GLASNIK MATEMATIČKI
Vol. 46(66)(2011), 311 – 323

THE $D(-k^2)$-TRIPLE $\{1, k^2+1, k^2+4\}$ WITH k PRIME

Alan Filipin and Yasutsugu Fujita
University of Zagreb, Croatia and Nihon University, Japan

Abstract. Let n be a nonzero integer. A set of m distinct positive integers is called a $D(n)$-m-tuple if the product of any two of them increased by n is a perfect square. Let k be a prime number. In this paper we prove that the $D(-k^2)$-triple $\{1, k^2+1, k^2+4\}$ cannot be extended to a $D(-k^2)$-quadruple if $k \neq 3$. And for $k = 3$ we prove that if the set $\{1, 10, 13, d\}$ is a $D(-9)$-quadruple, then $d = 45$.

1. Introduction

Let n be a nonzero integer. A set of m distinct positive integers $\{a_1, \ldots, a_m\}$ is called a Diophantine m-tuple with the property $D(n)$, or simply, a $D(n)$-m-tuple if $a_i a_j + n$ is a perfect square for each i, j with $1 \le i < j \le m$. Diophantus of Alexandria was the first who studied the existence of such sets, and he found $D(256)$-quadruple $\{1, 33, 68, 105\}$. For general n it is easy to see that there exist infinitely many $D(n)$-triples, so we can ask ourselves if there exists a $D(n)$-quadruple, or if some $D(n)$-triple can be extended to a quadruple. Brown ([4]) proved that if $n \equiv 2 \pmod 4$, then there does not exist a $D(n)$-quadruple. On the other hand, Dujella ([5]) proved that if $n \not\equiv 2 \pmod 4$, and if $n \notin S = \{-4, -3, -1, 3, 5, 8, 12, 20\}$, then there exists at least one $D(n)$-quadruple, and he conjectured that there does not exist a $D(n)$-quadruple for $n \in S$. Recently, there are numerous papers on this subject, specially in the cases $n = 1$, $n = -1$ and $n = 4$. In particular, Dujella ([8]) proved that there does not exist a $D(1)$-sextuple and the second author ([14]) proved that there are at most 10^{276} $D(1)$-quintuples. For the full list of references the reader can see http://web.math.hr/~duje/dtuples.html.

But except considering fixed n, there are also some results when n depends on some parameter k. Namely, the second author ([12]) proved that the

2010 *Mathematics Subject Classification.* 11D09, 11J68.
Key words and phrases. Diophantine tuples, simultaneous Diophantine equations.

$D(4k)$-triple $\{1, 4k(k-1), 4k^2+1\}$ with $|k|$ prime cannot be extended to a $D(4k)$-quadruple. Moreover, he ([13]) also proved that the $D(\mp k^2)$-triple $\{k^2, k^2 \pm 1, 4k^2 \pm 1\}$ cannot be extended to a $D(\mp k^2)$-quintuple. Usually for $n \notin \{\pm 1, \pm 4\}$, it is not easy to show either nonexistence or uniqueness of the extension of a $D(n)$-triple, unless some argument using congruences modulo a power of 2 works. That is why we have to know the fundamental solutions of at least two of the Pell equations

$$ay^2 - bx^2 = 1, \quad az^2 - cx^2 = 1, \quad bz^2 - cy^2 = 1.$$

In [12, 13], ab and ac are of Richaud-Degert type (see [16]) which gives the fundamental solutions of the corresponding Pell equations.

In this paper we consider a similar problem. Precisely, we consider the problem of the extension of $D(-k^2)$-triple $\{1, k^2+1, k^2+4\}$ for prime k. But our problem is more difficult to examine than that of $D(4k)$-triple $\{1, 4k(k-1), 4k^2+1\}$ which the second author considered in [12], because while for the $D(4k)$-triple $bc = 4k^2(2k-1)^2 - 4k$ is of RD type, for the $D(-k^2)$-triple $bc = (k^2+2)^2 + k^2$ is not of RD type and hence the solutions of the third equation cannot be expressed as well as the first and the second equations. In fact, in that paper the result [12, Theorem 1.3] for $k > 2$ is proved by comparing the fundamental solutions of the second and the third equations. Our main result is the following theorem.

THEOREM 1.1. *Let k be a prime number. Then, the $D(-k^2)$-triple $\{1, k^2+1, k^2+4\}$ cannot be extended to a $D(-k^2)$-quadruple if $k \neq 3$. Futhermore, if $\{1, 10, 13, d\}$ is a $D(-9)$-quadruple, then $d = 45$.*

In the proof of this theorem we use the standard methods when considering extension of a $D(n)$-triple. We transform our problem in solving a system of simultaneous Pellian equations. Here, we will consider k to be prime because it enables us to completely determine all fundamental solutions of one of the Pellian equations. Furthermore, we then transform our problem of solving a system of Pellian equations to finding intersection of two binary recurrence sequences. Then we get the contradiction for large parameter k, using congruence method together with Bennett's theorem on simultaneous approximation of the square roots of algebraic numbers which are close to 1. And at the end, for finitely many left k's we prove our theorem using Baker's theory on linear forms in logarithms of algebraic numbers. Even the methods here are standard, there is more technical work to be done than usually in such kind of problems. Let us also mention that we do not have to consider the case $k = 2$, because from [5, Remark 3] we know that $D(-4)$-quadruple must have all elements even.

We should mention another simple extension $\{1, k^2, k^2+1\}$ of the $D(-k^2)$-pair $\{1, k^2+1\}$. The second author and Togbé ([15]) recently showed that if $\{k^2, k^2+1, c, d\}$ is a $D(-k^2)$-quadruple for a positive integer k, then either

c or d equals one. Therefore, the $D(-k^2)$-triple $\{1, k^2, k^2+1\}$ for a positive integer k cannot be extended to a $D(-k^2)$-quintuple.

Let us also mention that it is easy to see that if $k = k_i$, where (k_i) is the sequence given by

$$k_1 = 3, \quad k_2 = 12, \quad k_i = 4k_{i-1} - k_{i-2}, \qquad i \geq 2,$$

then $\{1, k^2+1, k^2+4, 4k^2+9\}$ is a $D(-k^2)$-quadruple (k only has to satisfy the condition that $3k^2+9$ is a perfect square). So we see that for some k's that are not primes we have an extension of the $D(-k^2)$-triple $\{1, k^2+1, k^2+4\}$.

2. Lower bounds for solutions

Let k be a positive integer. Suppose that $\{1, k^2+1, k^2+4, d\}$ is a $D(-k^2)$-quadruple. Then there exist positive integers x, y, z such that

$$d - k^2 = x^2, \quad (k^2+1)d - k^2 = y^2, \quad (k^2+4)d - k^2 = z^2.$$

Eliminating d from these equations, we obtain the system of Pellian equations

$$y^2 - (k^2+1)x^2 = k^4, \tag{2.1}$$

$$z^2 - (k^2+4)x^2 = k^2(k^2+3), \tag{2.2}$$

$$(k^2+1)z^2 - (k^2+4)y^2 = 3k^2. \tag{2.3}$$

Note that equation (2.3) is linearly dependent on equation (2.1) and (2.2). In what follows, we assume that k is an odd prime number.

Lemma 2.1. *Let (y, x) be a positive solution of Pellian equation* (2.1). *Then, there exist a non-negative integer m and a solution (y_0, x_0) of* (2.1) *with*

$$(y_0, x_0) \in \{(k^2, 0),\ (k(k^2-k+1), \pm k(k-1)),\ (k^2+2, \pm 2)\}$$

such that

$$y + x\sqrt{k^2+1} = (y_0 + x_0\sqrt{k^2+1})(2k^2+1+2k\sqrt{k^2+1})^m.$$

Proof of Lemma 2.1. When k is odd prime number, because the right hand side of the equation (2.1) is of the form k^4 we know that there are at most 5 classes of solutions. We get this if we want to find primitive solutions of equations

$$y^2 - (k^2+1)x^2 = 1, k^2, k^4.$$

To grasp this one can see [17, Theorems 22 and 23] which yields that for k^2 and k^4 there are at most two classes of solutions and when the right hand side is equal to 1 there is only one class. And it is easy to check that for $k \geq 3$ the fundamental solutions given in this lemma are non-associated. $\square$

Let us also mention that for composite k, statement of Lemma 2.1 is not true in general. For example, even for small $k = 4$ we have seven classes of solutions of the equation (2.1). Also we can see that $D(-k^2)$-triples $\{1, k^2+1, c\}$ where $c = k^2, k^2+4, k^4-2k^3+2k^2$ correspond to the fundamental solutions from Lemma 2.1.

LEMMA 2.2. *Let (z, x) be a positive solution of Pellian equation (2.2). Then, there exist a non-negative integer n and a solution (z_1, x_1) of (2.2) with*

$$z_1 > 0 \quad \text{and} \quad |x_1| < \frac{k^2}{2} \tag{2.4}$$

such that

$$z + x\sqrt{k^2+4} = (z_1 + x_1\sqrt{k^2+4}) \left(\frac{k^2+2+k\sqrt{k^2+4}}{2} \right)^n . \tag{2.5}$$

PROOF OF LEMMA 2.2. The proof is similar to the one of Lemma 1 in [7]. Let (z, x) be a positive solution of (2.2), and (z^*, x^*) a pair of integers such that

$$z^* + x^*\sqrt{k^2+4} = (z + x\sqrt{k^2+4}) \left(\frac{k^2+2-k\sqrt{k^2+4}}{2} \right)^n$$

for an integer n (note that z^* and x^* are certainly integers, since z and x have the same parity). Then (z^*, x^*) is a solution of (2.2) and $z^* > 0$. Denote by (z_1, x_1) a pair (z^*, x^*) such that z^* is minimal, and define the integers z' and x' by

$$z' + x'\sqrt{k^2+4} = (z_1 + x_1\sqrt{k^2+4}) \left(\frac{k^2+2-\epsilon k\sqrt{k^2+4}}{2} \right),$$

where $\epsilon = 1$ if $x_1 \geq 0$ and $\epsilon = -1$ is $x_1 < 0$. Then the minimality of z_1 shows that

$$z' = \frac{(k^2+2)z_1 - \epsilon k(k^2+4)x_1}{2} \geq z_1,$$

and hence $kz_1 \geq \epsilon(k^2+4)x_1$. Squaring both sides of this inequality yields

$$k^2 z_1^2 \geq (k^2+4)^2 x_1^2 = (k^2+4)(z_1^2 - k^2(k^2+3)).$$

Thus we obtain $z_1^2 \leq k^2(k^2+3)(k^2+4)/4$. It follows that

$$x_1^2 = \frac{z_1^2 - k^2(k^2+3)}{k^2+4} \leq \frac{k^4(k^2+3)}{4(k^2+4)} < \frac{k^4}{4}. \tag{2.6}$$

Therefore, (z_1, x_1) satisfies (2.4) and (2.5) for some integer n.

Suppose that $n < 0$. Then one can express

$$\left(\frac{k^2+2+k\sqrt{k^2+4}}{2} \right)^n = \frac{\alpha - \beta\sqrt{k^2+4}}{2}$$

with positive integers α, β satisfying $\alpha^2-(k^2+4)\beta^2 = 4$. Hence $x = \alpha x_1 - \beta z_1$. Since $x > 0$, we see that $4x_1^2 > \beta^2(z_1^2 - (k^2+4)x_1^2) \geq k^2(k^2+3) > k^4$, which contradicts (2.6). ∎

By Lemmas 2.1 and 2.2, if x is a solution of the system of equations (2.1) and (2.2), then we may write $x = v_m = w_n$ with nonnegative integers m, n, where

$$v_0 = x_0,\ v_1 = (2k^2+1)x_0 + 2ky_0,\ v_{m+2} = 2(2k^2+1)v_{m+1} - v_m, \tag{2.7}$$

$$w_0 = x_1,\ w_1 = \frac{(k^2+2)x_1 + kz_1}{2},\ w_{n+2} = (k^2+2)w_{n+1} - w_n. \tag{2.8}$$

LEMMA 2.3. *Suppose that $x = v_m = w_n$ has a solution for some m and n. Then $(y_0, x_0) = (k(k^2-k+1), \pm(k(k-1)))$ or $(k^2+2, \pm 2)$. Moreover, if $(y_0, x_0) = (k(k^2-k+1), \pm k(k-1))$, then $(z_1, x_1) = (k\sqrt{2k^2+7}, \mp k)$.*

PROOF OF LEMMA 2.3. Suppose first that $(y_0, x_0) = (k^2, 0)$. By (2.7) and (2.8) we have $v_m \equiv 0 \pmod{k^2}$ and $w_n \equiv x_1 \pmod{k}$, and hence $x_1 \equiv 0 \pmod{k}$. Since (2.2) implies that $z_1^2 \equiv 0 \pmod{k^2}$, that is, $z_1 \equiv 0 \pmod{k}$, we see from (2.8) that $w_n \equiv x_1 \pmod{k^2}$. Hence by $v_m \equiv 0 \pmod{k^2}$ we obtain $x_1 \equiv 0 \pmod{k^2}$. It follows from (2.4) that $x_1 = 0$ and from (2.2) that $z_1 = k\sqrt{k^2+3}$, which cannot be an integer for $k \geq 2$.

Suppose next that $(y_0, x_0) = (k(k^2-k+1), \pm k(k-1))$. By (2.7) we have $v_m \equiv \mp k \pmod{k^2}$, and the same argument as above shows that $w_n \equiv x_1 \pmod{k^2}$ and from (2.4) we obtain $x_1 = \mp k$, and hence $z_1 = k\sqrt{2k^2+7}$. ∎

LEMMA 2.4. *If $x = v_m = w_n$ has a solution with $(y_0, x_0) = (k(k^2-k+1), \pm k(k-1))$ for some m and n, then*

$$m \geq \frac{(\sqrt{5}-2)k - 3}{4}.$$

PROOF OF LEMMA 2.4. Consider the relation $v_m \equiv w_n \pmod{(k^2+1)}$. We see from (2.7) that

$$v_m \equiv (-1)^{m-1}\{2mk \pm (k+1)\} \pmod{(k^2+1)}$$

and from (2.8) and Lemma 2.3 that

$$w_n \equiv \pm k,\ \frac{\pm k + \sqrt{2k^2+7}}{2} \text{ or } \frac{\pm k - \sqrt{2k^2+7}}{2} \pmod{(k^2+1)}.$$

Since k^2+1 is even, v_m is even. Hence we obtain

$$2mk \pm (k+1) \equiv \pm\frac{k \pm \sqrt{2k^2+7}}{2} \pmod{(k^2+1)},$$

and $Ak \pm 2 \equiv \pm\sqrt{2k^2+7} \pmod{(k^2+1)}$, where $A = 4m+3$ for the plus sign and $A = 4m-1$ for the minus sign. Squaring both sides of this congruence, we have $-A^2 \pm 4kA + 4 \equiv 5 \pmod{(k^2+1)}$, that is, $A^2 \pm 4kA + 1 \equiv 0 \pmod{(k^2+1)}$. Since $A^2 \pm 4kA + 1 = 0$ has no integer solution, we have

$|A^2 \pm 4kA + 1| \geq k^2 + 1$, which yields $(A \pm 2k)^2 \geq 5k^2$. By $A > 0$ we have $A + 2k \geq \sqrt{5}k$. Since $A = 4m \pm (2 \pm 1)$, we have $4m \pm (2 \pm 1) \geq (\sqrt{5} - 2)k$, and hence $m \geq \{(\sqrt{5} - 2)k - 3\}/4$. $\square$

In the case of $(y_0, x_0) = (k^2 + 2, \pm 2)$, we were not able to determine the fundamental solutions for equation (2.2) and hence to give a lower bound for m in terms of k. Instead, for small k we restrict the possibilities for the fundamental solutions, and for large k we give an absolute lower bound for m.

Lemma 2.5. *Suppose that $x = v_m = w_n$ has a solution with $(y_0, x_0) = (k^2 + 2, \pm 2)$ for some m and n.*

(1) *If $k \neq 3$, then m is odd.*
(2) *If $k < 5 \cdot 10^4$, then $(k, z_1, x_1) = (3, 11, \pm 1)$ or $(23, 1146, \pm 44)$.*
(3) *If $k > 5 \cdot 10^4$, then $m \geq 11$.*

Proof of Lemma 2.5. (1) By Lemma 2.1 with $(y_0, x_0) = (k^2 + 2, \pm 2)$, we may write $y = v'_m$, where

$$v'_0 = k^2 + 2,\ v'_1 = (k^2 + 2)(2k^2 + 1) \pm 4k(k^2 + 1),\ v'_{m+2} = 2(2k^2 + 1)v'_{m+1} - v'_m.$$

If m is even and $k \neq 3$, then $y \equiv 0 \pmod 3$, and equation (2.3) implies $z^2 \equiv 0 \pmod 3$ and $z^2 \equiv 0 \pmod 9$, which is a contradiction. Hence m is odd.

(2) By (2.7) with $(y_0, x_0) = (k^2 + 2, \pm 2)$ we have $v_m \equiv \pm 2 \pmod k$, and by (2.8) we have $w_n \equiv x_1 \pmod k$. Hence $x_1 \equiv \pm 2 \pmod k$. It is now easy to check by a computer that the only primes k for $3 \leq k < 5 \cdot 10^4$ such that $(k^2 + 4)x_1^2 + k^2(k^2 + 3)$ is a perfect square with $x_1 \equiv \pm 2 \pmod k$ and $|x_1| < k^2/2$ are 3, 5 and 23, and the corresponding pairs (z_1, x_1) are $(11, \pm 1)$, $(31, \pm 3)$ and $(1146, \pm 44)$, respectively. Suppose that $(k, z_1, x_1) = (5, 31, \pm 3)$. Since m is odd, we see from (2.7) with $(y_0, x_0) = (k^2 + 2, \pm 2)$ that $x = v_m \equiv 0 \pmod 4$. On the other hand, (2.8) with $(k, z_1, x_1) = (5, 31, \pm 3)$ shows that $w_n \not\equiv 0 \pmod 4$, which contradicts $v_m = w_n$.

(3) By (1), to prove the statement (3) we have to show that for $k > 5 \cdot 10^4$ there does not exist a solution of the equation (2.2) with $x = v_m$ where $m = 1, 3, 5, 7, 9$. To prove that we first express x in terms of parameter k and then we insert this into equation (2.2) to compute z. We check that in those cases z^2 is strictly between squares of two consecutive integers, which gives us a contradiction because z is an integer.

So, when $x = v_1 = 2k^3 \pm 4k^2 + 4k \pm 2$, we get

$$z^2 = 4k^8 \pm 16k^7 + 48k^6 \pm 104k^5 + 161k^4 \pm 176k^3 + 135k^2 \pm 64k + 16$$

and

$$(2k^4 \pm 4k^3 + 8k^2 \pm 10k + 4)^2 < z^2 < (2k^4 \pm 4k^3 + 8k^2 \pm 10k + 5)^2$$

for $k > 5 \cdot 10^4$.

When $x = v_3 = 32k^7 \pm 64k^6 + 96k^5 \pm 96k^4 + 70k^3 \pm 36k^2 + 12k \pm 2$, we have

$$\begin{aligned} z^2 =& 1024k^{16} \pm 4096k^{15} + 14336k^{14} \pm 34816k^{13} + 66944k^{12} \pm 103424k^{11} \\ & + 131968k^{10} \pm 140800k^9 + 126500k^8 \pm 95792k^7 + 60848k^6 \pm 32056k^5 \\ & + 13729k^4 \pm 4624k^3 + 1159k^2 \pm 192k + 16, \end{aligned}$$

which implies

$$\begin{aligned} &(32k^8 + 64k^7 + 160k^6 + 224k^5 + 198k^4 + 100k^3 + 88k^2 + 138k - 53)^2 < z^2 \\ &\quad < (32k^8 + 64k^7 + 160k^6 + 224k^5 + 198k^4 + 100k^3 + 88k^2 + 138k - 52)^2, \end{aligned}$$

for $k > 5 \cdot 10^4$ and $x_0 = 2$. And in the case $x_0 = -2$, we get

$$\begin{aligned} &(32k^8 - 64k^7 + 160k^6 - 224k^5 + 198k^4 - 100k^3 + 88k^2 - 138k - 52)^2 < z^2 \\ &\quad < (32k^8 - 64k^7 + 160k^6 - 224k^5 + 198k^4 - 100k^3 + 88k^2 - 138k - 51)^2, \end{aligned}$$

for $k > 5 \cdot 10^4$.

When $x = v_5 = 512k^{11} \pm 1024k^{10} + 2048k^9 \pm 2560k^8 + 2720k^7 \pm 2240k^6 + 1504k^5 \pm 800k^4 + 330k^3 \pm 100k^2 + 20k \pm 2$, we similarly get that for $k > 5 \cdot 10^4$,

$$(P_+(k) - 7469)^2 < z^2 < (P_+(k) - 7468)^2,$$

if $x_0 = 2$ and

$$(P_-(k) - 7468)^2 < z^2 < (P_-(k) - 7467)^2,$$

if $x_0 = -2$ and where

$$\begin{aligned} P_\pm(k) =& 512k^{12} \pm 1024k^{11} + 3072k^{10} \pm 4608k^9 + 5792k^8 \pm 5312k^7 \\ & + 4896k^6 \pm 4256k^5 + 970k^4 \mp 2780k^3 + 2408k^2 \pm 10634k. \end{aligned}$$

In cases $x = v_7$ and $x = v_9$ we get a contradiction in the same way, even there is more technical work to be done.

This argument was possible because the leading term in v_m considered as a polynomial in k, for m odd, is a perfect square. $\square$

The following lemma together with Lemmas 2.4 and 2.5 bounds x below in terms of k in case k is large.

Lemma 2.6. *Assume that $x = v_m$ with $m \neq 0$ and $k > 512$. Then $\log x > (2m - 0.21)\log(2k)$. Moreover, if $(y_0, x_0) = (k^2+2, \pm 2)$ and $k > 5 \cdot 10^4$, then $\log x > (2m + 0.87)\log(2k)$.*

Proof of Lemma 2.6. By (2.7) we have

$$v_m = \frac{1}{2\sqrt{k^2+1}}\Big\{(y_0 + x_0\sqrt{k^2+1})(k+\sqrt{k^2+1})^{2m} - (y_0 - x_0\sqrt{k^2+1})(k-\sqrt{k^2+1})^{2m}\Big\} \geq \frac{1}{2\sqrt{k^2+1}}(k+\sqrt{k^2+1})^{2m}$$
$$\times\Big\{(y_0 - |x_0|\sqrt{k^2+1}) - (y_0 + |x_0|\sqrt{k^2+1})(k+\sqrt{k^2+1})^{-4m}\Big\}. \tag{2.9}$$

We see from Lemma 2.1 that

$$y_0 - |x_0|\sqrt{k^2+1} \geq k(k^2-k+1) - k(k-1)\sqrt{k^2+1} = \frac{k^3}{k^2-k+1+(k-1)\sqrt{k^2+1}} > \frac{k}{2}$$

and for $m \geq 1$ we have

$$(y_0 + |x_0|\sqrt{k^2+1})(k+\sqrt{k^2+1})^{-4m} \leq \frac{k(k^2-k+1)+k(k-1)\sqrt{k^2+1}}{(2k^2+1+2k\sqrt{k^2+1})^2} < \frac{k}{2(2k^2+1+2k\sqrt{k^2+1})} < \frac{1}{8k}.$$

Hence for $k > 512$ we obtain

$$v_m > \frac{1}{2\sqrt{k^2+1}}(k+\sqrt{k^2+1})^{2m}\left(\frac{k}{2} - \frac{1}{8k}\right) > 0.249(k+\sqrt{k^2+1})^{2m} > (2k)^{2m-0.21}.$$

Suppose now that $(y_0, x_0) = (k^2+2, \pm 2)$ and $k > 5 \cdot 10^4$. Then, since

$$y_0 - |x_0|\sqrt{k^2+1} = k^2+2-2\sqrt{k^2+1} > (k^2+1)\left(1 - \frac{2}{\sqrt{k^2+1}}\right)$$

and

$$(y_0 + |x_0|\sqrt{k^2+1})(k+\sqrt{k^2+1})^{-4m} \leq \frac{k^2+2+2\sqrt{k^2+1}}{(2k^2+1+2k\sqrt{k^2+1})^2} < \frac{1}{2(2k^2+1+2k\sqrt{k^2+1})} < \frac{1}{8k^2},$$

it follows from (2.9) and $k > 5 \cdot 10^4$ that

$$v_m > \frac{k^2+1}{2\sqrt{k^2+1}}(k+\sqrt{k^2+1})^{2m}\left(1 - \frac{2}{\sqrt{k^2+1}} - \frac{1}{8k^2(k^2+1)}\right) > 0.499k(k+\sqrt{k^2+1})^{2m} > (2k)^{2m+0.87}.$$

$\square$

3. An upper bound for solutions

Let $\theta_1 = \sqrt{1+1/k^2}$ and $\theta_2 = \sqrt{1+4/k^2}$.

Lemma 3.1. *All positive solutions* (x, y, z) *of the system of Pellian equations* (2.1) *and* (2.2) *with* $k \geq 3$ *satisfy*

$$\max\left\{\left|\theta_1 - \frac{y}{kx}\right|, \left|\theta_2 - \frac{z}{kx}\right|\right\} < \frac{k^2}{\sqrt{3}x^2}.$$

Proof of Lemma 3.1.

$$\left|\theta_1 - \frac{y}{kx}\right| = \frac{1}{kx}|x\sqrt{k^2+1} - y| = \frac{1}{kx} \cdot \frac{k^4}{x\sqrt{k^2+1}+y} < \frac{k^3}{2kx^2} < \frac{k^2}{\sqrt{3}x^2}.$$

$$\left|\theta_2 - \frac{z}{kx}\right| = \frac{1}{kx}|x\sqrt{k^2+4} - z| = \frac{1}{kx} \cdot \frac{k^2(k^2+3)}{x\sqrt{k^2+4}+z}$$

$$< \frac{k(k^2+3)}{2x^2\sqrt{k^2+4}} < \frac{k^2}{2x^2}\sqrt{1+\frac{3}{k^2}} \leq \frac{k^2}{\sqrt{3}x^2}.$$

□

In our situation, Bennett's theorem ([2, Theorem 3.2]) can be rephrased as follows.

Theorem 3.2. *If* $k > 512$, *then the numbers* θ_1 *and* θ_2 *satisfy*

$$\max\left\{\left|\theta_1 - \frac{p_1}{q}\right|, \left|\theta_2 - \frac{p_2}{q}\right|\right\} > (2675k^2)^{-1}q^{-\lambda}$$

for all integers p_1, p_2, q *with* $q > 0$, *where*

$$\lambda = 1 + \frac{\log(660k^2)}{\log(0.0116k^4)} < 2.$$

Lemma 3.3. *Suppose that* $\{1, k^2+1, k^2+4, d\}$ *is a* $D(-k^2)$*-quadruple with* $k > 512$. *Then*

$$\log x < \frac{12\log(3.4k)\log(0.329k)}{\log(0.00419k)}.$$

Proof of Lemma 3.3. Applying Theorem 3.2 with $p_1 = y$, $p_2 = z$, $q = kx$, we see from Lemma 3.1 that

$$(2675k^2)^{-1}(kx)^{-\lambda} < \frac{k^2}{\sqrt{3}x^2}.$$

By $\lambda < 2$ we have

$$x^{2-\lambda} < \frac{2675}{\sqrt{3}}k^{4+\lambda} < \frac{2675}{\sqrt{3}}k^6 < (3.4k)^6.$$

Since

$$\frac{1}{2-\lambda} = \frac{1}{1-\frac{\log(660k^2)}{\log(0.0116k^4)}} = \frac{\log(0.0116k^4)}{\log\left(\frac{0.0116k^2}{660}\right)} < \frac{2\log(0.329k)}{\log(0.00419k)},$$

we obtain

$$\log x < \frac{12\log(3.4k)\log(0.329k)}{\log(0.00419k)}.$$

$\square$

We are now ready to prove Theorem 1.1 for all but finitely many k.

Proposition 3.4. *Let k be an odd prime number. Suppose that $x = v_m = w_n$ has a solution for some m and n.*

(1) *If $(y_0, x_0) = (k(k^2 - k + 1), \pm k(k-1))$, then $k = 3$, 19 or 53.*
(2) *If $(y_0, x_0) = (k^2 + 2, \pm 2)$, then $k = 3$ or $k = 23$.*

Proof of Proposition 3.4. Remark that a solution of $x = v_m = w_n$ gives a $D(-k^2)$-quadruple $\{1, k^2+1, k^2+4, d\}$ with $d = x^2 + k^2$. (1) Suppose that $k > 619$, that is, $k \geq 631$. Then, Lemmas 2.6 and 3.3 together imply

$$2m - 0.21 < \frac{12\log(3.4k)\log(0.329k)}{\log(2k)\log(0.00419k)} =: f(k).$$

Since $f(k)$ is a decreasing function, we have $f(k) \leq f(631) < 71$. On the other hand, Lemma 2.4 implies that

$$2m - 0.21 \geq \frac{(\sqrt{5}-2)k - 3}{2} - 0.21 > 72,$$

which is a contradiction. Therefore, $k \leq 619$. Since $z_1 = k\sqrt{2k^2+7}$ by Lemma 2.3 and z_1 is an integer, $2k^2 + 7$ has to be a square. The only primes k with $k \leq 619$ such that $2k^2 + 7$ is a square are 3, 19 and 53.

(2) Lemma 2.5 (2) implies that it suffices to show that assuming $k > 5 \cdot 10^4$ leads to a contradiction. Combining Lemmas 2.6 and 3.3 yields $2m + 0.87 < f(k)$. Hence we obtain $f(k) < f(5 \cdot 10^4) < 22.81$ and $m < 11$, which contradicts Lemma 2.5 (3). $\square$

4. The solutions for exceptional k

In this section we will consider the remaining cases. From Proposition 3.4 we know that we have to solve the equation $x = v_m = w_n$ when $(y_0, x_0) = (k(k^2 - k + 1), \pm k(k-1))$ and $k = 3$, 19 or 53. And we also have to solve the same equation in the case $(y_0, x_0) = (k^2 + 2, \pm 2)$ and $k = 3$ or $k = 23$. So, we actually have to find intersection of binary recurrence sequences. To do that we use the standard method in dealing with such kind of problems. We use Baker's theory on linear forms in logarithms, which yields us an upper bound for indices m and n. Then we reduce that bound using Baker-Davenport reduction. And at the end we check if we can have any intersection of those sequences with small indices. Because the procedure here is pretty standard we will not give all details.

Let us first consider the equation $x = v_m = w_n$ and $(y_0, x_0) = (k(k^2 - k+1), \pm k(k-1))$ for $k = 3, 19, 53$. From Lemma 2.3 we know $(z_1, x_1) = (k\sqrt{2k^2+7}, \mp k)$. So our sequences v_m and w_n are given by

$$v_0 = \pm k(k-1),\ v_1 = 2k^2(k^2-k+1) \pm (2k^2+1)(k^2-k),$$
$$v_{m+2} = 2(2k^2+1)v_{m+1} - v_m, \tag{4.1}$$

(4.2)

$$w_0 = \mp k,\ w_1 = \frac{k^2\sqrt{2k^2+7} \mp k(k^2+2)}{2},\ w_{n+2} = (k^2+2)w_{n+1} - w_n.$$

Because we will check what is happening for the small indices later, let us suppose $m, n > 2$. Then it is easy to see that if $v_m = w_n$ has a solution then $m \le n < 2m$. Let us define the linear form

$$\Lambda = m\log(2k^2+1+2k\sqrt{k^2+1}) - n\log\left(\frac{k^2+2+k\sqrt{k^2+4}}{2}\right)$$
$$+ \log\frac{\sqrt{k^2+4}(k(k^2-k+1) \pm k(k-1)\sqrt{k^2+1})}{\sqrt{k^2+1}(k\sqrt{2k^2+7} \mp k\sqrt{k^2+4})}.$$

Now we can prove the following lemma.

Lemma 4.1. *If $v_m = w_n$ for $m, n > 2$, then*

$$0 < \Lambda < 6k^2(2k^2+1+2k\sqrt{k^2+1})^{-2m}.$$

Proof of Lemma 4.1. Let

$$P = \frac{1}{\sqrt{k^2+1}}(k(k^2-k+1) \pm k(k-1)\sqrt{k^2+1})(2k^2+1+2k\sqrt{k^2+1})^m$$

and

$$Q = \frac{1}{\sqrt{k^2+4}}(k\sqrt{2k^2+7} \mp k\sqrt{k^2+4})\left(\frac{k^2+2+k\sqrt{k^2+4}}{2}\right)^n.$$

Then $v_m = w_n$ implies

$$P - \frac{k^4}{k^2+1}P^{-1} = Q - \frac{k^2(k^2+3)}{k^2+4}Q^{-1}.$$

It is easy to see that $P, Q > 1$. Then we have

$$P - Q = \frac{k^4}{k^2+1}P^{-1} - \frac{k^2(k^2+3)}{k^2+4}Q^{-1} > \frac{k^4}{k^2+1}(P^{-1} - Q^{-1})$$
$$= \frac{k^4}{k^2+1}(Q-P)P^{-1}Q^{-1},$$

which implies $P > Q$. Moreover, we have

$$\frac{P-Q}{P} < \frac{k^4}{k^2+1}P^{-2}.$$

Then we can conclude

$$
\begin{aligned}
0 < \log\frac{P}{Q} &= -\log\left(1-\frac{P-Q}{P}\right) \le \frac{-\log\left(1-\frac{1}{2}\right)}{\frac{1}{2}}\frac{P-Q}{P} < 1.39\frac{k^4}{k^2+1}P^{-2} \\
&= 1.39\frac{k^4}{k^2+1}\frac{(k^2+1)(2k^2+1+2k\sqrt{k^2+1})^{-2m}}{(k(k^2-k+1)\pm k(k-1)\sqrt{k^2+1})^2} \\
&= 1.39\frac{(k(k^2-k+1)\mp k(k-1)\sqrt{k^2+1})^2}{k^4} \\
&\quad \times (2k^2+1+2k\sqrt{k^2+1})^{-2m} \\
&< 6k^2(2k^2+1+2k\sqrt{k^2+1})^{-2m}.
\end{aligned}
$$

□

Now using Baker's theory on linear forms in logarithms ([1, Theorem]) we get a lower bound for our linear form Λ. Combining that with the upper bound from Lemma 4.1 we get that if $v_m = w_n$ with $m, n > 2$, then

$$\frac{n}{\log n} < 5.74 \cdot 10^{15}(\log k)^2.$$

Now using $m \le n$ and $k \le 53$, we conclude $m < 4 \cdot 10^{18}$.

Because this upper bound for m is rather large we should use Baker-Daveport reduction method ([11, Lemma 5a]) here. We have done this in Mathematica. And after at most three steps of reduction we get $m \le 2$ in all cases.

So at the end it is left to check what is happening for $m \le 2$. If $k = 19$ or $k = 53$ we do not get any solution. But in the case $k = 3$, for $x_0 = 6$, we get one solution $v_0 = w_1 = 6$. And if $x = 6$ we get an extension of $D(-9)$-triple $\{1, 10, 13\}$ to a $D(-9)$-quadruple $\{1, 10, 13, 45\}$.

We should now consider the case when $(y_0, x_0) = (k^2+2, \pm 2)$ and $k = 3$ or $k = 23$. But in that case, in exactly the same way, using mentioned methods we get that $v_m = w_n$ does not have any solution, which finishes the proof of Theorem 1.1.

References

[1] A. Baker and G. Wüstholz, *Logarithmic forms and group varieties*, J. Reine Angew. Math. **442** (1993), 19–62.

[2] M. A. Bennett, *On the number of solutions of simultaneous Pell equations*, J. Reine Angew. Math. **498** (1998), 173–199.

[3] M. A. Bennett, M. Cipu, M. Mignotte and R. Okazaki, *On the number of solutions of simultaneous Pell equations II*, Acta Arith. **122** (2006), 407–417.

[4] E. Brown, *Sets in which $xy + k$ is always a square*, Math. Comp. **45** (1985), 613–620.

[5] A. Dujella, *Generalization of a problem of Diophantus*, Acta Arith. **65** (1993), 15–27.

[6] A. Dujella, *Complete solution of a family of simultaneous Pellian equations*, Acta Math. Inform. Univ. Ostraviensis **6** (1998), 59–67.

[7] A. Dujella, *An absolute bound for the size of Diophantine m-tuples*, J. Number Theory **89** (2001), 126–150.
[8] A. Dujella, *There are only finitely many Diophantine quintuples*, J. Reine Angew. Math. **566** (2004), 183–214.
[9] A. Dujella, A. Filipin and C. Fuchs, *Effective solution of the $D(-1)$-quadruple conjecture*, Acta Arith. **128** (2007), 319–338.
[10] A. Dujella and C. Fuchs, *Complete solution of a problem of Diophantus and Euler*, J. London Math. Soc. **71** (2005), 33–52.
[11] A. Dujella and A. Pethő, *A generalization of a theorem of Baker and Davenport*, Quart. J. Math. Oxford Ser. (2) **49** (1998), 291–306.
[12] Y. Fujita, *The non-extensibility of $D(4k)$-triples $\{1, 4k(k-1), 4k^2+1\}$ with $|k|$ prime*, Glas. Mat. Ser. III **41** (2006), 205–216.
[13] Y. Fujita, *Extensions of the $D(\mp k^2)$-triples $\{k^2, k^2 \pm 1, 4k^2 \pm 1\}$*, Period. Math. Hungar. **59** (2009), 81–98.
[14] Y. Fujita, *The number of Diophantine quintuples*, Glas. Mat. Ser. III **45** (2010), 15–29.
[15] Y. Fujita and A Togbé, *The extension of the $D(-k^2)$-pair $\{k^2, k^2+1\}$*, Period. Math. Hungar., to appear.
[16] R. A. Mollin, Quadratics, CRC Press, Boca Ranton, 1996.
[17] J. E. Shockley, Introduction to Number Theory, Holt, Rinehart and Winston, 1967.

A. Filipin
Faculty of Civil Engineering
University of Zagreb
Fra Andrije Kačića-Miošića 26, 10 000 Zagreb
Croatia
E-mail: filipin@grad.hr

Y. Fujita
Department of Mathematics
College of Industrial Technology
Nihon University
2-11-1 Shin-ei, Narashino
Chiba
Japan
E-mail: fujita.yasutsugu@nihon-u.ac.jp

Received: 13.5.2010.
Revised: 24.7.2010.