SOME STRONGLY REGULAR GRAPHS AND SELF-ORTHOGONAL CODES FROM THE UNITARY GROUP $U_4(3)$

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Abstract. We construct self-orthogonal codes from the row span over $\mathbb{F}_2$ or $\mathbb{F}_3$ of the adjacency matrices of some strongly regular graphs defined by the rank-3 action of the simple unitary group $U_4(3)$ on the conjugacy classes of some of its maximal subgroups. We establish some properties of these codes and the nature of some classes of codewords.

1. Introduction

Codes spanned by the incidence matrix of combinatorial structures have been the subject of much research, see [1] for collected results. Although codes associated with graphs have not been extensively studied, some significant results concerning the p-rank of adjacency matrices of strongly regular graphs have been established: see Brouwer and van Eijl ([8, 24]). Recent studies on the interplay between codes and graphs have provided several useful and interesting results, see for example [16,28]. In particular, some special strongly regular graphs generate interesting codes, see [22, 23, 27]. Graph associated constructions of codes are also studied in [29].

The most symmetric strongly regular graphs are rank-3 graphs, that is graphs whose automorphism group acts transitively on the vertices, ordered pairs of adjacent vertices and ordered pairs of non-adjacent vertices. Rank-3 graphs play a significant role in group theory.

In this paper, using ideas similar to those described in [13,14] and those of [12] we construct rank-3 graphs from the conjugacy classes of some maximal subgroups of the simple unitary group $U_4(3)$. Furthermore, using the

adjacency matrices of the graphs we construct binary (resp. ternary) self-orthogonal codes, which are invariant under $U_4(3)$. The reader is encouraged to consult [18–20] for a construction of codes from groups, graphs or designs. We use the properties of the graphs and their geometry to gain some insight into the nature of possible codewords, particularly those of smallest and largest weight. Hence the paper uncovers the interplay between linear codes and graphs, in particular rank-3 graphs, associated with the simple unitary group $U_4(3)$ and provides a geometric description of some classes of codewords, helped by the geometry of the graphs. We used Magma ([3]) and GAP ([15]), and the GAP package GRAPE ([25]) to examine the graphs and codes described in this paper.

The paper is organized as follows: after a brief description of our terminology and some background, Section 3 gives a brief description of the simple unitary group $U_4(3)$; Section 4 outlines the construction of graphs from the conjugacy classes of maximal subgroups of $U_4(3)$ and in Section 5 we present our results.

2. Terminology and notation

We assume that the reader is familiar with some basic notions and elementary facts from strongly regular graphs, design and coding theory. Our notation for designs and codes are standard and follow that of $[1, 26]$ and for the structure of groups we follow the ATLAS ([11]) notation.

An incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$, with point set $\mathcal{P}$, block set $\mathcal{B}$ and incidence $\mathcal{P}$ is a t-(v, k, λ) design, if $|\mathcal{P}| = v$, every block $B \in \mathcal{B}$ is incident with precisely k points, and every t distinct points are together incident with precisely λ blocks. A 1-design is just an incidence structure in which the number of points incident with a block and the number of blocks incident with a point are constants. The complementary design of $\mathcal{D}$ is obtained by replacing all blocks of $\mathcal{D}$ by their complements. The design $\mathcal{D}$ is *symmetric* if it has the same number of points and blocks. An automorphism of a design $\mathcal{D}$ is a permutation on $\mathcal{P}$ which sends blocks to blocks. The set of all automorphisms of $\mathcal{D}$ forms its full automorphism group denoted by Aut $\mathcal{D}$.

The *code C_F of the design* $\mathcal{D}$ over the finite field F is the space spanned by the incidence vectors of the blocks over F, we denote this code by C.

The weight enumerator of C is defined as $W_C(x) = \sum_{i=0}^n A_i x^i$, where A_i denotes the number of codewords of weight i in C. The *dual* code $C^\perp$ is the orthogonal complement under the standard inner product $(,)$, i.e., $C^\perp = \{v \in F^n \mid (v, c) = 0 \text{ for all } c \in C\}$. A code C is *self-orthogonal* if $C \subseteq C^\perp$ and it is *self-complementary* if it contains the all-one vector. The all-one vector will be denoted by $\mathbf{1}$, and is the constant vector of weight the length of the code. A binary code C is *doubly-even* if all codewords of C have weight divisible

by four. An *automorphism* of a code is any permutation of the coordinate positions that maps codewords to codewords.

Terminology for *graphs* is standard: the graphs, $\mathcal{G} = (V, E)$ with vertex set V and edge set E, are undirected and the *degree* of a vertex is the number of edges containing the vertex. A graph is *regular* if all the vertices have the same degree; a regular graph is *strongly regular* of type (n, k, λ, μ) if it has n vertices, degree k, and if any two adjacent vertices are together adjacent to λ vertices, while any two non-adjacent vertices are together adjacent to μ vertices. A rank 3 graph is a graph that admits an automorphism group which is transitive on the vertices, edges, and nonedges. Note that any rank 3 graph is a strongly regular graph. The converse is not always true. The complementary graph of a strongly regular graph with parameters (n, k, λ, μ) is a strongly regular graph with parameters $(n, n - k - 1, n - 2k + \mu - 2, n - 2k + \lambda)$. A code $C_{\mathcal{G}}$ of a graph $\mathcal{G}$ is the code of its $(0, 1)$-adjacency matrix. The dimension of $C_{\mathcal{G}}$ is equal to the p-rank of its adjacency matrix, i.e., the rank of $\mathcal{G}$ regarded as a matrix over $GF(p)$. A connected strongly regular graph has diameter 2. If v and w are vertices of a connected strongly regular graph $\mathcal{G}$ such that $d(v, w) = i$, $i = 0, 1, 2$, then the number p_{ij} of neighbors of w whose distance from v is j, $j = 0, 1, 2$, are the intersection numbers of $\mathcal{G}$. The 3×3-matrix with entries p_{ij}, $i, j = 0, 1, 2$, is called the *intersection matrix* of $\mathcal{G}$.

It follows from [12], that from a conjugacy class of a maximal subgroup H of a simple group G one can construct a regular graph in the following way:

- the vertex set of the graph is $ccl_G(H)$,
- the vertex H^{g_i} is adjacent to the vertex H^{g_j} if and only if $H^{g_i} \cap H^{g_j} \cong G_i$, $i = 1, \ldots, k$, where $\{G_1, ..., G_k\} \subset \{H^x \cap H^y \mid x, y \in G\}$. We denote a regular graph constructed in this way by $\mathcal{G}(G, H; G_1, ..., G_k)$.

3. The group $U_4(3)$

The unitary group $U(n, q^2)$ is defined to be the set of linear transformations which leave a given non-degenerate hermitian form invariant. The *projective special unitary group* $PSU(n, q^2)$ is the group of all collineations of $\mathcal{P}(V)$ that commute with the polarity σ. Equivalently, if A is the matrix associated with the hermitian form $\langle , \rangle$ then $A = \overline{A}^t$ and the group $U(n, q^2)$ consists of all invertible matrices P which satisfy $\overline{P}^t A P = A$, and $PSU(n, q^2)$ is the factor group $SU(n, q^2)/SU(n, q^2) \cap Z$, where Z is the group of scalar matrices.

We take G to be the simple unitary group $PSU(4, 3^2)$ denoted in the ATLAS ([11]) as $U_4(3)$. Notice that G is the group of all non-singular 4×4 matrices over $\mathbb{F}_9$ preserving a unitary form, and whose determinant is 1. G has order 3265920 and its automorphism group is an extension of $U_4(3)$ by D_8 which is a group of order 26127360. The group $U_4(3)$ has sixteen maximal subgroups up to conjugation, namely of degrees 112, 126, 126, 162, 162, 280,

540, 567, 567, 1296, 1296, 1296, 1296, 2835, 4536, 4536 (see Table 1). The representations of degrees 112, 126, 162, and 280 are all of rank-3. The representations of degree 126, 162, 567, 1296 and 4356 are pairwise equivalent under an outer automorphism group (see $\mathbb{ATLAS}$ ([11])). In Table 1 which follows, the first column depicts the ordering of the primitive representations as given by the $\mathbb{ATLAS}$ ([11]) and as used in our computations; the second gives the maximal subgroups and the third gives the degrees (the number of cosets of the point stabilizer).

No.	Max. sub.	Deg.	No.	Max. sub.	Deg.
1	$3^4{:}A_6$	112	9	$2^4{:}A_6$	567
2	$U_4(2)$	126	10	A_7	1296
3	$U_4(2)$	126	11	A_7	1296
4	$L_3(4)$	162	12	A_7	1296
5	$L_3(4)$	162	13	A_7	1296
6	$3^{1+4}_+ \cdot 2S_4$	280	14	$2(A_4 \times A_4) \cdot 4$	2835
7	$U_3(3)$	540	15	M_{10}	4536
8	$2^4{:}A_6$	567	16	M_{10}	4536

TABLE 1. Maximal subgroups of $U_4(3)$

4. STRONGLY REGULAR GRAPHS FROM $U_4(3)$

As indicated above some representations are pairwise equivalent, so we only consider graphs, designs and codes obtained from one representation of that degree, since the structures obtained from both representations are isomorphic. For example, there are two rank-3 representations of degree 126, in that case we consider the first representation and construct from it the associated incidence structures, i.e., graphs and codes. The representations of degree 540, 567, 1296, 2835 and 4536 are not comfortably easy to compute with, as such we have not considered determining the incidence structures deriving from those. The conjugacy classes of the subgroups H_i, $i = 1, \ldots, 16$, in G are denoted by $ccl_G(H_i)$. Since G is a simple group and H_i are maximal subgroups we have that

$$N_G(H_i) = H_i \Rightarrow |ccl_G(H_i)| = [G : H_i].$$

For $i \in \{1, \ldots, 16\}$ we denote the elements of $ccl_G(H_i)$ by $H_i^{g_1}, H_i^{g_2}, \ldots, H_i^{g_j}$, where $j = [G : H_i]$.

5. STRONGLY REGULAR GRAPHS AND CODES FROM $U_4(3)$

In the sequel we construct binary or ternary codes from strongly regular graphs defined by the representations of degree at most 280. We denote

the graphs $\mathcal{G}_i$ where $i = 112, 126, 162, 280$ and their complements $\overline{\mathcal{G}}_i$ with i as above. The p-rank of some of the graphs, i.e., the dimension of their associated codes, that we will examine is known, in particular that of the graphs with parameters $(112, 30, 2, 10)$ and $(162, 56, 10, 24)$ and can be found in [8]. Furthermore, taking the row span of the adjacency matrix of $\mathcal{G}_i$ (resp. $\overline{\mathcal{G}}_i$) over $\mathbb{F}_2$ or $\mathbb{F}_3$ we construct binary (resp. ternary) codes denoted C_i or $\overline{C}_i$.

5.1. *A strongly regular* $(112, 30, 2, 10)$ *graph.* The cardinality of the conjugacy class $ccl_G(H_1)$ is 112. The intersection of two distinct elements of $ccl_G(H_1)$ is isomorphic to A_6 or $3_+^{1+4} : 4$.

The incidence structure $\mathcal{G}_{112} = (\mathcal{V}_1, \mathcal{E}_1, \mathcal{I}_1)$, where $\mathcal{V}_1 = \{V_1^{(1)}, \ldots, V_{112}^{(1)}\}$ and vertices $V_i^{(1)}$ and $V_j^{(1)}$ are adjacent if and only if $H_1^{g_i} \cap H_1^{g_j} \cong 3_+^{1+4} : 4$, i.e., the graph $\mathcal{G}(U_4(3), H_1; 3_+^{1+4} : 4)$ is a strongly regular graph with parameters $(112, 30, 2, 10)$ and spectrum $30^1 2^{90} (-10)^{21}$.

It is well-known that a graph with the parameters of $\mathcal{G}_{112}$ is unique (see [9,10]). In fact $\mathcal{G}_{112}$ is isomorphic to the collinearity graph of the unique generalized quadrangle $GQ(3, 9)$ and it is the first subconstituent of the McLaughlin graph. For a description of a construction of $\mathcal{G}_{112}$ using the unique Steiner system $S(4, 7, 23)$, see [9, Section C, p. 113]. The full automorphism group of $\mathcal{G}_{112}$ is isomorphic to the group $U_4(3) \cdot D_8$ and it has order 26127360 acting as a rank-3 group with point stabilizer isomorphic to the group $3^4 : ((2 \times A_6) \cdot 2^2)$. Hence we have a rank-3 graph. As described in Section 5, from the row span over $\mathbb{F}_2$ or $\mathbb{F}_3$ of the adjacency matrices of $\mathcal{G}_{112}$ (resp. $\overline{\mathcal{G}}_{112}$) we now construct, binary (resp. ternary) codes which are invariant under $U_4(3)$.

Note that an adjacency matrix for the graph constructed in the manner given above, will simply be an incidence matrix for a corresponding 1-design, so that the 1-design is necessarily self-dual, this follows from a construction outlined in [18]. Thus, we define self-dual symmetric 1-(i, k, k) designs $\mathcal{D}_i$ (resp. $\overline{\mathcal{D}}_i$) via the adjacency matrices of the graphs $\mathcal{G}_i$ (resp. $\overline{\mathcal{G}}_i$), where the i vertices serve as both point and blocks of the design and adjacency in the graph is incidence in the design. In this way using the representation of degree 112, we can construct a self-dual symmetric 1-$(112, 30, 30)$ design $\mathcal{D}_{112}$ (resp. a self-dual symmetric 1-$(112, 81, 81)$ design $\overline{\mathcal{D}}_{112}$). Furthermore, taking the row span of the incidence matrix of $\mathcal{D}_{112}$ (resp. $\overline{\mathcal{D}}_{112}$) over $\mathbb{F}_2$ or $\mathbb{F}_3$ one can construct binary (resp. ternary) codes C_{112} and $\overline{C}_{112}$, isomorphic to those given in Propositions 5.1 and 5.5. Using some properties of the graph we establish some facts on the codes, and by using the graph's geometry we describe some classes of codewords, in particular those of smallest and largest weight. In Proposition 5.1 and Corollary 5.3 below we examine codes obtained from $\mathcal{G}_{112}$.

PROPOSITION 5.1.   (i) C_{112} *is a* $[112, 22, 30]_2$ *self-orthogonal code.*
(ii) $\mathbf{1} \in \overline{C}_{112}$.

(iii) $C_{112}^{\perp}$ *is a self-complementary* $[112, 90, 6]_2$ *code with* 5040 *words of weight 6.*

(iv) $\mathrm{Aut}(\mathcal{G}_{112}) = \mathrm{Aut}(C_{112}) \cong U_4(3) \cdot D_8.$

PROOF. We will use the strong regularity of $\mathcal{G}_{112}$ to show that C_{112} is self-orthogonal. Notice first that C_{112} is obtained from the strongly regular graph $\mathcal{G}_{112}$ with parameters $(112, 30, 2, 10)$ and intersection matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 30 & 2 & 10 \\ 0 & 27 & 20 \end{bmatrix}.$$

Now, if we fix a vertex v in $\mathcal{G}_{112}$ we can divide the remaining vertices into two sets, namely $\mathcal{G}_{112}'$ of size 30 and $\mathcal{G}_{112}''$ of size 81, with $\mathcal{G}_{112}'$ being the set of vertices adjacent to v, and $\mathcal{G}_{112}''$ the set of vertices non-adjacent to v. As it can be seen Figure 1, each vertex in $\mathcal{G}_{112}'$ is adjacent to v and to 2 other vertices in $\mathcal{G}_{112}'$, thus to 27 vertices in $\mathcal{G}_{112}''$. Each vertex in $\mathcal{G}_{112}''$ is adjacent to 10 vertices in $\mathcal{G}_{112}'$, and so to 20 vertices in $\mathcal{G}_{112}''$.
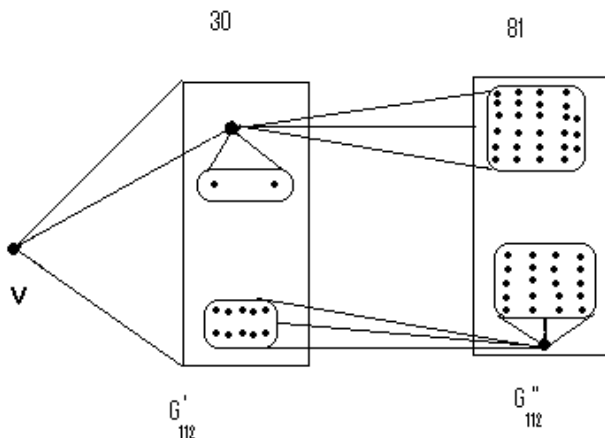

FIGURE 1. Number of joins between orbits of a stabilizer

The valency 30 ensures that generating codewords have weight zero (mod 2) and the 2 and the 10 ensure that (i) any two generating codewords have an even number of non-zero common entries, and (ii) that any two generating codewords are orthogonal to one another.

Since the blocks of $\mathcal{D}_{112}$ are of even size, we have that $\mathbf{1}$ meets evenly every vector of C_{112}, so $\mathbf{1} \in C_{112}{}^{\perp}$. The 2-rank 22 of $\mathcal{G}_{112}$, that is, the dimension of C_{112} is well known (see [8]), and also follows from [24, Lemma 6.1]. The

minimum distance 30 and the fact that $\mathbf{1} \in C_{112}$ can be deduced from the weight enumerator of this code which follows

$$
\begin{aligned}
W_{C_{112}} \ = \ & 1 + 112\ x^{30} + 1134\ x^{32} + 13608\ x^{40} + 5184\ x^{42} + 127008\ x^{46} \\
& + 394065\ x^{48} + 371952\ x^{50} + 544320\ x^{54} + 1279536\ x^{56} \\
& + 544320\ x^{58} + 371952\ x^{62} + 394065\ x^{64} + 127008\ x^{68} \\
& + 5184\ x^{70} + 13608\ x^{72} + 1134\ x^{80} + 112\ x^{82} + \ x^{112}.
\end{aligned}
$$

The minimum weight 6 of $C_{112}{}^\perp$ follows from computations with Magma ([3]). □

REMARK 5.2. (i) The 112 codewords of weight 30 in C_{112} can be described geometrically using either the graph $\mathcal{G}_{112}$ or the design $\mathcal{D}_{112}$. These codewords are the rows of the adjacency matrix of $\mathcal{G}_{112}$. Moreover, they can also be regarded as the incidence vectors of the blocks of the design $\mathcal{D}_{112}$. The stabilizer in $U_4(3)$ and in $\mathrm{Aut}(C_{112})$ of a codeword (or of the set of codewords) of weight 30 in C_{112} is a maximal subgroup of index 112 isomorphic to $3^4{:}A_6$ and $3^4{:}(2 \times A_6 \cdot 2^2)$ respectively. Hence $U_4(3)$ and $U_4(3){\cdot}D_8$ act primitively as rank-3 groups on points and blocks of the design $\mathcal{D}_{112}$. In addition to that we have that the codewords of weight 30 span C_{112}.

(ii) The 5040 codewords of minimum weight in $C_{112}{}^\perp$ form a single orbit stabilized by a non-maximal subgroup of $\mathrm{Aut}(C_{112}{}^\perp)$ of type $(((3 \times ((3 \times 3 \times 3){:}Q_8)){:}2){:}2){:}2$.

It can be deduced from the weight enumerator of C_{112} that this code contains a doubly-even subcode. We denote this code $\mathcal{L}$. Now, the image under $U_4(3)$ of the support of any non-zero codeword a_l of weight l in $\mathcal{L}$ form the blocks of a 1-$(112, l, k_l)$ design, where $k_l = |(a_l)^{\mathrm{Aut}(\mathcal{L})}| \times \frac{l}{112}$. We now prove

COROLLARY 5.3. *The non-zero codewords in C_{112} of weight divisible by four form a self-orthogonal doubly-even code $\mathcal{L}$ with parameters $[112, 21, 32]_2$. $\mathbf{1} \in \mathcal{L}$. The automorphism group of $\mathcal{L}$ is isomorphic to $U_4(3){\cdot}D_8$. The dual code $\mathcal{L}^\perp$ is a $[112, 91, 4]_2$ with 280 codewords of weight 4.*

PROOF. Let a_l be a non-zero codeword of weight l in $\mathcal{L}$. For each a_l, take the support of a_l and orbit that under the action of $\mathrm{Aut}(\mathcal{L})$ to form the A_l blocks of $1 - (112, l, k_l)$ designs $\mathcal{D}_{a_l}$ where $k_l = |(a_l)^{\mathrm{Aut}(\mathcal{L})}| \times \frac{l}{112}$ and on which $\mathrm{Aut}(\mathcal{L})$ acts transitively. Each of these designs is easily seen to be self-orthogonal. The remaining follows mutatis mutandis arguments similar to those used in Proposition 5.1 and so we omit the details. The weight enumerator of $\mathcal{L}$ is as follows:

$$
\begin{aligned}
W_{\mathcal{L}} \ = \ & 1 + 1134\ x^{32} + 13608\ x^{40} + 394065\ x^{48} + 1279536\ x^{56} \\
& + 394065\ x^{64} + 13608\ x^{72} + 1134\ x^{80} + \ x^{112}.
\end{aligned}
$$

□

REMARK 5.4. (i) The words of weight 32 in $\mathcal{L}$ can also be described geometrically: these are the $567 + 567 = 1134$ bases, of mutually orthogonal nonisotropic points all of the same kind, forming two orbits of sizes 567 under $U_4(3)$ with stabilizer isomorphic to $2^4{:}A_6$. However under $\mathrm{Aut}(\mathcal{L})$ these orbits fuse into a single orbit with codewords stabilized by non-maximal subgroups isomorphic to $(((2 \times 2 \times 2 \times 2){:}A_6){:}2){:}2$, (see [11]).

(ii) Finally, the words of weight 4 in $\mathcal{L}^\perp$ also have a geometric description: they form the 280 4-cocliques (namely, the totally isotropic lines), forming a single orbit. The stabilizer of one is a maximal subgroup in $U_4(3)$ of type $3_+^{1+4} \cdot 2S_4$ and a maximal subgroup of $\mathrm{Aut}(\mathcal{L})$ of type $3_+^{1+4} \cdot 2_-^{1+4} \cdot D_{12}$.

We now look at $\overline{C}_{112}$, i.e., the code obtained from the row span over $\mathbb{F}_3$ of the adjacency matrix of the unique strongly regular graph $\overline{\mathcal{G}}_{112}$, with parameters $(112, 81, 60, 54)$.

PROPOSITION 5.5.   (i) $\overline{C}_{112}$ is a $[112, 19, 42]_3$ self-orthogonal code.

(ii) $\overline{C}_{112}^{\perp}$ is a $[112, 93, 4]_3$ code with 560 words of weight 4.

(iii) $\mathbf{1} \in \overline{C}_{112}^{\perp}$.

(iv) $\mathrm{Aut}(\overline{\mathcal{G}}_{112}) = \mathrm{Aut}(\overline{C}_{112}) \cong U_4(3){\cdot}D_8$.

(v) $U_4(3){\cdot}D_8$ acts irreducibly on $\overline{C}_{112}$ as a $GF(3)$ module.

PROOF. Self orthogonality of $\overline{C}_{112}$ follows, since all its weights are divisible by 3. Since the block size is $81 \equiv 0 \ (mod \ 3)$ we have that $\mathbf{1} \in \overline{C}_{112}^{\perp}$. That the 3-rank of $\overline{\mathcal{G}}_{112}$ is 19, follows from [8, item 5, p. 339] [1]. The minimum distance 42 can be deduced from the weight enumerator of this code which follows

$$
\begin{aligned}
W_{\overline{C}_{112}} \ = \ & 1 + 9072\, x^{42} + 28350\, x^{48} + 386400\, x^{54} + 8436960\, x^{60} \\
& + 14333760\, x^{63} + 63141120\, x^{66} + 132269760\, x^{69} \\
& + 251376048\, x^{72} + 273248640\, x^{75} + 245216160\, x^{78} \\
& + 107610272\, x^{81} + 49503960\, x^{84} + 14152320\, x^{87} \\
& + 2322432\, x^{90} + 215460\, x^{96} + 9072\, x^{102} + 1680\, x^{108}.
\end{aligned}
$$

That $\overline{C}_{112}^{\perp}$ has minimum weight 4 was found using Magma ([3]).

Finally, notice that the 3-modular character table of the group $U_4(3)$ is completely known (see [17]) and follows from it that the irreducible 19-dimensional $GF(3)-$representation is unique. Since $\mathrm{Aut}(\overline{C}_{112})$ contains $U_4(3)$, by using the weight enumerator above we can easily see that $\overline{C}_{112}$, under the action of $U_4(3)$, does not contain an invariant subspace of dimension 1. So if $\overline{C}_{112}$ is reducible, it must contain an invariant subspace U of dimension m where $2 \leq m \leq 18$. However, calculations with MeatAxe within

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[1] For the benefit of the reader we point out here that there is a typing error in the TABLE on p. 340 of [8]: 20 should be corrected to 19.

Magma ([3]) show that the module $19 \cdot (1 \oplus 1 \oplus 1)$ (with no trivial submodules) occurs naturally as a submodule of $U_4(3)$ acting on the cosets of $3^4 : A_6$. In fact, according to [4] this module can be realizable as the non-trivial composition factor of $S^2(V_6)$ where V_6 denotes the natural $GF(3)-$module of $\Omega_6^-(3) \cong 2 \cdot U_4(3)$. Hence $\overline{C}_{112}$ is the 19-dimensional $GF(3)$ module on which $U_4(3) \cdot D_8$ acts irreducibly. □

REMARK 5.6. (i) The words of weight 42 in $\overline{C}_{112}$ can be described geometrically: these are the 9072 10-cocliques in the graph. These form a single orbit under the action of $\mathrm{Aut}(\overline{C}_{112})$ with stabilizer a maximal subgroup isomorphic to $4^3(2 \times S_4)$, (but split into two orbits of size 4536 each, with stabilizer isomorphic to M_{10} in $U_4(3)$). These 10-cocliques can be regarded as elliptic quadrics (see [6]).

(ii) The words of weight 48 in the code $\overline{C}_{112}$ split into three orbits of sizes 5670, 5670 and 17010 respectively under the action of $U_4(3)$ with stabilizers $2^4:A_6$, $2^4:A_6$ of orders 1152 and $((((2 \times 2 \times 2 \times 2):2):2):3):2$ of order 384 respectively. However, under the action of $\mathrm{Aut}(\overline{C}_{112})$ the set of codewords of weight 48 splits into two orbits of sizes 11340 and 17010 respectively. Notice that the two orbits of size 5670, obtained earlier under $U_4(3)$ now fuse into a single orbit of size 11340, under $\mathrm{Aut}(\overline{C}_{112})$. The stabilizer of a codeword in the first orbit is a non-maximal subgroup of $\mathrm{Aut}(\overline{C}_{112})$ of order 4608 isomorphic to $((2^3.12) : 6) : 2^3$, while that of a codeword in the second orbit is a non-maximal subgroup of order 3072 isomorphic to $(4^2 \times 2).(S_4 \times 2^2)$.

(iii) The words of weight 108 in $\overline{C}_{112}$ form a single orbit with stabilizer $3_+^{1+4} \cdot 2S_4$ in $U_4(3)$), but remain indivisible under the action of $\mathrm{Aut}(\overline{C}_{112})$ and are stabilized by $3_+^{1+4} \cdot 2_-^{1+4} \cdot D_{12}$.

(iv) The words of weight 4 in $\overline{C}_{112}^{\perp}$ can also be described geometrically: they form the 2-sets of 280 4-cocliques each (namely, the totally isotropic lines), forming two orbits. The stabilizer of a line is a maximal subgroup of $U_4(3)$ of type $3_+^{1+4} \cdot 2S_4$ and a maximal subgroup of $\mathrm{Aut}(\overline{C}_{112}^{\perp})$ of type $3_+^{1+4} \cdot 2_-^{1+4} \cdot D_{12}$.

5.2. *A strongly regular* $(126, 45, 12, 18)$ *graph.* The class $ccl_G(H_2)$ has 126 elements and the intersection of two distinct elements of that class is isomorphic to $((3^3 : 2) : 2) : 3$ or $(2^3 \cdot 12) : 6$.

The incidence structure $\mathcal{G}_{126} = (\mathcal{V}_2, \mathcal{E}_2, \mathcal{I}_2)$, where $\mathcal{V}_2 = \{V_1^{(2)}, \dots, V_{126}^{(2)}\}$ and vertices $V_i^{(2)}$ and $V_j^{(2)}$ are adjacent if and only if $H_2^{g_i} \cap H_2^{g_j} \cong (2^3 \cdot 12) : 6$, i.e., the graph $\mathcal{G}(U_4(3), H_2; (2^3 \cdot 12) : 6)$, is a strongly regular graph with parameters $(126, 45, 12, 18)$. The graph $\mathcal{G}_{126}$ is one of two strongly regular locally $GQ(4, 2)$ graphs, see [21]. $\mathcal{G}_{126}$ is isomorphic to $NO_6^-(3)$, the graph on one class of nonisotropic points of $PG(5, 3)$ provided with a nondegenerate

quadratic form, where two points are joined when they are orthogonal (when the connecting line is elliptic).

The full automorphism group of the graph $\mathcal{G}_{126}$ is isomorphic to the group $U_4(3) \cdot (2^2)_{122}$ and it has order 13063680 acting as a rank-3 group with point stabilizer isomorphic to $U_4(2)$. Thus we have a rank-3 graph. Unlike in Section 5.1, here we have ternary codes from the graph followed by binary codes from its complement. From the row span of the adjacency matrix of $\mathcal{G}_{126}$ over $\mathbb{F}_3$ and of its complement $\overline{\mathcal{G}}_{126}$ over $\mathbb{F}_2$ we construct a ternary (resp. binary) code which we denoted C_{126} (resp. $\overline{C}_{126}$). In Proposition 5.7 below, we list the main obvious properties of the codes obtained from $\mathcal{G}_{126}$ without providing a proof. A proof for this proposition follows mutatis mutandis arguments similar to those used in the proofs of the preceding results and so we omit it. Notice that the group H_3 is also a subgroup of index 126, so the conjugacy class $ccl_G(H_3)$ leads to a graph isomorphic to $\mathcal{G}_{126}$.

PROPOSITION 5.7.   (i) C_{126} is a $[126, 21, 36]_3$ self-orthogonal code.
(ii) $C_{126}^{\perp}$ is a $[126, 105, 6]_3$ code with 23250 words of weight 6.
(iii) $\mathbf{1} \in C_{126}$.
(iv) $\mathrm{Aut}(\mathcal{G}_{126}) = \mathrm{Aut}(C_{126}) \cong U_4(3) \cdot (2^2)_{122}$.

The weight enumerator for this code is as follows:

$$
\begin{aligned}
W_{C_{126}} \;=\; & 1 + 252\,x^{36} + 476\,x^{45} + 5670\,x^{48} + 27216\,x^{51} + 44940\,x^{54} \\
& + 158760\,x^{57} + 123606\,x^{60} + 1683360\,x^{63} + 13096566\,x^{66} \\
& + 33589080\,x^{69} + 190041600\,x^{72} + 567322056\,x^{75} \\
& + 1257407550\,x^{78} + 1958343884\,x^{81} + 2317461030\,x^{84} \\
& + 2059389360\,x^{87} + 1290308796\,x^{90} + 577943100\,x^{93} \\
& + 154487088\,x^{96} + 29820840\,x^{99} + 7994700\,x^{102} \\
& + 934416\,x^{105} + 113820\,x^{108} + 27216\,x^{111} \\
& + 17010\,x^{114} + 10080\,x^{117} + 730\,x^{126}.
\end{aligned}
$$

REMARK 5.8. (i) The codewords of weight 36 in C_{126} have a geometric description: they are the incidence vectors of the blocks of a 1-design $\mathcal{D}_{126}$ obtained by identifying the vertices of $\mathcal{G}_{126}$ with both points and blocks, and adjacency in the graph with incidence in the design, and their scalar multiples. Under the action of $\mathrm{Aut}(C_{126})$ the set of codewords of weight 36 splits into two orbits each of size 126. The stabilizer of a codeword in each set is a maximal subgroup of $\mathrm{Aut}(C_{112})$ of order 103680, isomorphic with $(U_4(2){:}2) \times 2$. Moreover, the support of the codewords of weight 36 in C_{126} span isomorphic self-dual symmetric 1-designs $\mathcal{T}$ with parameters 1-(126, 36, 36). The binary codes of these designs have parameters $[126, 70, d]_2$ where $d \leq 36$.

(ii) The codewords of weight 45 split into four orbits of types u, v, x, y of sizes $[112, 112, 126, 126]$. The orbit of type u consists of codewords with 45

(1) while that of type v consists of codewords with 45 (-1) as their non-zero coordinate positions. Furthermore, the orbit of type x consists of codewords with 45 (-1) and that of type y consists of codewords with 45 (1). The stabilizer of a codeword in an orbit of type u or v is a maximal subgroup of $\text{Aut}(C_{112})$ isomorphic with $3^4{:}(S_6 \times 2)$, while the stabilizer of a codeword in an orbit of type x or y is a maximal subgroup isomorphic to $U_4(2){:}2 \times 2$.

(iii) The 730 codewords of weight 126 in C_{126} split into eight orbits of types a, b, c, d, e, f, g, h of sizes $[1, 1, 112, 112, 126, 126, 126, 126]$, respectively, each stabilized by $U_4(3) \cdot (2^2)_{122}$. The orbit of type a consists of the all-ones vector, and the orbit of type b consists of its scalar multiple, i.e., a vector whose coordinate positions are all (-1). The orbits of types c and d consist of vectors with 81 (1) and 45 (-1), and vectors with 45 (1) and 81 (-1) as their coordinate positions. The orbits of types e and f consist of vectors with 90 (1) and 36 (-1), and vectors with 36 (1) and 90 (-1) respectively. Finally the orbits of types g and h consist of vectors with 45 (1) and 81 (-1) and vectors with 81 (1) and 45 (-1) respectively.

We now look at the code $\overline{C}_{126}$, i.e., the code obtained from the row span over $\mathbb{F}_2$ of the adjacency matrix of the strongly regular graph $\overline{\mathcal{G}}_{126}$, i.e., the graph with parameters $(126, 80, 52, 48)$.

PROPOSITION 5.9.   (i) $\overline{C}_{126}$ is a $[126, 34, 24]_2$ self-orthogonal doubly-even code.
(ii) $\overline{C}_{126}^{\perp}$ is a $[126, 92, 6]_2$ code with 567 words of weight 6.
(iii) $\mathbf{1} \in \overline{C}_{126}^{\perp}$.
(iv) $\text{Aut}(\overline{\mathcal{G}}_{126}) = \text{Aut}(\overline{C}_{126}) \cong U_4(3){\cdot}D_8$.
(v) $U_4(3)$ acts irreducibly on $\overline{C}_{126}$ as a $GF(2)$ module.

The weight enumerator for this code is as follows:

$$
\begin{aligned}
W_{C_{126}} = {} & 1 + 1680\, x^{24} + 42525\, x^{32} + 1998864\, x^{40} + 11249280\, x^{44} \\
& + 142165800\, x^{48} + 710881920\, x^{52} + 2256449904\, x^{56} \\
& + 4233325824\, x^{60} + 4797752715\, x^{64} + 3265194240\, x^{68} \\
& + 1384432560\, x^{72} + 324164736\, x^{76} + 46981368\, x^{80} \\
& + 4682880\, x^{84} + 544320\, x^{88} + 567\, x^{96}.
\end{aligned}
$$

REMARK 5.10. (i) The codewords of weight 24 in $\overline{C}_{126}$ form a single orbit with stabilizer a non-maximal subgroup of $U_4(3)$ of order 7776 which is isomorphic to $3^4 : (S_4 \times 2^2)$.

(ii) The 567 codewords of weight 96 form a single orbit stabilized by maximal subgroups of indices 567 in $U_4(3)$ and in $\text{Aut}(\overline{C}_{126})$ respectively. These maximal subgroups are respectively isomorphic with $2^4{:}A_6$ and $(4^2 \times 2)(2 \times S_4)$. The said maximal subgroups are also the stabilizers in $U_4(3)$ and

$\mathrm{Aut}(\overline{C}_{126}{}^{\perp})$ of the codewords in the single orbit formed by the 567 codewords of smallest weight in $C_{126}{}^{\perp}$.

5.3. *A strongly regular* $(162, 56, 10, 24)$ *graph.* The cardinality of the conjugacy class $ccl_G(H_4)$ is 162. The intersection of two distinct elements of $ccl_G(H_4)$ is isomorphic to A_6 or $(((D_8 \times 2) : 2) : 2) : 3$.

The incidence structure $\mathcal{G}_{162} = (\mathcal{V}_3, \mathcal{E}_3, \mathcal{I}_3)$, where $\mathcal{V}_3 = \{V_1^{(3)}, \dots, V_{162}^{(3)}\}$ and vertices $V_i^{(3)}$ and $V_j^{(3)}$ are adjacent if and only if $H_4^{g_i} \cap H_4^{g_j} \cong A_6$, i.e., the graph $\mathcal{G}(U_4(3), H_4; A_6)$, is a strongly regular graph with parameters $(162, 56, 10, 24)$ and spectrum $56^1 2^{140} (-16)^{21}$. Moreover $\overline{\mathcal{G}}_{162}$ has parameters $(162, 105, 72, 60)$ and spectrum $105^1 15^{21} (-3)^{140}$.

A graph with the parameters of $\mathcal{G}_{162}$ is unique by [9,10] and isomorphic to the second subconstituent of the McLaughlin graph. See [9, Section C, p. 113] for a construction using $L_3(4)$, the projective plane $PG(2, 4)$ and orbits of Fano subplanes. The full automorphism group of $\mathcal{G}_{162}$ is isomorphic to the group $U_4(3) \cdot (2^2)_{133}$ and it has order 13063680 acting as a rank-3 group with point stabilizer isomorphic to $L_3(4){:}2^2$. Thus we have a rank-3 graph. We now look at the codes C_{162} (resp. $\overline{C}_{162}$). The 2-rank 20 of $\mathcal{G}_{162}$ and the 3-rank 21 of $\overline{\mathcal{G}}_{162}$ that is, the dimension of C_{162} and $\overline{C}_{162}$ respectively are known (see [9]). Notice that the group $\mathrm{Aut}(\mathcal{G}_{162})$ is not isomorphic to the group $\mathrm{Aut}(\mathcal{G}_{126})$. The conjugacy class $ccl_G(H_5)$ leads to a graph isomorphic to $\mathcal{G}_{162}$. From the row span over $\mathbb{F}_2$ of the adjacency matrix of the strongly regular graph $\overline{\mathcal{G}}_{162}$, we obtain a binary code denoted C_{162} and whose properties we examine in

PROPOSITION 5.11.   (i) C_{162} *is a* $[162, 20, 56]_2$ *self-orthogonal doubly-even code.*
  (ii) $C_{162}^{\perp}$ *is a self-complementary* $[162, 142, 6]_2$ *code with* 86562 *words of weight 6. Moreover* $\mathbf{1} \in C_{162}^{\perp}$.
  (iii) $\mathrm{Aut}(\mathcal{G}_{162}) = \mathrm{Aut}(C_{162}) \cong U_4(3) \cdot (2^2)_{133}$.
  (iv) $U_4(3)$ *acts irreducibly on* C_{162} *as a* $GF(2)$ *module.*

The weight enumerator for C_{162} is as follows:

$$
\begin{aligned}
W_{C_{162}} = \ & 1 + 162\,x^{56} + 4536\,x^{60} + 8505\,x^{64} + 117936\,x^{72} + 258552\,x^{76} \\
& + 204120\,x^{80} + 181440\,x^{84} + 192780\,x^{88} + 78408\,x^{92} \\
& + 1134\,x^{96} + 840\,x^{108} + 162\,x^{120}.
\end{aligned}
$$

REMARK 5.12. (i) The words of weight 56 in C_{162} can be described geometrically: these are the rows of the adjacency matrix of $\mathcal{G}_{162}$, and also the incidence vectors of the blocks of the design $\mathcal{D}_{162}$ constructed as given above.

(ii) The stabilizer of a codeword (or of the set of codewords) of weight 56 in C_{162} is a maximal subgroup of index 162, isomorphic to $L_3(4)$ in $U_4(3)$ (of index 162 isomorphic to $L_3(4){:}2^2$ in $\mathrm{Aut}(C_{162})$). Hence $U_4(3)$ and $U_4(3){\cdot}(2^2)_{133}$

act primitively on points and blocks of the design $\mathcal{D}_{162}$. Moreover C_{162} is spanned by its minimum weight codewords.

(iii) Under the action of $U_4(3)$, the 4536 codewords of weight 60 in C_{162} form a single orbit stabilized by M_{10}. That orbit remains undivisible under $\mathrm{Aut}(C_{162})$ with the stabilizer of an element being isomorphic to $M_{10} \cdot 2^2$.

(iv) Under the action of $U_4(3)$, the set of codewords of weight 96 splits into two orbits of sizes 567 each, with stabilizer of a codeword in either orbit being isomorphic to $2^4{:}A_6$ with vertex orbit sizes $[1, 15, 30, 96, 96, 96, 120, 320, 360]$. However these orbits fuse under the action of $\mathrm{Aut}(C_{162})$ and the stabilizer of a codeword is a non-maximal subgroup isomorphic to $2^4{\cdot}S_6$.

(v) The codewords of weight 120 in C_{162} form a single orbit stabilized by $L_3(4){:}2^2$ with vertex orbit size $1, 56, 105$.

(vi) The 86562 codewords of minimum weight in $C_{162}^{\perp}$ split into three orbits of sizes $9072, 17010, 60480$, all stabilized by non-maximal subgroups of $\mathrm{Aut}(C_{162}^{\perp})$ isomorphic to $A_6{:}2^2$, $8.(S_4 \times 2^2)$, $3^3{:}2^3$, respectively.

(vii) Furthermore, notice that the 2-modular character table of $U_4(3)$ is completely known (see [17]) and follows from it that the irreducible 20-dimensional $GF(2)-$representation is unique and 20 is the smallest dimension for any non-trivial irreducible $GF(2)-$module. Since $\mathrm{Aut}(C_{162})$ contains $U_4(3)$, by using the above weight enumerator we can easily see that C_{162}, under the action of $U_4(3)$, does not contain an invariant subspace of dimension 1. So, if C_{162} is reducible, it must contain an invariant irreducible subspace $\mathcal{E}$ of dimension m with $2 \leq m \leq 19$, which is not possible. Hence C_{162} is irreducible and must be isomorphic to the 20-dimensional $GF(2)$-module on which $U_4(3)$ acts irreducibly.

Proposition 5.13 below deals with the code $\overline{C}_{162}$ obtained from the row span over $\mathbb{F}_3$ of the rows of the adjacency matrix of $\overline{\mathcal{G}}_{162}$.

PROPOSITION 5.13.   (i) $\overline{C}_{162}$ is a $[162, 21, 54]_3$ self-orthogonal code.
(ii) $\overline{C}_{162}^{\perp}$ is a $[162, 141, 6]_3$ code with 120960 words of weight 6.
(iii) $\mathbf{1} \in \overline{C}_{162}$.
(iv) $\mathrm{Aut}(\overline{\mathcal{G}}_{162}) = \mathrm{Aut}(\overline{C}_{162}) \cong U_4(3) \cdot (2^2)_{133}$.

The weight enumerator for $\overline{C}_{162}$ is as follows:

$$
\begin{aligned}
W_{\overline{C}_{162}} = \ & 1 + 1680\,x^{54} + 324\,x^{57} + 53298\,x^{66} + 27216\,x^{72} + 281880\,x^{78} \\
& + 1197952\,x^{81} + 2238840\,x^{84} + 3274992\,x^{87} + 26570754\,x^{90} \\
& + 59149440\,x^{93} + 321365394\,x^{96} + 720815760\,x^{99} \\
& + 1252507536\,x^{102} + 1751613012\,x^{105} + 2073483720\,x^{108} \\
& + 1827182448\,x^{111} + 1375043040\,x^{114} + 662872896\,x^{117}
\end{aligned}
$$

$$+292287528\ x^{120} + 72077040\ x^{123} + 11277144\ x^{126}$$
$$+5341140\ x^{129} + 1061424\ x^{132} + 366240\ x^{135} + 17010\ x^{138}$$
$$+145152\ x^{141} + 90720\ x^{144} + 9072\ x^{150} + 550\ x^{162}.$$

REMARK 5.14. (i) The words of weight 54 in $\overline{C}_{162}$ are stabilized by a non-maximal subgroup of $U_4(3)$ of order 3888 and isomorphic to $3_+^{1+4}{:}(Q_8.2)$. However, under the action of $\mathrm{Aut}(\overline{C}_{162})$ these codewords are stabilized by a group of order 15552, isomorphic to $(3_+^{1+4} : (Q_8.2)) : 2^2$.

(ii) The 324 codewords of weight 57 split into two orbits of size 162 each. The stabilizer in $U_4(3)$ and $U_4(3) \cdot (2^2)_{133}$ of a codeword of weight 57 are maximal subgroups isomorphic with $L_3(4)$ and $L_3(4){:}2^2$ respectively.

(iii) The 9072 codewords of weight 105 split into two orbits of sizes 4536 each stabilized by $A_6 \cdot 2^2 \times 2$.

(iv) The 550 codewords of weight 162 split into six orbits of sizes 1, 1, 112, 112, 162 and 162 respectively. Notice that the stabilizer of a codeword in each of the orbits is a group isomorphic to $U_4(3) \cdot (2^2)_{133}$. A description similar to that given in Section 5.2 for the codewords of largest weight in the code $[126, 21, 36]_3$ can also be given here, for the codewords of weight 550.

5.4. *A strongly regular* $(280, 36, 8, 4)$ *graph.* Finally we describe the strongly regular graph $(280, 36, 8, 4)$ and its binary code. The code of the complement of this graph possess large dimension and thus computationally uncomfortable to derive its parameters and properties. In the sequel we expound briefly on the graph and its binary code. The conjugacy class $ccl_G(H_6)$ has 280 elements. The intersection of two distinct elements from that class is isomorphic to $Q_8 \cdot S_3$ or $3^4 : 4$.

The incidence structure $\mathcal{G}_{280} = (\mathcal{V}_4, \mathcal{E}_4, \mathcal{I}_4)$, where $\mathcal{V}_4 = \{V_1^{(4)}, \ldots, V_{280}^{(4)}\}$ and vertices $V_i^{(4)}$ and $V_j^{(4)}$ are adjacent if and only if $H_6^{g_i} \cap H_6^{g_j} \cong 3^4 : 4$, i.e., the graph $\mathcal{G}(U_4(3), H_6; 3^4 : 4)$, is a strongly regular graph with parameters $(280, 36, 8, 4)$.

The full automorphism group of the graph $\mathcal{G}_{280}$ has 26127360 elements and is isomorphic to the group $U_4(3){\cdot}D_8$.

The graph $\mathcal{G}_{280}$ is not isomorphic to a strongly regular graph with the parameters $(280, 36, 8, 4)$ which can be constructed from the Janko group J_2 as $\mathcal{G}(J_2, 3.A_6.2; A_5)$. The graph $\mathcal{G}(J_2, 3.A_6.2; A_5)$ can also be obtained from the rank-4 primitive permutation representation on 280 points of the Janko group J_2, taking as a base block the suborbit of length 36. Further, $\mathcal{G}(J_2, 3.A_6.2; A_5)$ can be constructed from 280 dekades (10-cocliques) of the Hall-Janko graph, a strongly regular graph with parameters $(100, 36, 14, 12)$, as described in [6].

PROPOSITION 5.15.   (i) C_{280} *is a* $[280, 70, 36]_2$ *self-orthogonal with* 280 *words of weight 36.*

(ii) $C_{280}^{\perp}$ *is a* $[280, 210, 8]_2$.

(iii) **1** *is in* C_{280}.
(iv) $\text{Aut}(\mathcal{G}_{280}) = \text{Aut}(C_{280}) \cong U_4(3) \cdot D_8$.

REMARK 5.16. The 280 codewords of weight 36 in C_{280} can be described geometrically: these are the rows of the adjacency matrix of $\mathcal{G}_{280}$, these are also the incidence vectors of the blocks of the design $\mathcal{D}_{280}$. The stabilizer of one is a maximal subgroup in $U_4(3)$ of type $3_+^{1+4} \cdot 2S_4$ and a maximal subgroup in $\text{Aut}(C_{280})$ of type $3_+^{1+4} \cdot 2_-^{1+4} \cdot D_{12}$. Hence $U_4(3)$ and $U_4(3) \cdot D_8$ act primitively as rank-3 groups on points and blocks of the design $\mathcal{D}_{280}$. The codewords of weight 36 span C_{280}.

6. Concluding remarks

In general there are some limitations in the computations that can be carried on the codes whose dimension is large, when computing with Magma. For this reason the authors have restricted their work to those codes or dual codes, with a smaller dimension and thus manageable in Magma computing terms.

Although we were unable to provide the intrinsic parameters for codes associated with a strongly regular graph with parameters $(540, 224, 88, 96)$, due to the large dimension of these codes (in this representation $U_4(3)$ has three non-trivial orbits, and the smallest dimension a binary code can have is of size 120. The following size for the dimensions are: 140, 260, 280 and 441 respectively. The smallest dimension of a ternary code in this representation is 99), below we describe how one would obtain such graph from conjugacy classes of $U_4(3)$. The conjugacy class $ccl_G(H_7)$ has cardinality 540. The intersection of two distinct elements from that class is isomorphic either to $Q_8 : 3, 3^2 : 3$, or $(Q_8 : 3) : 4$.

The incidence structure $\mathcal{G}_{540} = (\mathcal{V}_5, \mathcal{E}_5, \mathcal{I}_5)$, where $\mathcal{V}_5 = \{V_1^{(5)}, \ldots, V_{540}^{(5)}\}$ and vertices $V_i^{(5)}$ and $V_j^{(5)}$ are adjacent if and only if $H_7^{g_i} \cap H_7^{g_j} \cong 3^2 : 3$, i.e., the graph $\mathcal{G}(U_4(3), H_7; 3^2 : 3)$, is a strongly regular graph with parameters $(540, 224, 88, 96)$.

The full automorphism group of the graph $\mathcal{G}_{540}$ has 26127360 elements and is isomorphic to the group $U_4(3) \cdot D_8$.

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