location (esp. decimal numeration level) on the cord; (c) the type of knot(s) on the cord (E = figure-eight knots, L = long knots, S = single knots); d) the total cord length; (e) the cord colour based on a standardised colour code; (f) a reading of all knots on the cord for their total decimal value; and (g) the number and location(s) of any subsidiaries that are attached to the cord.

Urton (2003, p. 153) has observed that all of these features (material, spin/ply, and decimal reading) in this sample (see Table 1) are coded in the same way (= 0).

### **Methodological Approach**

This paper describes a new approach for analysing the type of system of record keeping that is represented by the *khipu* sample. This new methodology, in which *tukapu* (see Table 2 and Figures 7 and 10) guides the creation of *khipu* through the description and careful study of variations and patterns in the construction of these devices, is the best way to approach the study of the *khipu*. One might presume that the *khipu* maker had a large, complicated body of information that was organised within a complex structural mental image and that the *tukapu* was an appropriate model to adopt for the “recording” system that is represented by the *khipu*, instead of pen, ink, and paper. We must thus approach studying *khipu* with the understanding that all of the structural and physical features interacted in the symbol-using processes of the *khipu* makers who consulted the *khipus*. If the *khipu* project was larger and more complicated, the *khipukamayuyq* would most likely have chosen to use a more formal, although rudimentary, outline for organising his ideas, points, and details and constructing his forthcoming *khipu*, because the record keepers could not have accomplished the same ends with a purely mental image.

This methodology thus gives us the opportunity and control to study the patterns and variations in *khipu* construction in new ways. We can proceed one step further; in doing so, we can sketch a structuring of the meanings in these devices, which allows us to become much closer to developing an approach that could help to decipher the *khipu* (see Figures 2 and 8).

Pendant Number	1	2	3	4	5	6	7	Decimal Value
↓								
1	o	o	o	?	—	—	—	—
2	o	o	o	•	—	—	—	—
3	o	o	o	•	o/o	•/•	o	10/8
3s1	o	⌘	o	—	o	o	o	9
4	o	o	o	•	•	o	o	1
5	o	o	o	o	—	—	—	—
6	o	•	o	?	o	o	o	10
6s1	o	o	o	—	—	—	—	—
7	o	o	o	o	o	o	o	1
7s1	o	•	o	—	o	•	o	2
8	o	o	o	•	—	—	—	—
9	o	⌘	o	•	o/o/o	•/o/•	o	10/10/8
9s1	o	o	o	—	o	o	o	3
10	o	o	o	•	o/o	•/o	o	10/10
10s1	o	⌘	o	—	o	•	o	8
11	o	o	o	•	—	—	—	—
12	o	•	o	•	—	—	—	—
13	o	•	o	o	—	—	—	—
								Total = 100

**Table 1.** Binary Coding of Khipu UR19 from Chachapoyas.

Note. The three colour/tone symbols have the following values:

- o = Red/Creator Rainbow or light hue (light wool/cotton colour tone)
- = Dark Rainbow or dark hue (dark wool/cotton colour tone)
- ⌘ = Bicolour (i.e., Red/Dark with light/dark)
- o/o = Two knots on the pendant (etc.)
- ? = Unable to determine the attachment direction

(Adapted from Urton, 2003, p. 130, with added elements).

In this study, we show how the Inca astronomers made measurements of astronomical distances, the size of the Moon and the distance to the Earth, and that they determined both quite accurately. According to the information that is shown in Table 3, we have a total of 18 mosaic pieces (13 pendant strings plus 5 subsidiaries), and we must arrange them to form a specific pattern. We can draw, for example, a rhombus (see Figures 7 and 10) that is composed of two triangles that are joined together along a side.

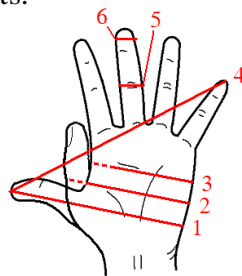
**Characterisations of the four-sided shape in the tukapu shown in Figure 10:**

Two pairs of opposite sides are equal in length.

Opposite angles are equal in measure.

Two pairs of opposite sides are parallel.

What did the *khipu* sample signs denote? The approach to interpreting the *khipu* sample sign system that is laid out in the present study (see Figures 7 and 8) appears to confirm that the string manipulations and their products might have constituted calculations on quantitative values that are recorded on this *khipu* (see Platt, 2002; Urton, 1998). As Figure 7 and Table 3 show, on the 5 subsidiary strings, with the exception of Subsidiary No. 6s1 (numerical value = 0), only long-knots (2 to 9 turns) are tied, i.e., 9, 2, 3, and 8 turns. Urton (2003, p. 91) clarified the distinction between a long knot and a figure-eight knot, noting that the *khipu* numbers were typically registered as completed counts that were broken into their constituent decimal-based units.



**Figure 3.** Three archaic hand units of measurement.

Courtesy: [http://fr.wikidia.org/wiki/Unit%C3%A9s\\_de\\_longueur\\_bas%C3%A9es\\_sur\\_la\\_main](http://fr.wikidia.org/wiki/Unit%C3%A9s_de_longueur_bas%C3%A9es_sur_la_main)

- 1: Manual foot (*pes manualis*; *makichaki*, in Quechua) = 1.524 m.
- 2: Hand (*maki*) = 4 inches = 10.16 cm
- 3: Palm (*k'apa*) = 3 inches = 7.62 cm
- 4: Span (*maki t'aqlla*) = 9 inches = 22.86 cm
- 5: Finger (1) a name for the middle finger (*chawpi ruk'ana*), a unit of distance equal to 3/4 inch = 19.05 mm
- 6: Finger (2) a traditional unit of distance equal to 2 nails or 4.5 inches = 11.43 centimeters.



**Figure 4.** An Inca astrologer (post-Conquest) holding a *cqta-k'aspi* (a measuring stick) in his right hand and a *khipu* in his left hand (Guaman Poma, 1980 [1615]: 829 [883]).

**RESULTS**

**Results considering fractional parts of real numbers recorded on the khipu sample**

A decimal number is a number that has integer and fraction parts, which are separated by a decimal point. Urton (2003, p. 87) raised the highly speculative question about the possibility that *khipu* could contain fractions: Could the ones values have represented decimals, the tens thus representing ones, and so on up the strings? Using this recording principle, in this specific sample, with the exception of the Subsidiary strings that have only long-knots (2 to 9 turns), i.e., 9, 2, 3, and 8 turns, we might read Pendant No. 9, which has the following knot quantities (from top to bottom), as the following: 10/10/8, as 2.8; Pendant No. 7 would represent 0.1; and so on.

Subsidiaries by Pendant strings	Data values are multiplied by their corresponding subsidiary strings**	Z-knots	S-knots
No. 3 by 3s1	$1.8 \times 9 = 16.2$	16.2	
No. 6 by 6s1	$0.1 \times (0) = 0.1^*$	0.1	
No. 7 by 7s1	$0.1 \times 2 = 0.2$		0.2
No. 9 by 9s1	$2.8 \times 3 = 8.4$		8.4
No. 10 by 10s1	$2 \times 8 = 16$		16
<b>Total</b>	<b>40.9</b>	<b>16.3</b>	<b>24.6</b>

**Table 2:** Bivariate plot determined by the distribution of pendant strings and subsidiaries.

*Note.* \* In this case, we do not have the zero-product property; instead, the decimal equivalent for 1/10 is 0.1, i.e., the 1 is in the tens place rather than the ones place, as tied on Pendant no. 6 (i.e.,  $1 \div 10 = 0.1$ ).

\*\* The 7 remaining strings (1, 2, 5, 8, 11, 12 and 13) have no numerical values.

The figure-eight knot tied on Pendant no. 4 (with the verso attachment mode to the primary cord) indeed appears to have conveyed some type of information (i.e., differences that resulted from the process of decision making), as is the case with the number sentence:  $1 \div 10 = 0.1$  (see Figure 7).

Z-Knots Strings (1 to 13)

S-Knots

Signatures	1	2	3	3s1	4	5	6	6s1	7	7s1	8	9	9s1	10	10s1	11	12	13
I	o	o	o	o	o	o	o	o	o	o	o	o	o	o	o	o	o	o
II	o	o	o	⌘	o	o	•	o	o	•	o	⌘	o	o	⌘	o	•	•
III	o	o	o	o	o	o	o	o	o	o	o	o	o	o	o	o	o	o
IV	?	•	•	—	•	o	?	—	o	—	•	•	—	•	—	•	•	o
V	—	—	o	o/o	•	—	o	—	o	o	—	o/o/o	o	o/o	o	—	—	—
VI	—	—	•/•	o	o	—	o	—	o	•	—	•/o/•	o	•/o	•	—	—	—
VII	—	—	o	o	o	—	o	—	o	o	—	o	o	o	o	—	—	—
Value	0	0	10/8	9	1	0	10	0	1	2	0	10/10/8	3	10/10	8	0	0	0
Spin/ply	Z/S/?	Z/S/v	Z/S/v	Z/S	Z/S/v	Z/S/r	Z/	Z/S	Z/S/r	S/Z	S/Z/v	S/Z/v	S/Z	S/Z/v	S/Z	S/Z/v	S/Z/v	S/Z/r
Knot/type	—	—	1S/L	L	E	—	-/?	—	E	L	—	2S/L	L	2S	L	—	—	—

**Table 3.** Binary Coding of Khipu UR19 from Chachapoyas (Data displayed in Table 1).  
(Drawn on the basis of data from Urton (2003, p. 130), with added elements).

*Note.* The three color/tone symbols have the following values:

- o = Red/Creator Rainbow or light hue (light wool/cotton color tone)
- = Dark Rainbow or dark hue (dark wool/cotton color tone)
- ⌘ = Bicolour (i.e., Red/Dark with light/dark)
- o/o = Two knots on pendant (etc.)
- ? = Unable to determine the attachment direction

**Results for long-knot values recorded on subsidiary strings: Properties of Addition and Multiplication**

The total sum of the five subsidiary string values (3s1, 6s1, 7s1, 9s1, 10s1), i.e., the Property of Addition, is the following:  $(9 + 0 + 0 + 2 + 3 + 8) = 22$ , which has a calendrical significance of the *huaca* (grave, in Quechua) numbers (Zuidema, 1989) in all four *suyus* (regions, in Quechua):  $(22 + 33 + 29 + 23) = (55 + 52)$ .

Focusing on the differences between the numerical values that are encoded in Z-knots, those calculated using the totals displayed in Table 3 (Pendant Nos. 1, 2, 3, 3s1, 4, 5, 6, 6s1, 7) and those from the S-knots, i.e., Pendant nos. 7s1, 8, 9, 9s1, 10, 10s1, 11, 12, 13 results in the following subtraction:  $(0+0+18+9+1+0+10+0+1=39) - (2+0+28+3+20+8+0+0+0=61)$ . Finally, we subtract one total from another to show the difference:  $61 - 39 = 22$ , which is the same as the total sum of the five subsidiary string values (see above).

According to the *Pachaquipu* (*khipu* of time), found in the *Exsul Immeritus Blas Valera Populo Suo* (see esp. Laurencich-Minelli and Magli, 2009), the period during which the Pleiades are invisible is reported to be 61 days. Curiously, we find a similar situation to that described above in the case of the total sum of the five subsidiary strings (i.e.,  $9 + 0 + 0 + 2 +$

3 + 8 = 22), which represents the same total sum as the differences between the S-knot string and those from the Z-knot string pairs (61 – 39 = 22). Conversely, the quantitative difference between 100 and 61, i.e., 39, could code for the Chachapoyas solar calendar, which belongs to early post-conquest times, i.e., (327 + 39) = (365 + 1). I should also point out that adding the long knots of pendants and subsidiary strings plus one figure-eight knot (8 + 9 + 1 + 2 + 8 + 3 + 8 = 39), their sum represents the same total sum of Z-knots: (0 + 0 + 18 + 9 + 1 + 0 + 10 + 0 + 1 = 39).

Museum identification: CMA-480/LC1-109.3 (Centro Mallqui, Leymebamba, Peru)

Material: Cotton; Main cord: Construction: Z-Spin/S-Ply; Color: W; Total length: 26.0 cm.

0.0 cm: beginning knot; space of 0.5 cm.

0.5 cm: group of 3 pendants (1–3); space of 1.0 cm.

2.5 cm: group of 4 pendants (4–7); space of 1.0 cm.

4.5 cm: group of 5 pendants (8–12); space of 2.0 cm.

7.5 cm: one pendant (13); space of 10.5 cm.

24.0 cm: end knot  $\phi$ ; space of 1.5 cm.

Pendant (spin/ply/attach.)	Knot (no./type/position/direction)	Length	Color	Value	Subsidiaries
1 (Z/S/?)	—	0.5b	AB	—	—
2 (Z/S/v)	—	16.0b	AB	—	—
3 (Z/S/v)	1S(10.5/Z); 8L(20.0/Z)	32.0 $\phi$	AB	18	1:5.0
3s1 (Z/S)	9L(13.5/Z)	31.0	KB:W	9	—
4 (Z/S/v)	1E(20.0/S)	39.5	MB 1	—	—
5 (Z/S/r)	—	21.0b	AB	—	—
6 (Z^/?)	1S(9.5/Z)	31.0	KB	10	1:2.0
6s1 (Z/S)	(untied knot)	32.0	W	—	—
7 (Z/S/r)	1E(12.0/Z)	18.5	MB	1	1:4.0
7s1 (S/Z)	2L(8.0/Z)	13.0	KB	2	—
8 (S/Z/v)	—	40.0 $\phi$	MB	—	—
9 (S/Z/v)	2S(9.0/Z); 8L(17.0/Z)	27.0	KB:W	28	1:4.5
9s1 (S/Z)	3L(11.5/Z)	17.0 $\phi$	W	3	—
10 (S/Z/v)	2S(9.0/Z)	33.0 $\phi$	MB	20	1:2.0
10s1 (S/Z)	8L(15.5/Z)	35.0	KB:W	8	—
11 (S/Z/v)	—	49.5 $\phi$	W	—	—
12 (S/Z/v)	—	29.0 $\phi$	BL	—	—
13 (S/Z/r)	—	36.0	BL	—	—

**Table 4.** Tabular Description of *Khipu* UR19 from Chachapoyas.

Note: Adapted from Urton (2003, pp. 180–181)

**Keys:**

*Main Cord Key*

W white

\$ beginning knot (twisted end of main cord)

$\phi$  end knot (on main cord and pendant strings)

*Pendant Key*

Z Z-spin or -ply

S S-spin or -ply

v verso attachment

r recto attachment

? unable to determine attachment direction

(Ex.: Z/S/v = Z-spun/S-plied/verso-attached pendant string)

*Knot Key*

- S single knot
- L long knot
- E figure-eight knot
- S S-knot
- Z Z-knot

**Results for the Inca method of calculating Pi**

The circle was considered by the ancient Incas to be the perfect shape. With regard to the general issue of the method of calculating  $\pi$ , I remind the reader that the calculation of  $\pi$  was revolutionised by the development of infinite series techniques in the 16th and 17th centuries. According to the concept suggested by Urton (2003, p. 159) and those published elsewhere (Saez Rodriguez, 2012), we can assume that the *khipu* recorded information (see Table 2) in a conventionalised manner, i.e., multiplying a pendant's value by its corresponding subsidiary.

Following that logic, the total sum is  $(9 \times 1.8) + (0 \times 1) + (8 \times 2) + (2 \times 0.1) + (3 \times 2.8) = 40.9$ . We can also calculate the total sum of the knot values on all 5 strings, including the subsidiaries that are attached to the cord (i.e., Nos. 8, 7s1, 7, 6s1, and 6), as follows:  $0 + 2 + 1 + 0 + 10 = 13$ . Using the number of arrangements that are in the calendrically significant patterns displayed in Figure 7 and Table 2, we can test this formula by knowing the circumference and diameter. Dividing  $C$  by  $d$ , our quotient should come close to Pi.

To clarify this alternative between a long knot and a figure-eight knot, Urton (2003, p. 91) noted that *khipu* numbers were typically registered as completed counts that were broken down into their constituent decimal-based units. This arrangement occurs on Pendant No. 4, where the figure-eight knot was tied onto a string (Spin/ply: Z/S/v) and no subsidiary strings were added.

We know that  $\frac{C}{d} = \pi$ , where  $C$  is the circumference and  $d$  is the diameter; thus,

$$\frac{40.9}{13} \approx 3.14\dots$$

or

$$(16.2+16+8.4+0+0.2+0.1) \div 13 \approx 3.14\dots$$

(which represents the first three digits in the approximation of Pi).

$\pi$

To identify the centre of a circle, we draw a line that passes through the centre point and cuts the circle on either end. This line is the diameter of the circle. Apparently, the Inca found that the mathematical constant Pi ( $\sim 3.14$ ). The calculations performed by the Inca astronomers were, thus, very accuracy. We also know that one foot is 30.48 centimeters, and one inch is



2.54 centimeters. According to Malpass (1996), 5 feet 4 inches converts to 1.6256 meters, i.e., a fathom (the standard measurement of land, *rikra*, in Quechua; see Figure 4) is equivalent to ~ 64 inches or 1.6256 meters (Rowe, 1946, pp. 323–324). There are 0.0254 metres in one inch. Therefore, 64 inches is equal to  $64 \times 0.0254 = 1.6256$  metres (a fathom). As proof of their great skill, another way to express such a relationships is the following:

$40.9 \times 13 = 531.7 \div 1.6256 \approx 327.079$ , which could code for a lunar year. ●

In other words, the result is an arrangement of numbers in calendrically significant patterns (e.g., Nordenskiöld 1925a, 1925b). In this specific arithmetic exercise, a lunar year, a value of ~ 327 days (see Saez Rodriguez, 2012), is equivalent to the circumference ( $C$ ) of the circle (= 40.9) 13 times its diameter (= 531.7), and this total is then divided by 1.6256 (i.e., the meters unit number 1.6256 m converts to 1 ftm, one fathom). Because the ratio of a circle's diameter and its circumference is the same for all of the circles, the limits are not involved in proving that  $\pi(C)$  is independent of circle  $C$ . In one particularly complex context, Zuidema (2004, p. 81) assigned to each *huaca* one degree in a circle of  $365^\circ$  (and thus not of  $327^\circ$ ). Zuidema (2004, p. 81) was able to reconstruct with confidence the precise calendar, with its sequence of 328 days. The issue in question is whether or not the Inca method of calculating Pi is compatible with the interpretation of astronomical *kipu* that is outlined in the present study. Although it is widely believed that the Inca *kipus* contain calendrical notations, there are three possible explanations: (a) that  $20 \times 20$  synodic months were intended, but the resulting 409 Synodic months (instead of 40.9) do not appear to have any significant relationship to the other numbers that are involved in sample UR19; (b) that they counted to twenty times twenty and added an extra month, just as they added an extra month to 36 months to make 37; or (c) that instead of  $20 \times 20$ , it might have been  $20 + 20$ , because 40 sidereal lunar months would equal  $3 \times$  solar years, which in turn would equal the 37 synodic months of the Incan calendar. (The real figure here should not be  $20 \times 20 = 400$ , but instead  $(20 + 20) + 0.9 = 40.9$ .) I remind the reader that long knots display a maximum of nine turns. According to Nordenskiöld (1925b), an extra month was inserted at the end of every three lunar years; as a result, they counted two  $\times$  lunar years of 12 months and then one of 13 months, where the thirteenth month was the extra month.

As the above formulas indicate, 327 (a lunar year) is the result of  $\frac{40.9 \times 13}{1.6256} \approx 327.07$ , i.e., a fathom (the largest measurement based on the human body, *rikra*, in Quechua, *braza*, in Spanish; see González Holguín, 1608) was equivalent to 1.6256 meters, as calculated today.

As Figure 3 shows, there are 0.1016 m (or 10.16 cm) in one hand (*k'apa*, in Quechua). Therefore, 16 hands is equivalent to  $16 \times 0.1016 = 1.6256$  metres (i.e., a fathom).

### Results for the necessity of handling large numbers

Extremely large numbers commonly appear in astronomy, which is one reason that astronomers and other scientists use scientific notation when handling large numbers. With regard to the general issue of metric number powers of 10, I remind the reader that "*myria*" (an obsolete metric unit) has been used during the first half of the nineteenth century as a metric prefix for 10,000, e.g., 10 kilometres = 1 *myriametre*. In this study, for the units to measure distance in Incan Astronomy, e.g.,  $1.6265 \times 1,000 = 1,626.5$ ;  $1.6265 \times 10,000 = 16,265$ , I have designated as one *kilo-rikra* and one *myria-rikra* (i.e., one *kilo-rikra* is equal to 1,000 ftm, and one *myria-rikra* is equal to 10,000 ftm). Because large numbers have an intimate connection with Incan Astronomy, we can thus have 1 *kilo-rikra* = 1,626.5 *kr* and 1 *myria-rikra* = 16,265 *mr* for the Incan astronomical units of measurement. This approach would have made it easier for the *kipukamayuk* to tie the knots in their appropriate place value. This Incan notation would have been a system for working with numbers that makes it easier to handle numbers that are too large to knot on a string. Because such notation relies on powers of ten, it is simple to convert a number from astronomical to standard notation. As Figure 6 shows, to convert a large number (with a positive power) from scientific notation to standard notation, we first identify the decimal point, and then, we shift the decimal to the right by the number indicated by the power. As Tables 2 and 4 and Figure 7 show, there are no S-knots tied on the pendant strings 11, 12 and 13 in the left-hand quadrant. The same circumstance occurs in the left-hand quadrant, where no Z-knots are tied on the pendant strings 1, 2 and 5 in the right-hand quadrant. As Table 3 shows, no knots are tied into eight pendant strings in the sample, i.e., 1, 2, 5, 6s1, 8, 11, 12, and 13, of which only one (string no. 6s1) is a subsidiary string. Assembled on both sides of a perpendicular axis, string nos. 1, 2, 5, and 6s1 are tied as Z-knots, while string nos. 8, 11, 12, and 13 are tied as S-knots.

The question that confronts us is thus as follows: What could the absence of knots on strings nos. 1, 2, 5, 6s1, 8, 11, 12, and 13 have to do with the type of construction that is considered here, i.e., the distance from the Earth to the Moon? What do these empty spaces mean? Do these empty spaces have any physical meaning or purpose? It is especially relevant to the issues that we are concerned with in this study to note that strings with the nos. 1, 2, 5, 6s1, 8, 11, 12, and 13 were attached to the primary cord on *Khipu* UR19 as "completed strings" in either the recto or the verso fashion, which accounts for any spacing between the strings or

groups of strings that the *kipukamayuk* might have wanted to maintain as a way of classifying the information in the *kipu*. Furthermore, the strings have different lengths.

The ancient Inca astronomers used naked eye observations and systematically recorded data over time, and they used the observed patterns to make accurate and precise climate predictions. In terms of the timing of the knotting of the strings, it is reasonable to conjecture that the *kipukamayuks* needed to construct an empty physical schema onto which data observed from astronomical sources were projected. Furthermore, it is reasonable to propose that such a (hypothetical) procedure could provide us with a potentially powerful strategy for charting out construction techniques that are involved in the fabrication of empty strings for later retrieval by trained *kipukamayuk*. Urton (2003, p. 42) has argued that a spun, plied, dyed, but unknotted pendant string (i.e., an empty string) represented a partial inscription (2003, p. 42). I thus suggest that an apt characterisation of what we are studying in *kipu* sample UR19 is that these values constitute an important component of calculating the distance from the Earth to the Moon. I thus wonder whether these empty strings (i.e., string nos. 1, 2, 5, 8, 11, 12, and 13) could have been ‘read’ in a system that allowed the Inca to grow their numbers exponentially, although we do not yet know how to interpret or read most of the information that is presumably encoded in the recording system that is described and analysed in this article. We now know how to derive thousands and even potentially millions (see Figures 6 and 7) of such astronomical data.

### **Results for Incan measurements of the diameter of the Moon and its distance.**

How large is the Moon? The history of ancient astronomy must be treated jointly with the history of astrology, whose contribution to the history of science has often been underestimated. The choice of a specific geometrical shape (e.g., a rhombus) for the forthcoming astrological *kipu* that was made by the Inca *kipukamayuk* in the woven geometrical *tuqapu* design shown in Figure 10 might have referred to *Wenu Mapu*, which is a word from the vocabulary of Mapudungun that signifies Sky and in Andean Cosmology represents a two-dimensional rhombus (see Micelli & Crespo, 2011, p. 15).

How did the Inca astronomers measure the distance from the Earth to the Moon? By considering the properties of multiplication with the long-knot values recorded on the subsidiary strings (see Table 2), it is reasonable to propose that the Inca astronomers might have found (according to the perspective proposed here) that the Earth’s shadow was ~2.5 Moon diameters wide at the Moon. It is a curious fact (see Tables 2 and 3 and Figure 7) that the sum of the Z-knots ( $16.2 + 0.1 = 16.3$ ) indicates the following ratio:

$$(9 \times 1.8 + 0.1) = 16.3 \times (2.5) = 40.75 \Leftrightarrow \frac{40.9}{16.3} = 2.5092\dots$$

As Figure 5 shows, the Earth's shadow is  $\sim 2.5$  Moon diameters wide at the Moon, thus  $40.9 \div 16.3 \approx 2.5$  (i.e., the ratio of the Moon's size to that of the Earth's shadow as it passes through), which is correct to two decimal places. From the material that is presented in this sample, I argue that there are weird similarities between the Inca astronomers' findings and the calculations that are used by the ancient Greeks ca. 250 B.C. for the size of the Moon as well as for its distance from the Earth. This determination of the relative sizes of the Earth and Moon predated the estimate of the absolute size of the Earth due to Eratosthenes and was first carried out by Aristarchus of Samos (310-230 BC).

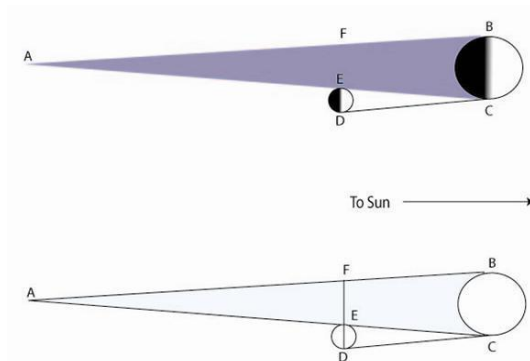
My hypothesis, as constructed by analogy with the conceptualisation and organisation of numbers, is that the Inca astronomers could have conceived of ways of representing the Moon's distance from Earth in their astronomical calculations. The other piece of evidence that I add is drawn from my analysis of the total sum of S-knots. According to the data displayed in the *khipu* sample, the Incas came up with a clever method of finding the diameter of the Earth, which is approximately 13,000 km ( $\sim 7,997$  kilo-rikra), probably by repeating Eratosthenes' experiment (circa 240 BC) to enable them to calculate the Moon's diameter (i.e., 3,640 km, equivalent to 2,236 kilo-rikra). I remind the reader that the total sum of the five subsidiary string values is the following:  $9 + 0 + 2 + 3 + 8 = 22$ .

According to the Greek method of calculation (El'natanov, 1983), because the Moon is 110 Moon diameters away, the Moon's distance from Earth is  $\sim 400,000$  km ( $400,000 \div 1.6256 \approx 246,063$  kilo-rikra). As Tables 2 and 3 and Figure 7 show, the total sum of S-knots ( $16 + 0.2 + 8.4 = 24.6$ ) equals the Moon's distance from Earth (246,062 kilo-rikra) divided by 10,000, which equals 24.6062 myria-rikra (with remarkable accuracy):

$$\frac{246062}{10000} = 24.6062$$

As stated above, without a symbol for zero, it became awkward for the *khipukamayuqs* to knot large numbers on a string. In this study, the values that were used to calculate both the Earth and Moon diameters are expressed as 13,000 km and 3,700 km converted to *rikras* (7,997 and 2,276, respectively), which was the standard unit of length in the Inca Empire and was kept for official use. The difference between the Earth's diameter (7,997) and the Moon's diameter (2,276) is 5,721 *Kr*. Dividing this difference (5,721 *Kr*) by 110 (or 110 Moon diameters) gives 52 (a calendrical value), i.e., 13 months  $\times$  4 weeks = 52 weeks or 328 days. In other words,  $328 \div 8 = 41$  or 41 eight-day weeks, which is 328 total days (see Zuidema,

1964, 2004). This result appears to suggest that the Sacaca *kipukamayuc* made calculations with data recorded on the *kipu* itself (Urton, 2003, p. 126), in calendrically significant patterns (Nordenskiöld, 1925a, 1925b). From the value obtained for the Moon’s diameter, knowing that the Moon is 110 Moon diameters from the Earth, Inca astronomers could have calculated the distance of the Moon. Multiplying the Moon’s diameter (5,721 *Kr*) by 110 gives 250,360 *Kr*, which is equivalent to 406,988 km ( $406,988 \div 1.6256 = 250,361$ ). As stated above, the Moon’s distance from Earth is ~ 400,000 km ( $246,062 \text{ kilo-rikra} \div 10,000 = 24.6 \text{ myria-rikra}$ , i.e., the total sum of S-knots), which indicates that the Inca astronomers could perform this measurement with a high degree of accuracy.



**Figure 5.** The Earth’s shadow is ~2.5 Moon diameters wide at the Moon

$$2 \times 10^9$$

2,000,000,000

1 2 3 4 5 6 7 8 9

$$2,000,000,000$$

**Figure 6.** Steps to convert a large number expressed from scientific notation into standard notation.

What does each quantitative value displayed in Table 2 (i.e., 16.2; 0.1; 0.2; 8.4; and 16) mean? The passage of the Moon through the Earth's shadow was the principal source of information that was used by Inca astronomers as they began to compile records about the fraction of the lunar diameter that is obscured by Earth's shadow. Carefully studying knotted-string records of numerical values, we can argue that these values are multiples of a fathom, i.e., a measuring stick (*cota-k'aspi*, in Quechua; see Figure 4), which equals 64 inches, i.e.,  $8.4 \div 13 \approx 0.64$ ;  $16 \div 2.5 = 6.4$ . According to the data displayed in Table 2 and Figure 7, dividing 110 (i.e., the Moon is 110 Moon diameters from the Earth) by 13 yields ~8.46...

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which is  $\frac{110}{13} \cong (8.46...) \equiv (3 \times 2.8)$ , which means that the knot values on subsidiary string No.

9s1 multiplied by the knot values on string No. 9 equals  $\sim 8.46$ .

### **Results for the “number labels” matching criteria.**

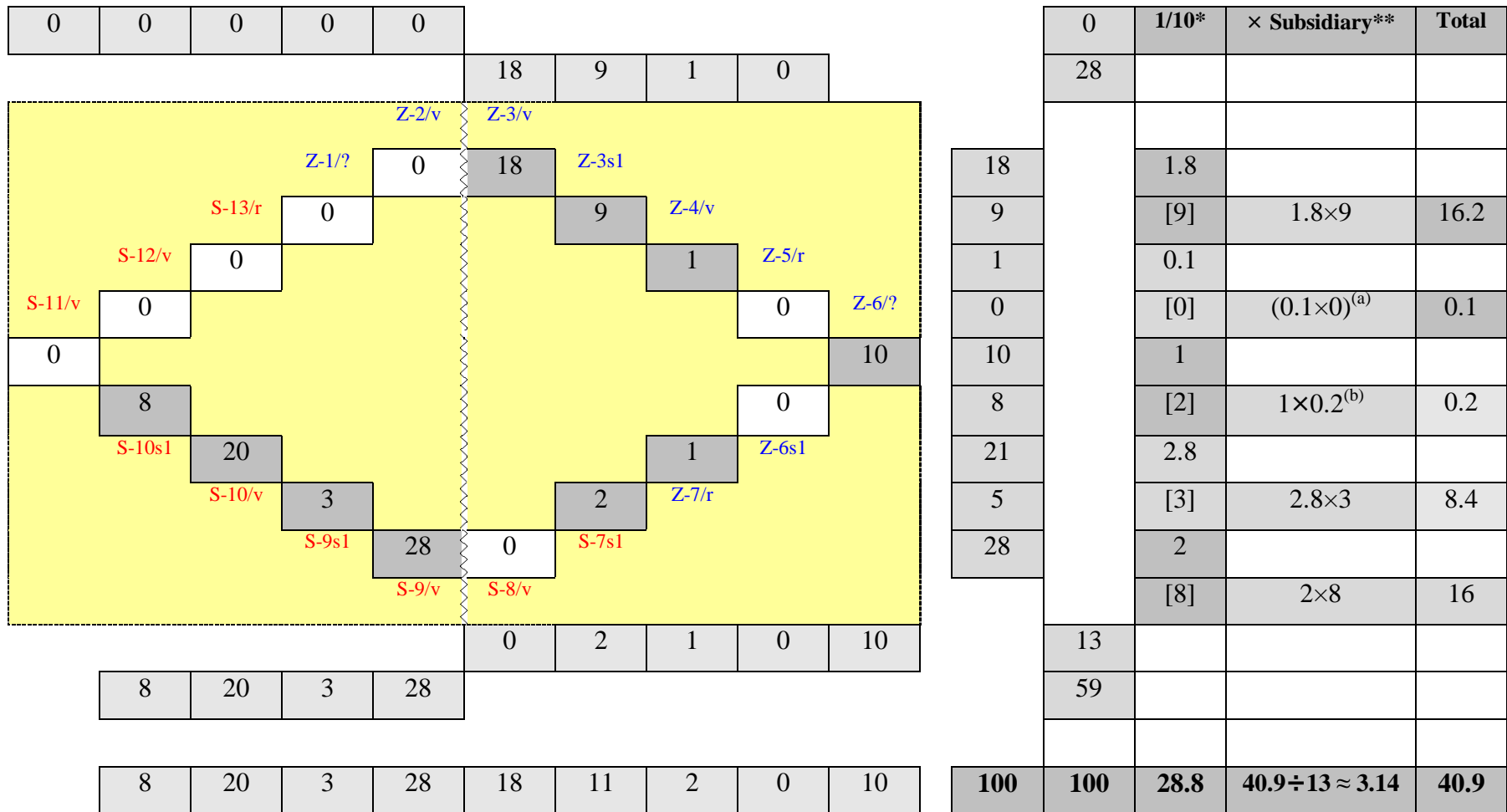
The question that arises next is the following: Could the observed patterning in the *khipu* sample (e.g., spinning, plying, attachment, knotting directionality, and possible significance of colours) be a direct reflection of the *khipukamayusqs* on the administrative class of the *khipu* sample or be somewhat more specific in relation to the general subject matter of the sample?

Urton (2003, p. 165) has also provided a more standard tabular description of the *khipu* sample. As employed in Table 4, the tabular description of Pendant No. 3 and Subsidiary No. 6s1 in the sample have the following values:

Pendant no. 3: 1S (10.5/Z), 8L(20.0/Z); Length: 32.0cm, colour: light brown.

Subsidiary no. 6s1 (Z/S), (untied knot); Length: 32.0cm, colour: W

As stated above, there are 0.1016 metres (or 10.16 cm) in one Hand (see Figure 3). To convert centimeters to Hands, we divide the centimeter measurement by 10.16. Thus, 32.0 centimeters is equivalent to  $\sim 3.14$  Hands ( $32 \div 10.16 \approx 3.14$ ). We have thus the same length for both strings:  $\sim 3.14$  Hands. Again, we could argue that the Length of this pair of Pendants reflects the link between Pi (i.e., the ratio of a circle's circumference to its diameter) and more specifically the concept of “number labels”. I would also raise the highly speculative question of whether or not the length of the strings could represent a convention for registering values that indicate the general subject matter for the *khipu* UR19.



**Figure 7.** Tukapu drawn based on data from the khipu sample UR19. (Such a geometric tukapu is shown in Figure 10.)

*Note.* Quantitative values recorded on each ‘cord’ (i.e., pendant, subsidiary, or top string), including: (a) the cord number; (b) the number of knots in each location (especially the decimal numeration level) on the cord; (c) the type of knot(s) on the cord (long knots, single knots, figure-eight knots); and (d) the number and location(s) of any subsidiaries that are attached to the cord. Orientations of the oblique axes of placement of the two knot types (i.e., **Zs** in the upper-right and lower-right; **Ss** in the upper-left and lower-left). \* The value of each of its digits decreases ten times; Values between square brackets [ ] correspond to subsidiaries attached to the cord. \*\* Multiplied by their corresponding subsidiary strings. <sup>(a)</sup> There is a missing factor in our multiplication (String no. Z-6s1 is empty), i.e., the numerical value for 0.1 corresponds to the ones place being tied on Pendant no. 6 (i.e., 1 ÷ 10 = 0.1). <sup>(b)</sup> Pendant string No. 7 and subsidiary string 7s1 are spun and plied in the opposite direction: Z/S vs. S/Z (i.e., 2 ÷ 10 = 0.2).

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**Figure 8.** Process of constructing and assembling the *khipu*. (Courtesy of Jean-JacquesQuisquater.pdf)



**Figure 9.** *Tukapu* divided into squares that bear geometrical designs. (Courtesy of <http://backstrapweaving.wordpress.com/2010/11/12/backstrap-weaving-tinkuy-de-tejedores-2010/>)



**Figure 10.** *Tukapu* composed of a rhombus that is split into two equilateral triangles. (Courtesy of [https://www.google.ch/search?q=inca+tocapu+textile&client=firefox-a&rls=org.mozilla:fr:official&noj=1&tbm=isch&tbo=u&source=univ&sa=X&ei=gt\\_kUc3SE6eM4AT E1YCwDg&ved=0CC8QsAQ&biw=1016&bih=566](https://www.google.ch/search?q=inca+tocapu+textile&client=firefox-a&rls=org.mozilla:fr:official&noj=1&tbm=isch&tbo=u&source=univ&sa=X&ei=gt_kUc3SE6eM4AT E1YCwDg&ved=0CC8QsAQ&biw=1016&bih=566))

## DISCUSSION

As Figure 7 and Tables 2 and 4 show, although the *khipu* sample records numerical values in a base-ten numeration system, the knots tied on subsidiary strings have been used to denote complementary actions, such as multiplication. We can thus ask whether long knots tied in different directions (S-knots vs. Z-knots), all of which can be read as comprising iterative



stroke-like units between two and nine lines, could have been used to denote complementary actions, such as addition/subtraction or multiplication/division. In the hypothesis developed by Urton (2003, p. 96), every knot of a decimal-registry on the *kipu* sample can be read as (a) numbers used as magnitudes and (b) a component of a sum that could have held administrative, calendrical, or purely mathematical value for state record keepers. We see on subsidiary string 7s1 (Z/S), which is tied to Pendant string No. 7 (Z/S), two numbers that are multiplied together:  $1 \times 2 = 2$ . Why would the *kipukamayuc* have needed or wanted to tie the subsidiary string 7s1 (Z/S) to Pendant string No. 7 (S/Z) spun and plied in the opposite direction, i.e., (Z/S) vs. (S/Z)? The point on which differentiation between Pendant string No. 7 and its subsidiary string are tied (according to the perspective proposed here) is that of *need*:  $1 \times 0.2 = 0.2$  (see Table 3 and Figure 7).

In our attempts to interpret the *kipu*, in this specific sample (see Figure 7 and Table 2), decimal fractions (see Ascher and Ascher, 1997, pp. 151–152; Urton, 2003) are expressed as an improper, top-heavy fraction, with a denominator of 10, e.g.,  $3 \times 2.8$ ,  $2 \times 0.1$ , and  $9 \times 1.8$ , and were expressed as  $3 \times \frac{28}{10}$ ,  $2 \times \frac{1}{10}$ , and  $9 \times \frac{18}{10}$ , respectively. In English-speaking countries, some Latin American countries and many Asian countries, a period (.) or raised period (•) is used as the decimal separator, e.g.,  $3 \times \frac{28}{10} = 8.4$ ,  $2 \times \frac{1}{10} = 0.2$ , and  $9 \times \frac{18}{10} = 16.2$ . In this respect, the 22nd General Conference on Weights and Measures declared in 2003 that “the symbol for the decimal marker shall be either the point on the line or the comma on the line”. Alternatively, somewhat more specifically in relation to the Inca accounting practices, the decimal separator is inserted into the numerator (with leading zeros added if needed) at a position from the right that corresponds to the power of ten in the denominator. A decimal number with an absolute value that is less than one often has a leading zero. In *kipu* sample UR19, Incan fractions are expressed as an improper, top-heavy fraction (see above), with a denominator of 10, i.e., the number 1.6 is  $\frac{16}{10}$ , or sixteen tenths. As Table 3 shows, subsidiary string 7s1 and Pendant string No. 6 have the same distinctive signature (° • °). We can propose that this same signature could have such translation values. In other words, this array could represent a certain "class" of information. For example, in the case of subsidiary string 7s1 and Pendant string No. 6, both strings are valued as 2 (long knot of only two turns) and 10 (one knot in the tens position), respectively. As a suggestion, it could be the representation of fractions or ratios (Ascher and Ascher 1997: 143–151).

The ancient Incas seemingly knew that not all of the numbers were rational. They could calculate  $\pi$  (Pi), the ratio of the circumference of a circle to its diameter, which is an irrational number, as 3.14; this circumstance does not imply that the Incas studied the existence of irrational numbers. As Urton (2003, p. 79) argued, variations in knot directionality were the results of deliberate, conscious, and meaningful actions, but the body of knowledge and habitus that produced them would have been central or core elements in the arts of recording performed by the *khipukamayuuq* using the *khipu*. I argue that this specific sample records values that could constitute an important component of calculating the distance from the Earth to the Moon. Thus, the sample was used as a structure for memorising and recalling information; the essential components would have been noted by *khipukamayuuq* and used as the framework (mnemonic) for scientific notation. In astronomy, the appearance of such extremely large numbers is common. For this reason, Incan astronomers would have used scientific notation when working with extremely large numbers. The question that confronts us is, thus, the following: What could the absence of knots on strings nos. 1, 2, 5, 6s1, 8, 11, 12, and 13 have to do with the type of construction that is considered here, i.e., the ratio of the circumference of a circle to its diameter? I suggest that an apt characterisation of what we are studying in *khipu* sample UR19 is that these values constitute an important component for calculating the ratio of the circumference of a circle to its diameter. The potentially differing signatures of adjacent pendants/arrays could indicate that the same value, i.e., zero, is likely not what the *khipukamayuuq* would have constructed on other strings. The constructor might have wanted to reach the requisite number of strings (13) without changing the total sum (40.9), and as a result, he used the ratio of a circle's circumference to its diameter:  $40.9 \div 13 \approx 3.14$ . If so, bearing in mind the large range of colours and other features that were linked and therefore provided cues for formulaic patterns, as suggested by Garcilaso de la Vega and Calancha for the colours in the *khipu*, what type of memory scheme was it?

To provide an example of the hypothetical mode of constructing a coded sequence of stones as outlined in Table 3, take the sequence:  $\circ \times \circ$  This array constitutes a hypothetical "signature" of strings that are directly involved in calculating the Moon's diameter and its distance from the Earth. This signature is distinctive and unique to any knot that has all of the relevant qualities directly involved in calculating the Moon's diameter and its distance from Earth, i.e., strings numbered 3s1; 9; and 10s1.

What about the remaining five strings, i.e., Pendants Nos. 1, 5, 6s1, 12, and 13? In this specific sample, strings that have no recorded quantitative values appear to be tied in ways

that violate or are contrary to decimal registration procedures. These empty strings (spun, plied, dyed, but unknotted pendant strings) appear to have been tied as a partial inscription.

Why did the chroniclers not mention the Pi value if it was, as in Europe, such an integral part of many calculations, including in subjects such as architecture and astronomy? This question is difficult to answer, especially for Spanish chroniclers. Cusco was the former capital of the Inca Empire and was a heavily Hispanicised city (MacCormack, 2001). In the 1580s, the Third Council of Lima declared *kipus* to be idolatrous objects and ordered them to be burned (Vargas Ugarte, 1959; Urton, 1998, 2003, p. 22). Spanish chroniclers, thus, most likely never witnessed *kipu* being handled, much less interpreted, in circumstances that were not fraught with considerable tension or negative, censorious attitudes.

### SUMMARY AND CONCLUSIONS

Can we confirm that the Incas determined the mean distance of the Moon and that they recorded measurements of the ratio of the circumference of a circle to its diameter as an irrational number with sufficient accuracy to find Pi ( $\pi \approx 3.14\dots$ )? While many colonial writers in Peru left accounts of the *kipu* that inform us about the intellectual, technological, political, and ideological transformation for the native Americans after the Conquest (Rappaport and Cummins, 1994, 1998), no account is sufficiently extensive or detailed to put us on solid ground in our current attempts to understand exactly what the Inca learned from Spaniards about arithmetic relations between functions, especially those that involve trigonometry and knowledge of the angle of elevation of the Sun. A valid assumption for us to make regarding the nature of knowledge and practice that the Incas could have learned from Spaniards in all domains of Astronomy would include the calculations used by ancient Greeks ca. 250 B.C. for the size of the Moon as well as its distance from the Earth. If such had been the case, we would have to accept that a typical unit for astronomical distances should be (and was) the League (6,687 km), which is a traditional Spanish unit of distance (see Cantera Burgos, 1931). Instead, although the Inca astronomers needed a truly large unit to measure distances to the Moon, they worked with *rikra* (fathoms). Nonetheless, according to Laurencich-Minelli and Magli (2009, p. 123), the Chachapoyas calendar describes months of either 30 or 31 days with one of 29 days, which the European calendar could have influenced. Introducing alphabetic literacy into the Andes during the conquest represented a force for significant intellectual, technological, political, and ideological transformations for the native Americans. For the values themselves, I suggest that the information that was recorded in the *kipu* sample had a somewhat more complicated nature, more so than suggested by Sarmiento

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de Gamboa (1999, p. 41). In my view, the evidence strongly supports the notion that the units of length recorded in the *khipu* sample UR19 were based on the human body, which appears to confirm Rowe's point of view (1946, p. 323; Malpass, 1996). I find it suggestive that there were units that equal the distance between the outstretched thumb and forefinger (5-6 inches); palm (8 inches); forearm or cubit (18 inches); and fathom (64 inches). The latter unit (*rikra*) was used as a standard, and sticks (*cqta-k'aspi*, in Quechua) of this length were kept for official use (see Figure 4). Conklin (2002), among others, indicated that all script systems represent mnemonic recording devices to one degree or another. The overall question that we address here is, thus, the following: What type of recording system could the *khipu* UR19 have represented? In addressing this question, we must begin by sorting the differences and relationships between recording both calendrical and geometrical information on the *khipu* sample. For *khipu* UR19, the evidence strongly supports the notion that these numerical values indicate that the Incas calculated the ratio of the circumference of a circle to its diameter as well as the distance from the Earth to the Moon.

I conclude this study by noting that this *khipu* incorporates the results from lunar eclipse observations made by Inca astronomers, i.e., the diameter of the shadow cone (~2.5 lunar diameters; see Figure 5). With these values and simple geometry (see Paolantonio and Pintado, 2006), the Incas could determine the mean distance to the Moon. I believe that it is not rash to claim that this new perspective on the *khipu* provides us with a methodological and theoretical grounds to begin to identify something that resembles an organisation of (and for) *tukapu* structures in the Inca *khipu*. This finding moves us slightly closer to deciphering the *khipu*.

As Urton (2003, p. 136) has noted, even as the conventional recording units (the numerical values) have gone unremarked over the centuries since literate Europeans first encountered *khipu*, the numerical values have unquestionably existed in the very fabric of these devices, and these values are there (for anyone who cares to do so) to see, study, and record today. It is a curious fact and probably no more than a coincidence that the Inca astronomers' findings coincide with the respective calculations used by the ancient Greeks.

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