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A NOTE ON ULTRAPRODUCTS OF VELTMAN MODELS

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Abstract. We consider ultraproducts of Veltman models, and show that a version of Łos theorem is true.

1. Introduction

Interpretability logic is an extension of provability logic GL (Gödel, Löb). For precise definitions and details, see e.g. [5] or [6]. We are only interested in interpretability logic as a system of modal logic, i.e., we are interested in semantics for interpretability logic. There are several kinds of semantics for interpretability logic. The basic semantics is given by Veltman models. In [1] ultraproducts are used for a proof of the existence of countably saturated models and for modal definability. In [4] various preservation results, i.e., versions of Łos theorem, for ultraproducts of Kripke models are considered.

We define an ultraproduct of Veltman models over a countably complete ultrafilter. Then we show that the ultraproduct is a Veltman model. Using the standard translation for interpretability logic we prove a version of Łos theorem for Veltman models.

2. Ultraproducts of Veltman models

The notion of Veltman model is defined in [3].

Definition 2.1. *An ordered triple* $\langle W, R, \{S_w : w \in W\}\rangle$ *is called a Veltman frame if it satisfies the following conditions:*

a) $\langle W, R\rangle$ *is a GL–frame, i.e.,* W *is a non-empty set, and* R *is transitive and reverse well–founded relation on* W*;*
b) *For every* $w \in W$ *is* $S_w \subseteq W[w] \times W[w]$*, where* $W[w] = \{u : wRu\}$*;*

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c) *The relation* S_w *is reflexive and transitive for every* $w \in W$*;*

d) *If* $wRuRv$ *then* uS_wv*.*

An ordered quadruple $\langle W, R, \{S_w : w \in W\}, \Vdash\rangle$ *is called a Veltman model if it satisfies the following conditions:*

1) $\langle W, R, \{S_w : w \in W\}\rangle$ *is a Veltman frame;*

2) $\Vdash$ *is a forcing relation. We emphasize only the definition*

$$w \Vdash A \rhd B \text{ if and only if } \forall u((wRu \ \& \ u \Vdash A) \Rightarrow \exists v(uS_wv \ \& \ v \Vdash B)).$$

We denote a Veltman model $\langle W, R, \{S_w : w \in W\}, \Vdash\rangle$ shortly by W.

Definition 2.2 (see [2]). *An ultrafilter* U *is countably complete if for each sequence* $(A_n)_{n\in\mathbb{N}}$ *in* U *we have* $\cap A_n \in U$*.*

Lemma 2.3. *An ultrafilter* U *is countably complete if and only if there is no sequence* $(B_n)_{n\in\mathbb{N}}$ *in* U *such that* $\cap B_n = \emptyset$*.*

Proof of Lemma 2.3. If U is an ultrafilter such that there is a sequence (B_n) such that $\cap B_n = \emptyset$, then $\cap B_n \notin U$.

Let (A_n) be a sequence in an ultrafilter U such that $\cap A_n \notin U$. Then we have $(\cap A_n)^c \in U$. Let us define a sequence (B_n) by: $B_0 = (\cap A_n)^c$, and $B_{n+1} = A_n$. Obviously, $B_n \in U$ for each $n \in \mathbb{N}$, and $\cap B_n = \emptyset$. ∎

Let $\{W_i : i \in I\}$ be a set of Veltman models, $W_i = (W_i, R_i, \{S^{(i)}_w : w \in W_i\}, \Vdash)$. Let U be a countably complete ultrafilter over the set I, and $W = \prod_U W_i$. We define a relation $R \subseteq W \times W$ in the following way:

$$\overline{f} R \overline{g} \text{ if and only if } \{i \in I : f(i) R_i g(i)\} \in U.$$

Let us denote $W[\overline{g}] = \{\overline{f} : \overline{g} R \overline{f}\}$, for each $g \in \prod_{i\in I} W_i$. For every $\overline{g} \in W$ we define a relation $S_{\overline{g}} \subseteq W[\overline{g}] \times W[\overline{g}]$ by

$$\overline{f} S_{\overline{g}} \overline{h} \text{ if and only if } \{i \in I : f(i) S^{(i)}_{g(i)} h(i)\ \} \in U.$$

Finally, we define a forcing relation on the set W by:

$$W, \overline{f} \Vdash p \text{ if and only if } \{i \in I : W_i, f(i) \Vdash p\} \in U.$$

Proposition 2.4. *The ultraproduct of Veltman models over a countably complete ultrafilter is a Veltman model.*

Proof of Proposition 2.4. It is easy to check that the relation R is transitive. Let us suppose that the relation R is not reverse well–founded. Let $(\overline{f}_n)$ be a sequence in W such that $\overline{f}_n R \overline{f}_{n+1}$. Then for each $n \in \mathbb{N}$ we have $A_n := \{i \in I : f_n(i) R_i f_{n+1}(i)\} \in U$. Because U is a countably complete ultrafilter, then $\cap A_n \neq \emptyset$. Let i be an arbitrary element of the set $\cap A_n$. Then we have $f_1(i) R_i f_2(i) R_i f_3(i) \ldots$. So, we have that the relation R_i is not reverse well–founded, which is a contradiction.

It is easy to check that the relation $S_{\overline{g}}$ is reflexive and transitive, for each $g \in \prod_{i\in I} W_i$. Let us show that $\overline{g}R\overline{f}R\overline{h}$ implies $\overline{f}S_{\overline{g}}\overline{h}$. Let f, g, and h be elements of $\prod_{i\in I} W_i$ such that $\overline{g}R\overline{f}R\overline{h}$. Then we have

$$\{i \in I : g(i)R_i f(i)\} \in U \qquad \text{and} \qquad \{i \in I : f(i)R_i h(i)\} \in U,$$

This implies $\{i \in I : g(i)R_i f(i)\} \cap \{i \in I : f(i)R_i h(i)\} \in U$, i.e., $\{i \in I : g(i)R_i f(i)R_i h(i)\} \in U$. Because W_i is a Veltman model, then we have

$$\{i \in I : g(i)R_i f(i)R_i h(i)\} \subseteq \{i \in I : f(i)S^{(i)}_{g(i)}h(i)\}.$$

So, $\{i \in I : f(i)S^{(i)}_{g(i)}h(i)\} \in U$, i.e., $\overline{f}S_{\overline{g}}\overline{h}$. □

3. Standard translations

Standard translation is a function that maps each modal formula to a first–order formula. In [1] standard translation of "standard" modal logic is considered. Interpretability logics are nonstandard logics. We define a standard translation for interpretability logics, and show basics results. Let $\sigma = \{P_0, P_1, \ldots\} \cup \{R, S\}$ be a first–order signature, where P_i is an unary relation symbol, R is a binary relation symbol, and S is a ternary relation symbol.

Definition 3.1. *Let x be a first–order variable. The standard translation ST_x taking modal formulas to first–order σ–formulas is defined as follows:*

$ST_x(p_i) = P_i(x)$,
$ST_x(\neg\varphi) = \neg ST_x(\varphi)$,
$ST_x(\varphi \wedge \psi) = ST_x(\varphi) \wedge ST_x(\psi)$,
$ST_x(\Box\varphi) = \forall y(xRy \to ST_y(\varphi))$,
$ST_x(\varphi \rhd \psi) = \forall y(xRy \wedge ST_y(\varphi), \ \to \exists z(S(x,y,z) \wedge ST_z(\psi)))$.

The following proposition is easy to prove by induction on complexity of modal formula.

Proposition 3.2. *Let φ be a modal formula, W a Veltman model and $w \in W$. Then we have:*

$$W, w \Vdash \varphi \qquad \text{if and only if} \qquad W \models ST_x(\varphi)[w].$$

By using the previous proposition and Łos theorem (see [2]), we get the following result.

Corollary 3.3. *Let $\{W_i : i \in I\}$ be a set of Veltman models, U a countably complete ultrafilter over a set I, and φ a modal formula. For each $f \in \prod_{i\in I} W_i$ we have:*

$$\prod_U W_i, \overline{f} \Vdash \varphi \qquad \text{if and only if} \qquad \{i \in I : W_i, f(i) \Vdash \varphi\} \in U.$$

Corollary 3.4. *Let* $(W, R, \{S_w : w \in W\}, \Vdash)$ *be a Veltman model,* U *a countably complete ultrafilter over a set* I*, and* $w \in W$*. Let* $f_w : I \to W$ *be a function that is defined by* $f_w(i) = w$*. For each modal formula* φ *we have:*

$$W, w \Vdash \varphi \qquad \textit{if and only if} \qquad \prod_U W, \overline{f_w} \Vdash \varphi.$$

Remark 3.5. We would like to mention that we have not obtained analogous results for generalized Veltman semantics. The definition of generalized Veltman model is given for example in [7]. It is easy to define a notion of ultraproduct for generalized Veltman model, and prove quasi–reflexivity of the relation $S_{\overline{g}}$. However we have not been able to produce a proof that the relation $S_{\overline{g}}$ is quasi–transitive.

Remark 3.6. We have mentioned in introduction that ultraproducts are useful for a proof of existence of countably saturated models. Each countably saturated model is a modal saturated Kripke model (see [1]). We try to prove van Benthem characterization theorem for interpretabilty logics. A big problem is to define a notion of modal saturated Veltman model, and to prove that an ultrafilter extension of Veltman model is Veltman model.

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