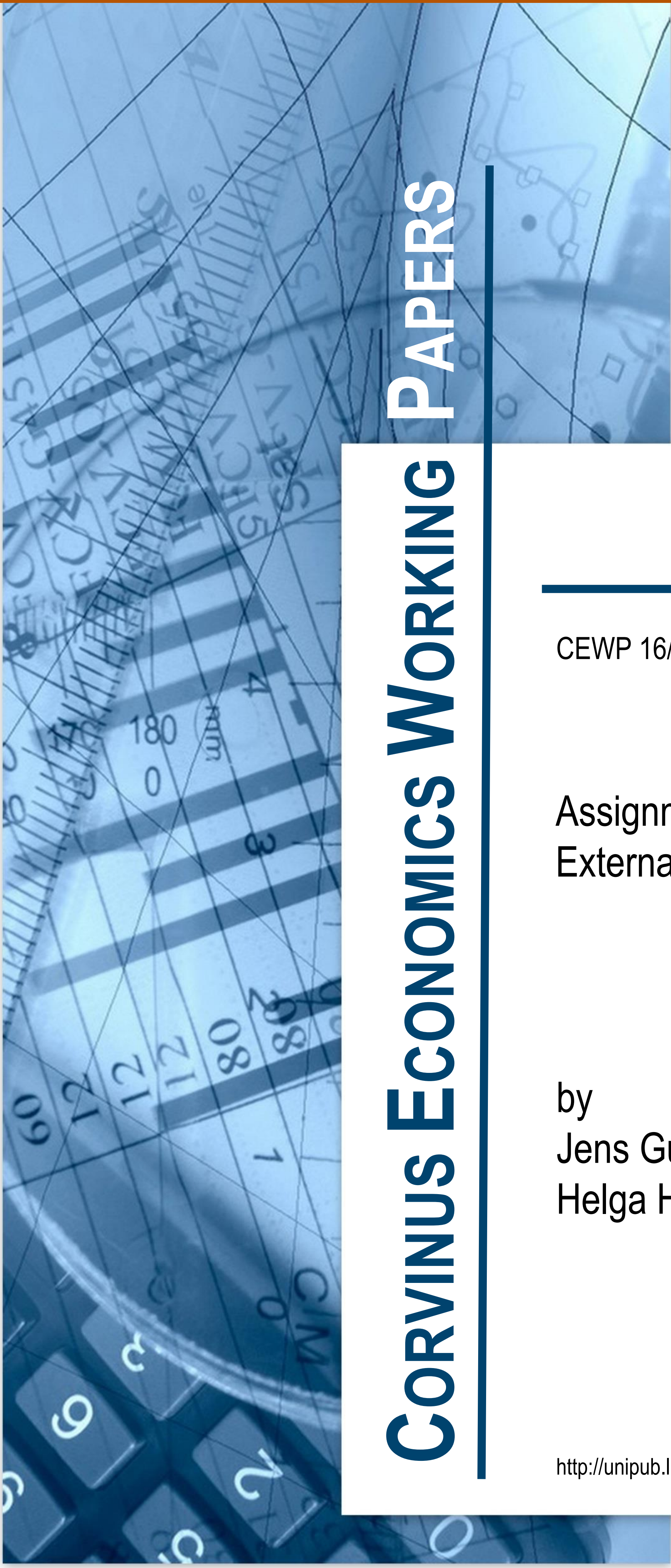
View metadata, citation and similar papers at core.ac.uk
brought to you by CORE
provided by Corvinus Research Archive



CORVINUS ECONOMICS WORKING PAPERS

CORVINUS UNIVERSITY of BUDAPEST

Faculty of Economics

CEWP 16/2015

Assignment Games with Externalities

by
Jens Gudmundsson and
Helga Habis

http://unipub.lib.uni-corvinus.hu/2063

Assignment Games with Externalities☆

Jens Gudmundsson

Department of Economics, Lund University, Box 7082, SE-222 07 Lund, Sweden

Helga Habis

Department of Microeconomics, Corvinus University of Budapest, 1093 Budapest, Fôvám tér 8., Hungary. E-mail: helga.habis@uni-corvinus.hu

Abstract

We examine assignment games, where matched pairs of firms and workers create some monetary value to distribute among themselves and the agents aim to maximize their payoff. In the majority of this literature, externalities - in the sense that a pair's value depends on the pairing of the others - have been neglected. However, in most applications a firm's success depends on, say, the success of its rivals and suppliers. Thus, it is natural to ask how the classical results on assignment games are affected by the introduction of externalities? The answer is – dramatically. We find that (i) a problem may have no stable outcome, (ii) stable outcomes can be inefficient (not maximize total value), (iii) efficient outcomes can be unstable, and (iv) the set of stable outcomes may not form a lattice. We show that stable outcomes always exist if agents are "pessimistic." This is a knife-edge result: there are problems in which the slightest optimism by a single pair erases all stable outcomes.

Keywords: Two-sided matching, assignment game, externalities, stability, efficiency
JEL: C71, C78, D62

1. Introduction

We consider assignment games in which firms hire workers after negotiating salaries. They were first studied by Koopmans and Beckmann (1957) and Shapley and Shubik (1971) and provide a primitive model of the job market; see Roth and Sotomayor (1992) for a survey. We introduce *externalities* into the model: the success of a firm and a worker depends on the pairing of the other firms and workers. Thus, it is natural to ask how the classical results on assignment games are affected by the introduction of externalities? Our study suggests

☆We would like to thank Tommy Andersson and Flip Klijn as well as participants at the Workshop on Networks and Externalities, the Summer School on Matching Problems in Budapest, 2013, the UECE Lisbon Meetings, 2013, and at Social Choice and Welfare, 2014, for valuable comments. Gudmundsson gratefully acknowledges the finanical support of the Jan Wallander and Tom Hedelius Foundation. Habis would like to thank the financial support of OTKA-112266 through Corvinus University of Budapest.

September 10, 2015

– dramatically. As we will see, all of the classical findings of Shapley and Shubik (1971) are overturned.

In most of the literature, it is implicitly assumed that agents' preferences are independent of how the other agents are matched. However, it is not difficult to see that externalities may play an important role. Consider for instance a firm that produces phones. Surely, its success stems mainly from the competence of its employees. But the success of firms that produce complementary goods (say network capacity and signal) may also have some importance, and the lack of success of its rival firms that produce similar phones could be influential as well. Therefore, the question we ask is a highly relevant one.

In the related literature, Li (1993) was the first to introduce externalities into the one-to-one, two-sided matching market by assuming that each agent has strict preferences over the set of all possible matchings. He finds that equilibrium may not exist in general, but it does if the externalities are small enough: more specifically, if an agent's preferences over matchings is lexicographically determined, first and foremost by his partner and then by how the other agents are matched. Similarly, Sasaki and Toda (1996) also find non-existence when expectation about residual behavior is determined endogenously. They show that there is always a stable matching if estimation functions on the set of possible outcomes are exogenously given. They are the first to examine assignment games with externalities, and find that a stable matching exists if agents find all matchings to be possible. Taking this approach one step further, Hafalir (2008) introduces endogenous beliefs depending on the preferences. He confirms the anticipation of Sasaki and Toda (1996) that rational expectations do not guarantee existence. He introduces the notion of sophisticated expectations, determined via an algorithm, inducing a game without externalities at the end, and shows that the resulting set of stable matchings is non-empty. To achieve non-emptiness, he assumes that there is no commitment; that is, a blocking pair can split up if they can get better off by blocking again through a different pair. Eriksson et al. (2011) consider assignment games where agents experience negative externalities from the payoffs of the agents on the same side of the market in form of ill will. They define a new, weaker notion of stability assuming bounded rationality, and show that such stable outcomes always exist. In a recent discussion paper, Chen (2013) examines a model similar to ours. Some of the results overlap (also noted in his paper) though for instance our negative result (Theorem 2) is stronger than his as he uses estimation functions that are not matching dependent. Furthermore, he focuses more on applications and less on properties of the set of stable outcomes than we do.

It is clear from the results summarized above that the introduction of externalities causes many issues which need to be resolved. For instance, it is not unambiguous how to generalize the notion of blocking and how to define stability. We introduce externalities into assignment games by allowing the values of matched pairs to depend on how the rest of the agents are matched. We look for stable outcomes in the standard sense: an outcome is *stable* if it is individually rational and has no blocking pairs. However, it is not straightforward how one should define the notion of blocking in this environment.

Formally, we model the externalities as follows. The value a firm and a worker create in a matching - that is, the amount of money they divide into profit and salary - depends on how the other agents are organized. An outcome specifies a matching as well as a profit for each firm and a salary for each worker. A firm f and a worker w *block* an outcome if they are both better off when f lays off its current employee and hires w instead, who resigns from her current firm. However, f and w have to take into account the fact that if they deviate from the current outcome, then this changes the values and hence the behavior of the other matched pairs as well.[1] As a consequence, others may re-match, changing the value of f and w.

Here, we aim to cover as many behavioral assumptions or beliefs as possible by applying a very *general definition of blocking*. When agents are deciding whether to form a blocking pair, they take the values for all contingencies into account. According to their attitude towards risk or beliefs about the other agents, they calculate a threshold based on the possible outcomes and form a blocking pair whenever this threshold exceeds the sum of their current payoffs. By using this general definition, we avoid imposing any initial assumption on beliefs or residual behavior. In turn, we can distinguish different types of agents based on how they determine their threshold.

Our first and positive finding is that, if all agents are pessimistic, then there always exists a stable outcome (Proposition 1).

The main result of the paper, Theorem 2, shows the necessity of pessimism. If there is just one pair that at one matching is slightly optimistic, then there are values such that the corresponding assignment game with externalities lacks a stable outcome.[2] This finding is very strong as it requires only the smallest conceivable form of optimism. Although, it exploits that externalities may be arbitrarily large. However, even if externalities are reasonably bounded, the negative result is still preserved (Example 5). Furthermore, we can find assignment games with vanishingly small externalities that lack stable outcomes if agents are sufficiently optimistic (Example 2).

In contrast to problems without externalities, stability and efficiency no longer go hand in hand. Specifically, when there are no externalities, stable matchings maximize the total value generated. In Example 3, we provide a simple problem in which an inefficient matching is stable whereas an efficient matching is unstable. However, if all agents are pessimistic - which guarantees that there exists a stable outcome - then there always exists a *Pareto optimal* stable outcome (Proposition 2). In Example 4 we highlight another discrepancy compared to games without externalities; namely, that the set of stable outcomes does not form a complete lattice.

The outline of the paper is as follows. Section 2 introduces the model. In Section 3, we first discuss existence of stable outcomes, then efficiency, and then the structure of the set of stable outcomes. In Section 4, we examine a more restrictive setting and show that our negative results persist. We conclude in Section 5.

[1] These considerations are typically referred to as *residual behavior* in the cooperative game theory literature. Whereas these reactions do not play a role in problems without externalities, different assumptions and expectations about residual behavior lead to different outcomes being stable when externalities are present. They range from the pessimistic approach of Aumann and Peleg (1960) to the optimistic one of Shenoy (1980).

[2] Results of a similar nature have been found for cooperative games (Funaki and Yamato, 1999; Kóczy, 2007) and housing markets (Mumcu and Saglam, 2007).

2. Model

2.1. Preliminaries

There are m firms F and $n = m$ workers W, with $N \equiv F \cup W$.[3] A firm can employ at most one worker and no two can employ the same. A **matching** is a bijection $\mu : F \to W$. The **set of matchings** is $\mathcal{M}$. We use the notation that, if $\mu(f) = w$, then $(f, w) \in \mu$. Moreover, let $\mathcal{M} = \mathcal{M}_{fw} \cup \mathcal{M}_{-fw}$ where, for each $\mu \in \mathcal{M}$, we have $\mu \in \mathcal{M}_{fw}$ if and only if $\mu(f) = w$. All firm-worker pairs generate a monetary **value** which they divide in the form of profits and salaries. For each $f \in F$ and $w \in W$, $\alpha_{fw} : \mathcal{M}_{fw} \to \mathbb{R}_+$ maps a value to each matching. Note that we may have $\alpha_{fw}(\mu) \neq \alpha_{fw}(\mu')$. Let $A \equiv (\alpha_{fw})_{f \in F, w \in W}$ and let $\mathscr{A}$ be the collection of all such lists of value functions A.

A **payoff vector** for F is $u \in \mathbb{R}^m_+$, where u_f is firm f's profit. Likewise, a payoff vector for W is $v \in \mathbb{R}^n_+$, where v_w is worker w's salary. Payoff vectors $(u, v) \in \mathbb{R}^m_+ \times \mathbb{R}^n_+$ are **compatible** with $\mu \in \mathcal{M}$ in $A \in \mathscr{A}$ if, for all $f \in F$ and $w = \mu(f)$, $u_f + v_w = \alpha_{fw}(\mu)$. An **outcome** of $A \in \mathscr{A}$ is $(\mu, u, v) \in \mathcal{M} \times \mathbb{R}^m_+ \times \mathbb{R}^n_+$ such that (u, v) are compatible with μ in A. Given $A \in \mathscr{A}$, $\mu \in \mathcal{M}$ is **efficient** if, for all $\mu' \in \mathcal{M}$,

$$\sum_{(f,w)\in\mu} \alpha_{fw}(\mu) \geq \sum_{(f,w)\in\mu'} \alpha_{fw}(\mu').$$

If, for all $f \in F$, $w \in W$, and $\{\mu, \mu'\} \subseteq \mathcal{M}$, we have, with some abuse of notation, $\alpha_{fw}(\mu) = \alpha_{fw}(\mu') \equiv \alpha_{fw}$, we say *there are no externalities*. Denote the collection of assignment games without externalities with $\mathscr{A}^0$. A pair $(f, w) \in F \times W$ **blocks** the outcome (μ, u, v) of $A \in \mathscr{A}^0$ if $u_f + v_w < \alpha_{fw}$. An outcome (μ, u, v) is **stable** in $A \in \mathscr{A}^0$ if no pair blocks it.

Theorem 1 (Shapley and Shubik, 1971)**.** *Let $A \in \mathscr{A}^0$ and $(\mu, u, v) \in \mathcal{M} \times \mathbb{R}^m_+ \times \mathbb{R}^n_+$.*

1. *If (μ, u, v) is stable in A, then μ is efficient in A.*
2. *If (μ, u, v) is stable in A and $\mu' \in \mathcal{M}$ is efficient, then (μ', u, v) is stable in A.*
3. *The set of stable outcomes of A forms a non-empty complete lattice with respect to the firms' profits and the workers' salaries.*

We ask if any of these results still hold on the larger domain $\mathscr{A}$, that is, when externalities are present. Before we can even start answering that we first have to define stable outcomes in games with externalities. The following example shows that this is not straightforward.

Example 1: What outcomes are stable? Consider the assignment game with externalities with agents $F = \{f_1, f_2, f_3\}$ and $W = \{w_1, w_2, w_3\}$. Table 1 displays the values created by the different pairs; all other values are zero. Is the outcome (μ_1, u, v) with $u = (1, 1, 1)$ and $v = (1, 1, 0)$ stable?

Except for f_2 and w_3, no pair has anything to gain from deviating from (μ_1, u, v). The stability of the outcome therefore boils down to whether f_2 and w_3 object to it. We have $u_{f_2} + v_{w_3} = 1 + 0 = 1$ and

$$\alpha_{f_2 w_3}(\mu_2) = 0 < 1 < 2 = \alpha_{f_2 w_3}(\mu_3).$$

[3] That $n = m$ is without loss of generality as we can create "null-agents" to balance the count, that is, agents that create no value in any pair.

Matching	Pair 1	Pair 2	Pair 3
$\mu_1 = \{(f_1, w_1), (f_2, w_2), (f_3, w_3)\}$	2	2	1
$\mu_2 = \{(f_1, w_2), (f_2, w_3), (f_3, w_1)\}$	2	0	2
$\mu_3 = \{(f_1, w_1), (f_2, w_3), (f_3, w_2)\}$	2	2	1

Table 1: Values for Example 1. For instance, the "0" indicates that f_2 and w_3 are matched at μ_2 and generate a value of 0.

In words, it is sensible for f_2 and w_3 to break up their current partnerships and match with one another if the resulting matching formed thereupon is μ_3, but not if it is μ_2. However, this is out of their control. Hence, whether they will block the outcome depends on whether they are optimistic (expect μ_3 to be formed) or pessimistic (expect μ_2). ∘

We wish to formalize the insights of Example 1. Agents have expectations on what will occur as a consequence of them deviating from an outcome. A pair of agents **block** an outcome if their "blocking threshold" exceeds their joint payoffs. We need three objects to formalize this. For each $(f, w) \in F \times W$, define the optimistic value ω_{fw} as the largest value the pair can achieve. Conversely, define the pessimistic value π_{fw} as the smallest value the pair can achieve (that is, the value the pair can guarrantee itself). In Example 1, we have $\omega_{f_2 w_3} = 2$ and $\pi_{f_2 w_3} = 0$.

$$\omega_{fw} = \alpha_{fw}(\mu) \text{ for some } \mu \in \mathcal{M}_{fw} \text{ and, for each } \mu' \in \mathcal{M}_{fw},\ \omega_{fw} \geq \alpha_{fw}(\mu')$$
$$\pi_{fw} = \alpha_{fw}(\mu) \text{ for some } \mu \in \mathcal{M}_{fw} \text{ and, for each } \mu' \in \mathcal{M}_{fw},\ \alpha_{fw}(\mu') \geq \pi_{fw}$$

Certainly, f and w have strong reasons to object to an outcome in which $u_f + v_w < \pi_{fw}$ – if they break up their current partnerships and form one together, they are better off no matter how the other agents react. For each $f \in F$ and $w \in W$, $\lambda_{fw} : \mathcal{M}_{-fw} \to [0,1]$ is used to determine the **blocking threshold**:

$$b_{fw}(\mu) = \lambda_{fw}(\mu) \cdot \omega_{fw} + \big(1 - \lambda_{fw}(\mu)\big) \cdot \pi_{fw}.$$

A pair $(f, w) \in F \times W$ is **optimistic** at $\mu \in \mathcal{M}_{-fw}$ if $\lambda_{fw}(\mu) = 1$. Conversely, the pair is **pessimistic** if $\lambda_{fw}(\mu) = 0$.

Definition 1. *An* assignment game with externalities *is* (N, A, λ), *where* $A \in \mathcal{A}$ *and* $\lambda \equiv (\lambda_{fw})_{f \in F, w \in W}$. *The collection of assignment games with externalities is* $\mathcal{E}$.

A pair $(f, w) \in F \times W$ **blocks** the outcome (μ, u, v) of $(A, \lambda) \in \mathcal{E}$ if $u_f + v_w < b_{fw}(\mu)$. An outcome (μ, u, v) is **stable** in $(A, \lambda) \in \mathcal{E}$ if no pair blocks it.

3. Results

We first observe that the set of stable outcomes for an assignment game with externalities is related to that of a particular game without externalities. The finding is straightforward and essentially follows from the definitions.

Observation 1. *An outcome* $(\mu, u, v) \in \mathscr{M} \times \mathbb{R}_+^m \times \mathbb{R}_+^n$ *is stable in* $(A, \lambda) \in \mathscr{E}$ *if and only if* (μ, u, v) *is stable in* $B \in \mathscr{A}^0$*, where* $B = (\beta_{fw})_{f \in F, w \in W}$ *and*

$$\beta_{fw} = \begin{cases} \alpha_{fw}(\mu) & \textit{if} \;\; (f, w) \in \mu \\ b_{fw}(\mu) & \textit{if} \;\; (f, w) \notin \mu. \end{cases}$$

Proof. Let (μ, u, v) be stable in (A, λ). We proceed in two steps. First, we show that (μ, u, v) is an outcome of B. Then we show that (μ, u, v) is stable in B.

Step 1: As (μ, u, v) is an outcome of (A, λ), for each $f \in F$ and $w = \mu(f)$, $u_f + v_w = \alpha_{fw}(\mu)$. By construction, $\alpha_{fw}(\mu) = \beta_{fw}$. Therefore $u_f + v_w = \beta_{fw}$ for all $f \in F$ and $w = \mu(f)$. Then (μ, u, v) is an outcome of B.

Step 2: As (μ, u, v) is stable in (A, λ), for each $f \in F$ and $w \in W$ such that $w \neq \mu(f)$, $u_f + v_w \geq b_{fw}(\mu)$. By construction, $b_{fw}(\mu) = \beta_{fw}$. Therefore $u_f + v_w \geq \beta_{fw}$ for all $f \in F$ and $w \neq \mu(f)$. Then (μ, u, v) is stable in B. □

This result *does not* imply that there always exist stable outcomes in games with externalities. Surely, $B \in \mathscr{A}^0$ has stable outcomes as it has no externalities (Theorem 1). However, if all of these outcomes are based on matchings other than μ, then none of them needs to be stable in (A, λ).

3.1. Existence

The next result provides a sufficient condition for the existence of a stable outcome in assignment games with externalities. Namely, there are stable outcomes if agents are pessimistic regarding residual behavior and hence careful in forming blocking pairs.

Proposition 1. *Fix* λ *such that pairs always are pessimistic: for each* $f \in F$*,* $w \in W$*, and* $\mu \in \mathscr{M}_{-fw}$*,* $\lambda_{fw}(\mu) = 0$*. Then, for all* $A \in \mathscr{A}$*, there is a stable outcome in* (A, λ)*.*

Proof. We reverse engineer an outcome $(\tilde{\mu}, u, v)$ as follows. Let $\tilde{\mu} \in \mathscr{M}$ be efficient in (A, λ). Construct $B \in \mathscr{A}^0$ as in Observation 1 but based on $\tilde{\mu}$, that is, $B = (\beta_{fw})_{f \in F, w \in W}$ where

$$\beta_{fw} = \begin{cases} \alpha_{fw}(\tilde{\mu}) & \text{if} \;\; (f, w) \in \tilde{\mu} \\ b_{fw}(\tilde{\mu}) & \text{if} \;\; (f, w) \notin \tilde{\mu}. \end{cases}$$

Then $\tilde{\mu}$ is efficient in B. By Theorem 1, there is a stable outcome $(\tilde{\mu}, u, v)$ in B. By Observation 1, $(\tilde{\mu}, u, v)$ is stable in (A, λ). □

Our main result says that even the "slightest" optimism can deter the existence of stable outcomes. This negative result is strong: it requires *one* pair to be arbitrarily close to pessimistic at *one* matching.

Theorem 2. *Fix* λ *such that pairs are* not *always pessimistic: there is* $f \in F$*,* $w \in W$*, and* $\mu \in \mathscr{M}_{-fw}$ *such that* $\lambda_{fw}(\mu) = \varepsilon > 0$*. Then there is* $A \in \mathscr{A}$ *such that* (A, λ) *has no stable outcome.*

Proof. Denote the agents $F = \{f_1, f_2, \dots, f_n\}$ and $W = \{w_1, w_2, \dots, w_n\}$. Assume the pair $f_1 \in F$ and $w_1 \in W$ is not pessimistic at $\hat{\mu} \in \mathcal{M}_{-f_1 w_1}$, say $\lambda_{f_1 w_1}(\hat{\mu}) = \varepsilon$. Without loss, define $\hat{\mu}$ and $\tilde{\mu} \in \mathcal{M}$ as follows.

$$\hat{\mu} = \{(f_1, w_2), (f_2, w_3), \dots, (f_{n-1}, w_n), (f_n, w_1)\}$$
$$\tilde{\mu} = \{(f_1, w_1), (f_2, w_2), \dots, (f_n, w_n)\}$$

Define $A \in \mathscr{A}$ such that, for each $f_i \in F$ and $\mu \in \mathcal{M}_{f_i w_{i+1}}$, $\alpha_{f_i w_{i+1}}(\mu) = 1 \pmod n$. Let also $\alpha_{f_1 w_1}(\tilde{\mu}) = 2/\varepsilon + 1$, and let all other values be 0. Then

$$b_{f_1 w_1}(\hat{\mu}) = \left[\lambda_{f_1 w_1}(\hat{\mu})\right] \cdot \omega_{f_1 w_1} + \left[(1 - \lambda_{f_1 w_1}(\hat{\mu}))\right] \cdot \pi_{f_1 w_1} = \varepsilon \cdot (2/\varepsilon + 1) + (1 - \varepsilon) \cdot 0 = 2 + \varepsilon.$$

Moreover, for each $f_i \in F$ and $\mu \in \mathcal{M}_{-f_i w_{i+1}}$, we have $\omega_{f_i w_{i+1}} = \pi_{f_i w_{i+1}} = 1$, hence $b_{f_i w_{i+1}}(\mu) = 1$.

Let $u \in \mathbb{R}^m_+$ and $v \in \mathbb{R}^n_+$ be arbitrary. **Case 1:** Consider $(\hat{\mu}, u, v)$. As $\hat{\mu}(f_1) = w_2$, $u_{f_1} + v_{w_2} = \alpha_{f_1 w_2}(\hat{\mu}) = 1$. As $\hat{\mu}(f_n) = w_1$, $u_{f_n} + v_{w_1} = \alpha_{f_n w_1}(\hat{\mu}) = 1$. As $u_{f_n} \geq 0$ and $v_{w_n} \geq 0$, $u_{f_1} + v_{w_1} \leq 2 < b_{f_1 w_1}(\hat{\mu})$. Hence, f_1 and w_1 block $(\hat{\mu}, u, v)$.

Case 2: Consider (μ, u, v) for $\mu \neq \hat{\mu}$. Then there is $f_i \in F$ such that $\mu(f_i) \neq w_{i+1}$. Without loss, assume $\mu(f_i) = w_j$ and $\mu(f_k) = w_{i+1}$ As $\mu(f_i) = w_j$, $u_{f_i} + v_{w_j} = \alpha_{f_i w_j}(\mu) = 0$. As $\mu(f_k) = w_{i+1}$, $u_{f_k} + v_{w_{i+1}} = \alpha_{f_k w_{i+1}}(\mu) = 0$. Then $u_{f_i} = u_{f_k} = v_{w_j} = v_{w_{i+1}} = 0$, and therefore, $u_{f_i} + v_{w_{i+1}} = 0 < b_{f_i w_{i+1}}(\mu)$. Hence, f_i and w_{i+1} block (μ, u, v).

As this exhausts all possibilities, there is no stable outcome. □

Example 2: Minimal externalities, no stable outcome. Consider an assignment game with externalities where pairs are "sufficiently" optimistic (n is the number of firms):

$$\forall f \in F,\ \forall w \in W,\ \forall \mu \in \mathcal{M}_{-fw},$$
$$\lambda_{fw}(\mu) > (n-1)/n.$$

Values are symmetric and externalities are vanishingly small: for some $\varepsilon > 0$, let

$$\forall f \in F,\ \forall w \in W,$$
$$\left\{\alpha_{fw}(\mu) : \mu \in \mathcal{M}_{fw}\right\} = \left\{1, 1 + \varepsilon\right\}.$$

Then, for all $f \in F$, $w \in W$, and $\mu \in \mathcal{M}_{-fw}$, we have $\omega_{fw} = 1 + \varepsilon$. Lastly, for each $\mu \in \mathcal{M}$, assume there is at least one pair $(f, w) \in \mu$ such that $\alpha_{fw}(\mu) = 1$. Then there is no stable outcome.[4] The externalities vanish both in relative and absolute terms:

$$\frac{\omega_{fw}}{\pi_{fw}} = \frac{1+\varepsilon}{1} \to 1 \text{ as } \varepsilon \to 0 \qquad\qquad \omega_{fw} - \pi_{fw} = (1+\varepsilon) - 1 \to 0 \text{ as } \varepsilon \to 0.$$

Hence, even the smallest of externalities can be problematic if agents are optimistic. ∘

[4] The smallest value needed to satisfy all pairs exceeds $n\big([(n-1)/n] \cdot (1+\varepsilon) + [1/n] \cdot 1\big)$, which simplifies to $n(1+\varepsilon) - \varepsilon$. But this is also an upper bound on the total value available.

3.2. Efficiency

In contrast to the case without externalities, efficiency and stability now no longer go hand in hand.

Example 3: Inefficient stable matching, unstable efficient matching. Consider the assignment game with externalities with agents $F = \{f_1, f_2, f_3\}$ and $W = \{w_1, w_2, w_3\}$ and values $\alpha_{fw}(\mu) = 0$ except the following:

$$\alpha_{f_1 w_1}(\mu_1) = 2 \text{ for } \mu_1 = \{(f_1, w_1), (f_2, w_2), (f_3, w_3)\}$$
$$\alpha_{f_2 w_3}(\mu_2) = 1 \text{ for } \mu_2 = \{(f_1, w_1), (f_2, w_3), (f_3, w_2)\}$$
$$\alpha_{f_2 w_3}(\mu_3) = 1 \text{ for } \mu_3 = \{(f_1, w_2), (f_2, w_3), (f_3, w_1)\}.$$

Then μ_1 is efficient, though agents f_2 and w_3 block any outcome (μ_1, u, v). The inefficient outcome (μ_2, u, v) is stable for $u = (0,1,0)$ and $v = (0,0,0)$. ∘

If all pairs are pessimistic, we can still attain a form of efficiency. The outcome $(\mu', u', v') \in \mathscr{M} \times \mathbb{R}^m_+ \times \mathbb{R}^n_+$ is a **Pareto improvement** over (μ, u, v) if, for all $f \in F$ and $w \in W$, $u'_f \geq u_f$ and $v'_w \geq v_w$ with at least one strict inequality. An outcome is **Pareto optimal** if it cannot be Pareto improved.

Proposition 2. *Fix λ such that pairs always are pessimistic: for each $f \in F$, $w \in W$, $\mu \in \mathscr{M}_{-fw}$, $\lambda_{fw}(\mu) = 0$. Let $A \in \mathscr{A}$ and $(\mu, u, v) \in \mathscr{M} \times \mathbb{R}^m_+ \times \mathbb{R}^n_+$ be stable in (A, λ). Then any Pareto improvement $(\mu', u', v') \in \mathscr{M} \times \mathbb{R}^m_+ \times \mathbb{R}^n_+$ of (μ, u, v) is stable in (A, λ). As a consequence, (A, λ) has a stable Pareto optimal outcome.*

Proof. **Part 1:** Suppose, to obtain a contradiction, $(f, w) \notin \mu'$ block (μ', u', v'). As f and w block (μ', u', v'), $b_{fw}(\mu') > u'_f + v'_w$. As (μ', u', v') Pareto improves (μ, u, v), $u'_f \geq u_f$ and $v'_w \geq v_w$. Therefore, $u'_f + v'_w \geq u_f + v_w$. **Case 1:** Suppose first $(f, w) \notin \mu$. As f and w always are pessimistic, $b_{fw}(\mu) = b_{fw}(\mu')$. But then $b_{fw}(\mu) = b_{fw}(\mu') > u'_f + v'_w \geq u_f + v_w$. Then f and w block the stable outcome (μ, u, v). This is a contradiction. **Case 2:** Suppose instead $\mu(f) = w$. Then $u_f + v_w = \alpha_{fw}(\mu)$. But then $b_{fw}(\mu') > u'_f + v'_w \geq u_f + v_w = \alpha_{fw}(\mu) \geq \pi_{fw}$. Then f and w are not pessimistic. This is a contradiction.

Part 2: By Proposition 1, (A, λ) has a stable outcome (μ, u, v). If (μ, u, v) is Pareto optimal, the proof is completed. So, suppose (μ, u, v) is not Pareto optimal. To obtain a contradiction, suppose there is no Pareto improvement of (μ, u, v) that itself cannot be Pareto improved. Then we can produce a sequence of outcomes, each Pareto improving its predecessor, that is cyclical (that is, the first outcome of the sequence Pareto improves upon the last). This is a contradiction as agents' payoffs are increasing throughout the sequence. Hence, there is a Pareto optimal outcome (μ', u', v') that Pareto improves (μ, u, v). By Part 1, (μ', u', v) is stable. □

3.3. The structure of the set of stable outcomes

As noted in Theorem 1, without externalities the set of stable outcomes forms a complete lattice with respect to the firms' profits and the workers' wages. It has two extreme points:

Matching	Pair 1	Pair 2	Pair 3
$\mu_1 = \{(1,1),(2,2),(3,3)\}$	8	6	4
$\mu_2 = \{(1,1),(2,3),(3,2)\}$	3	4	5
$\mu_3 = \{(1,2),(2,1),(3,3)\}$	6	4	4
$\mu_4 = \{(1,2),(2,3),(3,1)\}$	4	8	6
$\mu_5 = \{(1,3),(2,1),(3,2)\}$	5	2	5
$\mu_6 = \{(1,3),(2,2),(3,1)\}$	5	2	5

Table 2: Values for Example 4.

one which all firms prefer to all other stable outcomes, and one which all workers prefer to all other stable outcomes. We retrieve an immediate corollary of this result if we restrict attention to *individual* matchings in the following sense. Formally, for each $(A,\lambda) \in \mathscr{E}$ and each $\tilde{\mu} \in \mathscr{M}$, the set

$$\left\{(u,v) \in \mathbb{R}^m_+ \times \mathbb{R}^n_+ : (u,v) \text{ is compatible with } \tilde{\mu} \text{ and } (\tilde{\mu},u,v) \text{ is stable in } (A,\lambda)\right\}$$

forms a complete lattice with respect to the firms' profits and the workers' wages. Hence, if the problem has a unique stable matching, then the set of stable outcomes forms a lattice. However, the following example shows that, if there are multiple stable matchings, then the *full* set of stable outcomes does not form a lattice.

Example 4: The stable set may not form a lattice. Consider an assignment game with externalities $(A,\lambda) \in \mathscr{E}$ with agents $F = \{f_1, f_2, f_3\}$ and $W = \{w_1, w_2, w_3\}$. All pairs are pessimistic; values are in Table 2.

There are two stable matchings: μ_1 and μ_4. The set of stable payoffs compatible with μ_1 forms a complete lattice, with (firm-) minimal element $u = (1,0,0)$ and $v = (7,6,4)$. For μ_4, the stable outcomes form a lattice disjoint from the former set. The minimal element is $u' = (0,0,1)$ with $v' = (5,4,8)$. For neither matching, the meet $u'' = (0,0,0)$ is stable. Hence, the *full* set of stable outcomes does not form a lattice. See Figure 1 for a graphical illustration.

As agents get more optimistic, they object to more outcomes, and the set of stable outcomes shrinks. For concreteness, suppose all $\lambda_{fw}(\mu) = 0.4$. Then only μ_4 is stable. The smaller set of stable outcomes is marked in black in Figure 1. ∘

Without externalities, we can "swap" stable matchings: formally, if the outcomes (μ,u,v) and (μ',u',v') are stable in $A \in \mathscr{A}^0$, then so is (μ',u,v). It follows immediately from Example 4 that this no longer generally is valid for assignment games with externalities.

4. Imposing more structure

In Example 2 and in the proof of Theorem 2, we exploit either (i) large externalities when agents are optimistic, or (ii) (almost) optimistic agents when externalities are small. In this section, we impose more structure on the values and the externalities and ask whether this

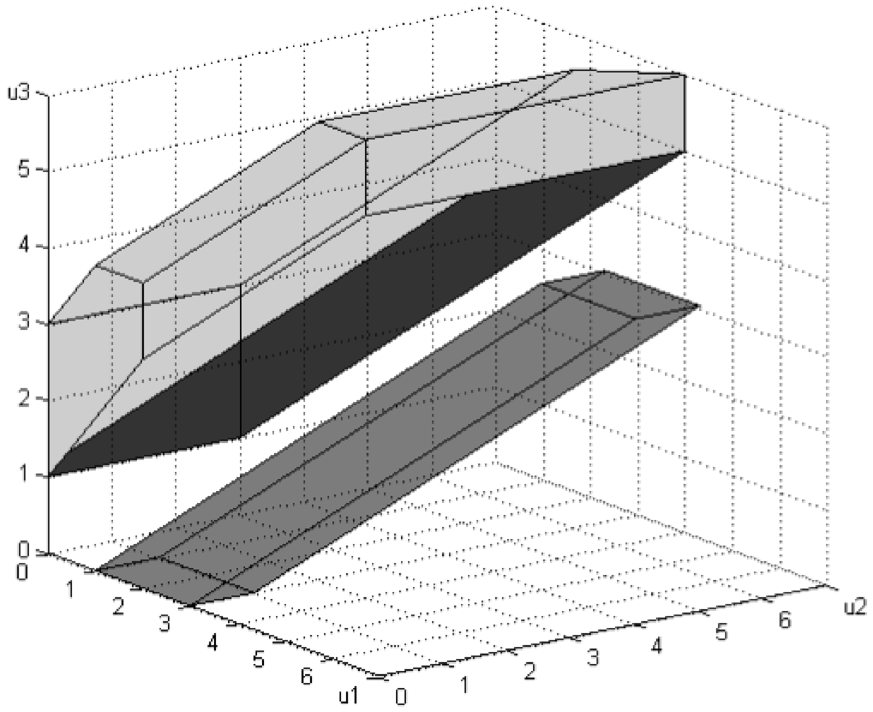


Figure 1: The dark grey area shows the lattice structure of the stable payoffs compatible with the matching μ_1 when agents are pessimistic in the example of Example 4. The light grey area consists of stable payoffs compatible with μ_4. The black area consists of stable payoffs compatible with μ_4 when agents are more optimistic. Importantly, the two grey areas are disjoint.

overturns the negative findings. The setting is highly restrictive on purpose. Our aim is to show that, even if values are constructed from a "realistic" problem, our negative results still apply.

The skill of worker w is measured by $s(w)$: no matter where w is employed, she will add a value of $s(w)$. Firms produce different products, some of which complement each other, some substitute. The "degree of complementarity" is captured by $\kappa : F \times F \to \mathbb{R}$. If firms f and g produce complementary goods, $\kappa(f,g) > 0$. If they produce substitutes, $\kappa(f,g) < 0$. For each $f \in F$, $\kappa(f,f) = 1$. The value created by $f \in F$ and $w \in W$ at $\mu \in \mathcal{M}_{fw}$ is given by

$$\alpha_{fw}(\mu) = \sum_{f' \in F} \kappa(f,f') \cdot s\big(\mu(f')\big).$$

Example 5: Negative results not overturned by more structure. Let $F = \{f_1, f_2, f_3\}$ and $W = \{w_1, w_2, w_3\}$. For simplicity, the skill of $w_i \in W$ is given by $s(w_i) = 2i$. That is, w_1 is the least productive worker, w_3 the most. Firms f_1 and f_3 produce substitutable goods (different brands of phones), say $\kappa(f_1, f_3) = -0.5$. Firm f_2 produces a complementary good (phone

Matching	**Pair 1**	**Pair 2**	**Pair 3**
$\mu_1 = \{(1,1),(2,2),(3,3)\}$	1	8	7
$\mu_2 = \{(1,1),(2,3),(3,2)\}$	3	9	6
$\mu_3 = \{(1,2),(2,1),(3,3)\}$	2	7	5
$\mu_4 = \{(1,2),(2,3),(3,1)\}$	6	9	3
$\mu_5 = \{(1,3),(2,1),(3,2)\}$	5	7	2
$\mu_6 = \{(1,3),(2,2),(3,1)\}$	7	8	1

Table 3: Values for Example 5.

accessories), say $\kappa(f_1, f_2) = \kappa(f_2, f_3) = 0.5$.[5] Therefore, at $\mu_1 = \{(f_1, w_1), (f_2, w_2), (f_3, w_3)\}$, the pair (f_1, w_1) creates a value of 1:

$$\begin{aligned} \alpha_{f_1 w_1}(\mu_1) &= \kappa(f_1, f_1)\cdot s\big(\mu_1(f_1)\big) + \kappa(f_1, f_2)\cdot s\big(\mu_1(f_2)\big) + \kappa(f_1, f_3)\cdot s\big(\mu_1(f_3)\big) \\ &= \underbrace{1\cdot s(w_1)}_{2} + \underbrace{1/2\cdot s(w_2)}_{2} + \underbrace{(-1/2)\cdot s(w_3)}_{-3} = 1. \end{aligned}$$

The values are summarized in Table 3.

If pairs always are pessimistic, then every matching is stable. However, if we consider the case where no pair is pessimistic, that is, when there is $\varepsilon > 0$ such that, for all $f \in F$, $w \in W$, and $\mu \in \mathcal{M}_{fw}$, we have $\lambda_{fw}(\mu) \geq \varepsilon$, then there is no stable outcome – no matter how small we make ε. We only give proof to this claim for μ_1. The exercise can be repeated for the other matchings.

Suppose, to obtain a contradiction, that (μ_1, u, v) is stable. For f_1 and w_2 not to block,

$$u_{f_1} + v_{w_2} = u_{f_1} + (8 - u_{f_2}) \geq b_{f_1 w_2}(\mu_1) \geq \varepsilon\cdot\omega_{f_1 w_2} + (1-\varepsilon)\cdot\pi_{f_1 w_2} > \pi_{f_1 w_2} = 2.$$

Then $u_{f_2} - u_{f_1} < 6$. For f_2 and w_1 not to block,

$$u_{f_2} + v_{w_1} = u_{f_2} + (1 - u_{f_1}) \geq b_{f_2 w_1}(\mu_1) = 7 \Leftrightarrow u_{f_2} - u_{f_1} \geq 6.$$

This is contradictory to the strict inequality just found. Thus, no matter the outcome (μ_1, u, v), one of the pair blocks. ∘

5. Conclusion

We have shown that the introduction of externalities into the assignment game overturns all the classical results. In particular, we show that (i) there are problems that have no stable outcome, (ii) stable outcomes need not be efficient (maximize total value created), (iii) efficient outcomes need not be stable, and (iv) the set of stable outcomes need not form a lattice.

[5] Number are chosen to simplify the example; more "varied" example are available upon request.

The positive finding of paper is the following knife-edge result: if all agents are pessimistic, then there are stable outcomes in all assignment games with externalities. The result is complemented by a strong negative finding that shows that even the slightest optimism can be complemented by values that yield an assignment game with externalities that lacks stable outcomes.

Aumann, R. J., Peleg, B., 1960. Von Neumann-Morgenstern solutions to cooperative games without side payments 66, 173–179.

Chen, B., 2013. Assignment games with externalities and matching-based cournot competition. Bonn Econ Discussion Papers 08/2013.

Eriksson, K., Jansson, F., Vetander, T., 2011. The assignment game with negative externalities and bounded rationality 13 (04), 443–459.

Funaki, Y., Yamato, T., 1999. The core of an economy with a common pool resource: A partition function form approach 28 (2), 157–171.
URL http://ideas.repec.org/a/spr/jogath/v28y1999i2p157-171.html

Hafalir, I. E., 2008. Stability of marriage with externalities 37 (3), 353–369.

Kóczy, L., August 2007. A recursive core for partition function form games 63 (1), 41–51.

Koopmans, T., Beckmann, M., 1957. Assignment problems and the location of economic activities 25, 53–76.

Li, S., 1993. Competitive matching equilibrium and multiple principal-agent models. Tech. rep., Center for Economic Research, Department of Economics, University of Minnesota.

Mumcu, A., Saglam, I., 2007. The core of a housing market with externalities 3 (57), 1–5.
URL http://ideas.repec.org/a/ebl/ecbull/eb-07c70026.html

Roth, A. E., Sotomayor, M. A. O., 1992. Two-sided matching: A study in game-theoretic modeling and analysis. Vol. 18. Cambridge University Press.

Sasaki, H., Toda, M., 1996. Two-sided matching problems with externalities 70 (1), 93–108.

Shapley, L. S., Shubik, M., 1971. The assignment game i: The core 1 (1), 111–130.

Shenoy, P. P., October 1980. A dynamic solution concept for abstract games 32 (2), 151–169.