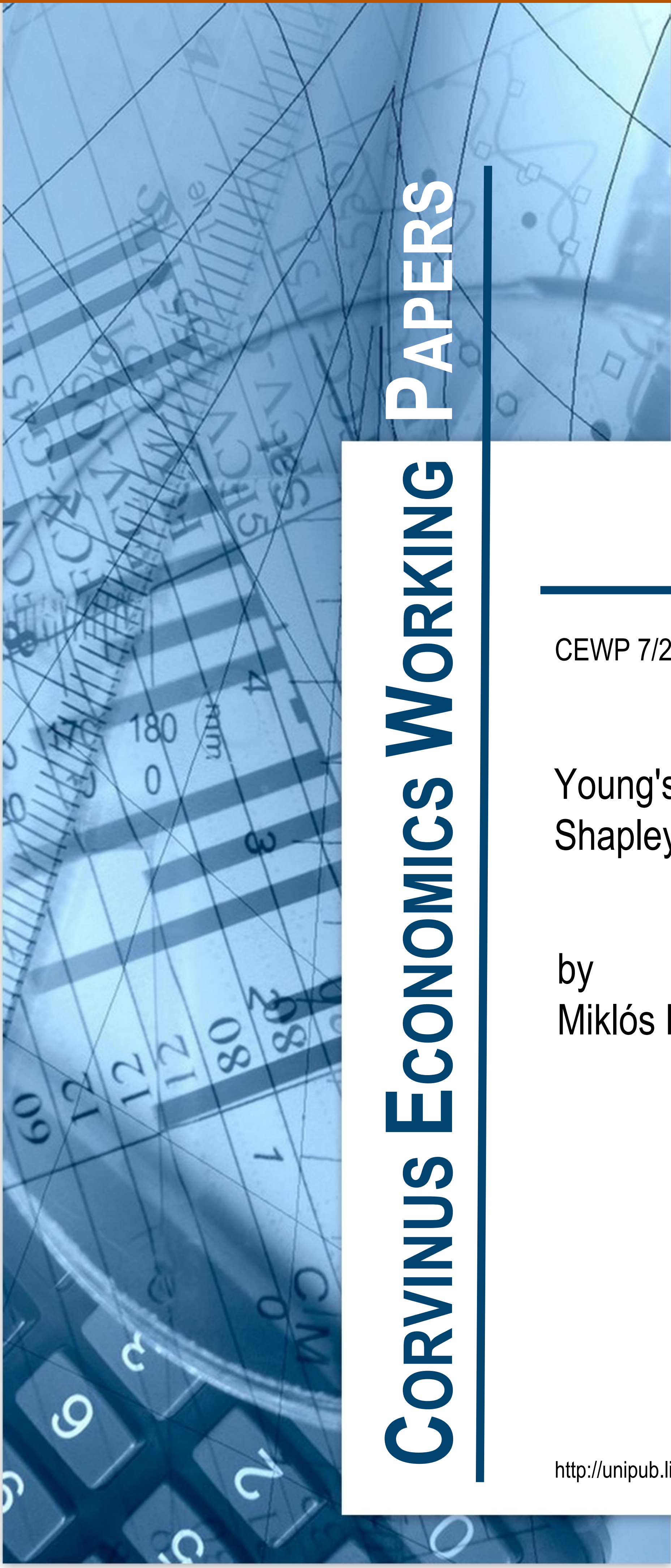

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Young's axiomatization of the Shapley value - a new proof*

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April 2, 2015

Abstract

We give a new proof of Young's characterization of the Shapley value. Moreover, as applications of the new proof, we show that Young's axiomatization of the Shapley value is valid on various well-known subclasses of *TU* games.

Keywords: *TU* games; Shapley value; Young's axiomatization.
JEL Classification: C71.

1 Introduction

In this paper we consider one of the well-known characterizations of the Shapley value (Shapley, 1953), the axiomatization of Young (1985). The Shapley value is probably the most popular one-point solution (value) of transferable utility (*TU*) cooperative games (henceforth games). It is applied in various fields ranging from medicine to statistics, from engineering to accounting etc. Therefore a solid characterization could well serve, among others, applications by helping in understanding its very nature.

Young (1985) axiomatizes the Shapley value with three axioms: *Efficiency* (*Pareto optimality PO*), *Equal Treatment Property ETP* (*Symmetry*)[1], and *Marginality.* Moulin (1988) suggests an alternative proof for Young's result

*I thank the AE and the two anonymous referees, Ferenc Forgó, Anna Khmelnitskaya, Zsófia Széna and William Thomson for their suggestions and remarks. Financial support by the Hungarian Scientific Research Fund (OTKA) and the János Bolyai Research Scholarship of the Hungarian Academy of Sciences is also gratefully acknowledged.

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[1] In brackets the original, stronger axiom used by Young (1985).

in the three player setting. Both Young (1985) and Moulin (1988) consider the whole class of TU games, however, Young (1985) also shows that this characterization is valid on other classes of games, he specifies the class of superadditive games.

With respect to other subclasses on which Young's axiomatization works we have to mention three other papers. Neyman (1989) shows that a solution defined on the additive group generated by a game is Efficient (PO), Symmetric (or ETP) and Strongly Monotonic if and only if it is the Shapley value. Khmelnitskaya (2003) proves that Young's axiomatization works on the class of non-negative constant-sum games with non-zero worth of grand coalition and on the (entire) class of constant-sum games. Furthermore, Mlodak (2003) applies the same method as that Khmelnitskaya does to characterize the Shapley value á la Young (1985) on the class of non-negative bilateral games.

It is well-known that the validity of an axiomatization can vary from subclass to subclass, e.g. Shapley's axiomatization of the Shapley value is valid on the class of monotone games but not valid on the class of strictly monotone games. Therefore, we must consider each subclass of games one by one.

By this new proof there are already three different methods for checking the validity of Young's axiomatization of the Shapley value (on subclasses of games): Young's, Moulin's and ours. We emphasize these three methods are not comparable, for each of them there are cases where the one works and the others do not and vice versa, and naturally there are cases where all work and where none of them works.

The setup of the paper is as follows. In Section 2 we introduce the terminology used throughout the paper. The last section discusses our main result.

2 Preliminaries

Notation: let $|N|$ and 2^N denote the cardinality of set N and the set of all subsets of N respectively. Moreover, $A \subset B$ means $A \subseteq B$, but $A \neq B$, and we also use $|a|$ for the absolute value of real number a. Furthermore, $\sum_{i \in \emptyset} x_i = 0$, that is, the empty sum is zero.

Let $N \neq \emptyset$, $|N| < \infty$ and $v : 2^N \to \mathbb{R}$ be a function such that $v(\emptyset) = 0$. Then N and v are called set of players and *transferable utility cooperative game* (henceforth *game*) respectively. The class of games with player set N is denoted by $\mathcal{G}^N$.

Let $v \in \mathcal{G}^N$, $i \in N$, and for each $S \subseteq N$: let $v'_i(S) = v(S \cup \{i\}) - v(S)$.

Then v_i' is called Player i's *marginal contribution function* in game v. In other words, $v_i'(S)$ is Player i's marginal contribution to coalition S in game v.

In this paper, along with $\mathcal{G}^N$, we consider also subclasses of games defined below. A game $v \in \mathcal{G}^N$ is

- *essential*, if $v(N) > \sum\limits_{i \in N} v(\{i\})$,
- *convex*, if for all $S, T \subseteq N$: $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$,
- *strictly convex*, if for all $S, T \subseteq N$, $S \nsubseteq T$, $T \nsubseteq S$: $v(S) + v(T) < v(S \cup T) + v(S \cap T)$,
- *superadditive*, if for all $S, T \subseteq N$, $S \cap T = \emptyset$: $v(S) + v(T) \leq v(S \cup T)$,
- *strictly superadditive*, if for all $S, T \subseteq N$, $S, T \neq \emptyset$, $S \cap T = \emptyset$: $v(S) + v(T) < v(S \cup T)$,
- *weakly superadditive*, if for all $S \subseteq N$, $i \in N \setminus S$: $v(S) + v(\{i\}) \leq v(S \cup \{i\})$,
- *strictly weakly superadditive*, if for all $S \subseteq N$, $S \neq \emptyset$, $i \in N \setminus S$: $v(S) + v(\{i\}) < v(S \cup \{i\})$,
- *monotone*, if for all $S, T \subseteq N$, $S \subseteq T$: $v(S) \leq v(T)$,
- *strictly monotone*, if for all $S, T \subseteq N$, $S \subset T$: $v(S) < v(T)$,
- *additive*, if for all $S, T \subseteq N$, $S \cap T = \emptyset$: $v(S) + v(T) = v(S \cup T)$,
- *weakly subadditive*, if for all $S \subseteq N$, $i \in N \setminus S$: $v(S) + v(\{i\}) \geq v(S \cup \{i\})$,
- *strictly weakly subadditive*, if for all $S \subseteq N$, $S \neq \emptyset$, $i \in N \setminus S$: $v(S) + v(\{i\}) > v(S \cup \{i\})$,
- *subadditive*, if for all $S, T \subseteq N$, $S \cap T = \emptyset$: $v(S) + v(T) \geq v(S \cup T)$,
- *strictly subadditive*, if for all $S, T \subseteq N$, $S, T \neq \emptyset$, $S \cap T = \emptyset$: $v(S) + v(T) > v(S \cup T)$,
- *concave*, if for all $S, T \subseteq N$: $v(S) + v(T) \geq v(S \cup T) + v(S \cap T)$,
- *strictly concave*, if for all $S, T \subseteq N$, $S \nsubseteq T$, $T \nsubseteq S$: $v(S) + v(T) > v(S \cup T) + v(S \cap T)$.

For the definition of essential games see e.g. von Neumann and Morgenstern (1953), and for other types of games see e.g. Peleg and Sudhölter (2003).

The following alternative definitions of (strictly) convex and (strictly) concave games are well known:

$$\begin{array}{l} \text{Game } v \in \mathcal{G}^N \text{ is (strictly) convex, if for each } i \in N,\, T, Z \subseteq N \setminus \{i\} \\ \text{such that } Z \subset T\text{: } v_i'(Z) \leq v_i'(T)\ (v_i'(Z) < v_i'(T))\ , \\ \text{and } v \in \mathcal{G}^N \text{ is (strictly) concave, if for each } i \in N,\, T, Z \subseteq N \setminus \{i\} \\ \text{such that } Z \subset T\text{: } v_i'(Z) \geq v_i'(T)\ (v_i'(Z) > v_i'(T))\ . \end{array} \tag{1}$$

The *dual* of game $v \in \mathcal{G}^N$ is the game $\bar{v} \in \mathcal{G}^N$ such that for each $S \subseteq N$: $\bar{v}(S) = v(N) - v(N \setminus S)$.

For any game $v \in \mathcal{G}^N$, players $i, j \in N$ are *equivalent*, $i \sim^v j$, if for each $S \subseteq N$ such that $i, j \notin S$: $v_i'(S) = v_j'(S)$. It is easy to verify that for any game $v \in \mathcal{G}^N$ $\sim^v$ is a binary equivalence relation on N.

Furthermore, if $S \subseteq N$ is such that for all $i, j \in S$: $i \sim^v j$, then we say that S is an *equivalence set* in game v.

Next we summarize some important properties of dual games. For any game $v \in \mathcal{G}^N$:

$$\begin{array}{l} \text{If } i \sim^v j, \text{ then } i \sim^{\bar{v}} j. \\ \text{If } w_i' = v_i', \text{ then } \bar{w}_i' = \bar{v}_i'. \\ \text{The dual of a (strictly) convex game is a (strictly) concave game.} \\ \text{The dual of a (strictly) concave game is a (strictly) convex game.} \end{array} \tag{2}$$

A function $\psi : A \to \mathbb{R}^N$, defined on set $A \subseteq \mathcal{G}^N$, is a *solution* on A. Throughout the paper we consider single-valued solutions (values).

For any game $v \in \mathcal{G}^N$ the *Shapley solution* ϕ is given by

$$\phi_i(v) = \sum_{S \subseteq N \setminus \{i\}} v_i'(S) \frac{|S|!(|N \setminus S| - 1)!}{|N|!}\ , \quad i \in N,$$

where $\phi_i(v)$ is also called Player i's *Shapley value* (Shapley, 1953).

The solution ψ on class of games $A \subseteq \mathcal{G}^N$ satisfies

- *Pareto optimality* (*PO*), if for each $v \in A$: $\sum_{i \in N} \psi_i(v) = v(N)$,
- *Equal Treatment Property* (*ETP*), if for all $v \in A$, $i, j \in N$: $i \sim^v j$ implies $\psi_i(v) = \psi_j(v)$,

- *Marginality* (M), if for all $v, w \in A$, $i \in N$: $v'_i = w'_i$ implies $\psi_i(v) = \psi_i(w)$.

Remark 2.1. Notice that the Shapley solution is (completely) determined by the players' marginal contribution functions. Therefore for any solution ψ meeting axiom M: if $\psi_i(v) = \phi_i(v)$ and $v'_i = w'_i$, then $\psi_i(w) = \phi_i(w)$.

It is well known and not difficult to check that the Shapley solution meets axioms PO, ETP and M (see e.g. Young (1985)).

3 The main result

In this section we present our main result. The following example illustrates the idea behind our result (Theorem 3.4). This example shows that we can construct chains of games such that in any chain the elements are connected by axiom M and in the terminal games all players are equivalent.

Example 3.1. Let $N = \{1, 2, 3\}$ and $v = (0, 0, 0, 3, 1, 2, 3) \in \mathcal{G}^N$, where $v = (v(\{1\}), v(\{2\}), v(\{3\}), v(\{1, 2\}), v(\{1, 3\}), v(\{2, 3\}), v(N))$. Then v is a superadditive but not convex game, and $1 \nsim^v 2$, $1 \nsim^v 3$, $2 \nsim^v 3$.

Furthermore, let ψ be a PO, ETP and M solution on $\mathcal{G}^N$. We show that $\psi_2(v) = \phi_2(v)$ (the solid path from vertex v to vertex z in Figure 1.).

Take Player 1 as a singleton equivalence set in game v and choose Player 2. Then there is a game $w = (0, 0, 0, 3, 2, 2, 4)$ such that $w'_2 = v'_2$ and $1 \sim^w 2$ (it is clear that w is not the only game in which players 1 and 2 are equivalent and $w'_2 = v'_2$).

Next take equivalence set $\{1, 2\}$ in game w and choose Player 3. Then there exists a game $z = (0, 0, 0, 2, 2, 2, 3)$ such that $z'_3 = w'_3$ and $1 \sim^z 2 \sim^z 3$.

Then axioms PO and ETP imply that $\psi(z) = \phi(z)$. Moreover, by axiom M (and Remark 2.1), $\psi_3(w) = \phi_3(w)$. Since ψ is PO and ETP, $1 \sim^w 2$, therefore $\psi(w) = \phi(w)$.

By applying axiom M (and Remark 2.1) again, we get $\psi_2(v) = \phi_2(v)$.

From Example 3.1 it is clear that we can deduce $\psi_i = \phi_i$ for any i (player), see the dashed paths in Figure 1. In other words, we can show that $\psi(v) = \phi(v)$. All we need is that ψ must be defined on the paths from v to the leafs in Figure 1. Notice that Figure 1 is only an illustration. For example considering the dashed path on the left we see that we copy Player 1 and make the equivalence set $\{1, 2\}$ first, then we copy Player 3, but we could make the equivalence set $\{1, 3\}$ first instead and copy Player 2 in the last step. Therefore there are more alternatives not only those in Figure 1.

The next notion is an important ingredient of our main theorem.

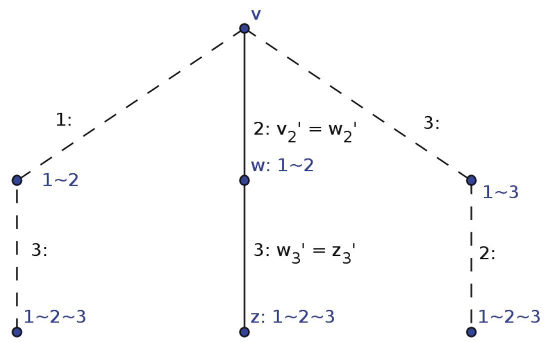

Figure 1: The overview of Example 3.1

Definition 3.2. *Class $A \subseteq \mathcal{G}^N$ is M-closed, if for each game $v \in A$, equivalence set in v $S \subseteq N$, and Player $k \in N \setminus S$ there exists $w \in A$ such that $S \cup \{k\}$ is an equivalence set in w and $w'_k = v'_k$.*

Remark 3.3. Notice that from (2), if $A \subseteq \mathcal{G}^N$ is an M-closed class of games, then $\bar{A} = \{\bar{v} \in \mathcal{G}^N : v \in A\}$ is also M-closed.

The following theorem is our main result.

Theorem 3.4. *Let class $A \subseteq \mathcal{G}^N$ be M-closed. Then solution ψ, defined on class A, satisfies axioms PO, ETP and M, if and only if $\psi = \phi$, that is, if and only if, it is the Shapley solution.*

Proof. If : It is well-known.

Only if: Take a game $z \in A$ and a player $i_1 \in N$. Class A is M-closed, therefore for any player $i_2 \in N \setminus \{i_1\}$ there exists $z(1) \in A$ such that $z(1)'_{i_2} = z'_{i_2}$ and $\{i_1, i_2\}$ is an equivalence set in $z(1)$. Let $i_3 \in N \setminus \{i_1, i_2\}$.

Class A is M-closed therefore there exists $z(2) \in A$ such that $z(2)'_{i_3} = z(1)'_{i_3}$ and $\{i_1, i_2, i_3\}$ is an equivalence set in $z(2)$. Let $i_4 \in N \setminus \{i_1, i_2, i_3\}$.

$\vdots$

Class A is M-closed therefore there exists $z(n-1) \in A$ such that $z(n-1)'_{i_n} = z(n-2)'_{i_n}$ and $\{i_1, i_2, \ldots, i_n\}$ $(= N)$ is an equivalence set in $z(n-1)$.

By axioms PO and ETP, $\psi(z(n-1)) = \phi(z(n-1))$, since all the players $i_1, i_2, \ldots, i_n$ are equivalent in $z(n-1)$.

Since solution ψ meets axiom M, and by construction $z(n-1)'_{i_n} = z(n-2)'_{i_n}$, it follows that (see Remark 2.1) $\psi_{i_n}(z(n-2)) = \phi_{i_n}(z(n-2))$. Next, all the players $i_1, i_2, \ldots, i_{n-1}$ are equivalent in game $z(n-2)$, whence by axioms ETP and PO we get $\psi(z(n-2)) = \phi(z(n-2))$.

Since solution ψ meets axiom M, and by construction $z(n-2)'_{i_{n-1}} = z(n-3)'_{i_{n-1}}$, it follows that (see Remark 2.1) $\psi_{i_{n-1}}(z(n-3)) = \phi_{i_{n-1}}(z(n-3))$.

Player i_{n-1} was arbitrarily chosen, so for any player not $i_1, i_2, \ldots, i_{n-2}$ we have the player gets its Shapley value by solution ψ in game $z(n-3)$. Next, all the players $i_1, i_2, \ldots, i_{n-2}$ are equivalent in game $z(n-3)$, whence by axioms ETP and PO we get $\psi(z(n-3)) = \phi(z(n-3))$.

$\vdots$

Since solution ψ meets axiom M, and by construction $z(1)'_{i_2} = z'_{i_2}$, it follows that (see Remark 2.1) $\psi_{i_2}(z) = \phi_{i_2}(z)$. Player i_2 was arbitrarily chosen, so for any player not i_1 we have the player gets its Shapley value by solution ψ in game z. Therefore by the axiom PO we get $\psi(z) = \phi(z)$. $\square$

Next, we show that the above theorem implements Young's result.

Theorem 3.5 (Young (1985))**.** *A solution ψ on $\mathcal{G}^N$ satisfies axioms PO, ETP and M, if and only if $\psi = \phi$, that is, if and only if, it is the Shapley solution.*

To prove Theorem 3.5 it is enough to show that $\mathcal{G}^N$ is M-closed.

Proposition 3.6. *The class of games $\mathcal{G}^N$ is M-closed.*

First we prove the following lemma.

Lemma 3.7. *Let $v \in \mathcal{G}^N$. Then $S \subseteq N$ is an equivalence set in game v, if and only if for all $T, Z \subseteq N$ such that $T \setminus S = Z \setminus S$ and $|T| = |Z|$: $v(T) = v(Z)$.*

Proof. If: It is left for the reader.

Only if: W.l.o.g. we can assume that $T \setminus Z \neq \emptyset$ (if $T \setminus Z = \emptyset$, then the proof ends), and let $T \setminus Z = \{l_1, \ldots, l_m\}$ and $Z \setminus T = \{q_1, \ldots, q_m\}$. Here, $T \setminus Z \subseteq S$ and $Z \setminus T \subseteq S$, S is an equivalence set in v, hence for each player i, $1 \leq i \leq m$:

$$\begin{aligned} &v((T \cap Z) \cup \{l_1, \ldots, l_i\}) \\ &= v((T \cap Z) \cup \{l_1, \ldots, l_{i-1}\}) + v'_{l_i}((T \cap Z) \cup \{l_1, \ldots, l_{i-1}\}) \\ &= v((T \cap Z) \cup \{q_1, \ldots, q_{i-1}\}) + v'_{q_i}((T \cap Z) \cup \{q_1, \ldots, q_{i-1}\}) \\ &= v((T \cap Z) \cup \{q_1, \ldots, q_i\}) \end{aligned}$$

Therefore $v(T) = v(Z)$. $\square$

Next, we consider a direct corollary of Lemma 3.7.

Corollary 3.8. *Let $v \in \mathcal{G}^N$, $S \subseteq N$ be an equivalence set in v, and $k \in N \setminus S$. Then for all $T, Z \subseteq N$ such that $T \setminus S = Z \setminus S$ and $|T| = |Z|$: $v'_k(T) = v'_k(Z)$.*

Proof of Proposition 3.6. Let $v \in \mathcal{G}^N$ be such that $S \subset N$ is an equivalence set in v, and $k \in N \setminus S$.

Let game $w \in \mathcal{G}^N$ be defined as follows for each coalition $T \subseteq N$: if $T \cap (S \cup \{k\}) = \emptyset$, then let $w(T)$ be arbitrarily defined such that $w(\emptyset) = 0$. In the other cases ($T \cap (S \cup \{k\}) \neq \emptyset$), let

$$w(T) = w(T \setminus (S \cup \{k\})) + \sum_{i=1}^{m} v'_k((T \setminus (S \cup \{k\})) \cup \{l_1, \dots, l_{i-1}\}) \ , \quad (3)$$

where $m = |(S \cup \{k\}) \cap T|$, and $l_i \in S \cap T$, $i = 1, \dots, m-1$.

Notice that from Corollary 3.8 $\sum_{i=1}^{m} v'_k((T \setminus (S \cup \{k\})) \cup \{l_1, \dots, l_{i-1}\})$ does not depend on the ordering of the elements of $S \cap T$, that is, w is well-defined.

It is easy to verify that $w'_k = v'_k$. Furthermore, notice that by equation (3) if coalitions $T, Z \subseteq N$ are such that $T \setminus (S \cup \{k\}) = Z \setminus (S \cup \{k\})$ and $|T| = |Z|$ then we have $w(T) = w(Z)$, hence from Lemma 3.7 $S \cup \{k\}$ is an equivalence set in w. $\square$

Proof of Theorem 3.5. See Theorem 3.4 and Proposition 3.6. $\square$

Next, we show that Young's axiomatization is also valid on some considered subclasses of games.

Theorem 3.9. *A solution ψ defined on the class of either (strictly) convex, (strictly) weakly superadditive, (strictly) monotone, additive, (strictly) weakly subadditive or (strictly) concave games satisfies axioms PO, ETP and M, if and only if $\psi = \phi$, that is, if and only if it is the Shapley solution.*

Proof. We show that all considered subclasses of games are M-closed.

Let $v \in \mathcal{G}^N$ be such that $S \subset N$ is an equivalence set in v, and $k \in N \setminus S$. The proof of Proposition 3.6 shows that there exists $w \in \mathcal{G}^N$ such that $S \cup \{k\}$ is an equivalence set in w, $w'_k = v'_k$, and for each coalition T such that $T \cap (S \cup \{k\}) = \emptyset$, $T \neq \emptyset$: $w(T)$ is arbitrarily defined. Therefore, the only thing we have to do is to show that we can give values to these coalitions such that w be in the considered class of games.

(I) The class of additive games: It is well known that game $z \in \mathcal{G}^N$ is additive, if and only if for each Player $i \in N$ there exists $c_i \in \mathbb{R}$ such that for each coalition $T \subseteq N \setminus \{i\}$: $z'_i(T) = c_i$.

Let $c^* = v'_k(\emptyset)$, and for for each $T \subseteq N$ let

$$w(T) = c^* |T| \ . \quad (4)$$

Then, it is easy to see that $w'_k = v'_k$, w is additive and N is an equivalence set in w.

(II) The classes of (strictly) convex, (strictly) weakly superadditive and (strictly) monotone games: Let $M > \max_{T \subset N} |v'_k(T)|$, and for the coalitions on which w is arbitrarily defined ($T \cap (S \cup \{k\}) = \emptyset$, $T \neq \emptyset$, see Proposition 3.6): let

$$w(T) = M|N|(|T|+1)^{|T|} \; . \tag{5}$$

(A) If v is a (strictly) weakly superadditive game, then $v'_k(Z) \geq v'_k(\emptyset)$ ($v'_k(Z) > v'_k(\emptyset)$), $k \notin Z \subseteq N$ therefore, by equations (3) and (5) we have that w is also a (strictly) weakly superadditive game.

If v is a (strictly) monotone game, then $v'_k(Z) \geq 0$ ($v'_k(Z) > 0$), $k \notin Z \subseteq N$, therefore, by equations (3) and (5) we have that w is also a (strictly) monotone game.

For the other properties see Proposition 3.6.

(B) Next we show that, if v is a (strictly) convex game, then so is w. First notice that game $z \in \mathcal{G}^N$ is strictly convex, if and only if for each $i \in N$, $T, Z \subseteq N \setminus \{i\}$ such that $Z \subset T$: $z'_i(Z) < z'_i(T)$ (see (1)).

Let $l \in N \setminus (S \cup \{k\})$ and $T, Z \subseteq N \setminus \{l\}$ be such that $Z \subset T$, then

$$\begin{aligned} & w'_l(T) = w(T \cup \{l\}) - w(T) \\ &= w((T \cup \{l\}) \setminus (S \cup \{k\})) + \sum_{i=1}^{m} w'_{l_i}(((T \cup \{l\}) \setminus (S \cup \{k\})) \cup \{l_1, \dots, l_{i-1}\}) \\ & -w(T \setminus (S \cup \{k\})) - \sum_{i=1}^{m} w'_{l_i}((T \setminus (S \cup \{k\})) \cup \{l_1, \dots, l_{i-1}\}) \; , \end{aligned}$$

and

$$\begin{aligned} & w'_l(Z) = w(Z \cup \{l\}) - w(Z) \\ &= w((Z \cup \{l\}) \setminus (S \cup \{k\})) + \sum_{i=1}^{n} w'_{l_i}(((Z \cup \{l\}) \setminus (S \cup \{k\})) \cup \{l_1, \dots, l_{i-1}\}) \\ & -w(Z \setminus (S \cup \{k\})) - \sum_{i=1}^{n} w'_{l_i}((Z \setminus (S \cup \{k\})) \cup \{l_1, \dots, l_{i-1}\}) \; , \end{aligned}$$

where $m = |(S \cup \{k\}) \cap T|$, $n = |(S \cup \{k\}) \cap Z|$ and $\{l_1, \dots, l_n\} = (S \cup \{k\}) \cap Z = (S \cup \{k\}) \cap (Z \cup \{l\}) \subseteq (S \cup \{k\}) \cap (T \cup \{l\}) = (S \cup \{k\}) \cap T = \{l_1, \dots, l_m\}$.

Notice that, if $T \setminus (S \cup \{k\}) = Z \setminus (S \cup \{k\})$, then the proof is complete. Therefore, w.l.o.g. we can assume that $Z \setminus (S \cup \{k\}) \subset T \setminus (S \cup \{k\})$. Game v is a (strictly) convex game, $S \cup \{k\}$ is an equivalence set in w and $n < |N|$ so (the inequality is by simple counting the players (notice that M is an upper estimation for the marginal contributions of players l_i) and by considering the signs of the terms)

$$
\begin{aligned}
&\sum_{i=1}^{m} w'_{l_i}(((T \cup \{l\}) \setminus (S \cup \{k\})) \cup \{l_1, \ldots, l_{i-1}\}) \\
&\quad - \sum_{i=1}^{m} w'_{l_i}((T \setminus (S \cup \{k\})) \cup \{l_1, \ldots, l_{i-1}\}) \\
&- \sum_{i=1}^{n} w'_{l_i}(((Z \cup \{l\}) \setminus (S \cup \{k\})) \cup \{l_1, \ldots, l_{i-1}\}) \\
&+ \sum_{i=1}^{n} w'_{l_i}((Z \setminus (S \cup \{k\})) \cup \{l_1, \ldots, l_{i-1}\}) > -2M|N| \ .
\end{aligned} \tag{6}
$$

On the other hand, from (5) (the inequality is by $Z \setminus (S \cup \{k\}) \subset T \setminus (S \cup \{k\})$)

$$
\begin{aligned}
& w((T \cup \{l\}) \setminus (S \cup \{k\})) - w(T \setminus (S \cup \{k\})) \\
& -w((Z \cup \{l\}) \setminus (S \cup \{k\})) + w(Z \setminus (S \cup \{k\})) \\
&= M|N| \big((|(T \cup \{l\}) \setminus (S \cup \{k\})| + 1)^{|(T \cup \{l\}) \setminus (S \cup \{k\})|} \\
&\quad - (|T \setminus (S \cup \{k\})| + 1)^{|T \setminus (S \cup \{k\})|} \\
&\quad - (|(Z \cup \{l\}) \setminus (S \cup \{k\})| + 1)^{|(Z \cup \{l\}) \setminus (S \cup \{k\})|} \\
&\quad + (|Z \setminus (S \cup \{k\})| + 1)^{|Z \setminus (S \cup \{k\})|} \big) > 2M|N| \ .
\end{aligned} \tag{7}
$$

Summing up (6) and (7)

$$
w'_l(T) - w'_l(Z) > 0 \ . \tag{8}
$$

Since $l \in N \setminus (S \cup \{k\})$ and $T, Z \subseteq N \setminus \{l\}$, $Z \subset T$ were arbitrarily chosen, w is (strictly) convex (for the other properties see Proposition 3.6).

(III) The class of (strictly) concave and (strictly) subadditive games.

(A) Notice that a game v is (strictly) concave game, if and only if $\bar{v}$ is (strictly) convex (see (2)). Therefore, see Remark 3.3, from Point (II) the class of (strictly) concave games is an M-closed class of games.

(B) The class of (strictly) weakly subadditive games: It is worth noticing that the dual of a (strictly) subadditive or a (strictly) weakly subadditive game is not necessarily (strictly) superadditive or (strictly) weakly superadditive respectively, e.g. $v = (4, 4, 4, 4, 4, 4, 7)$ is strictly subadditive, but $\bar{v}$ is not weakly superadditive. Moreover, the dual of a (strictly) superadditive or a (strictly) weakly superadditive game is not necessarily (strictly) subadditive or (strictly) weakly subadditive, e.g. $v = (0, 0, 0, 3, 1, 2, 4)$ is strictly superadditive, but $\bar{v}$ is not weakly subadditive.

Let $M = -\max_{T \subset N} |v'_k(T)|$, and for the coalitions on which w is arbitrarily defined ($T \cap (S \cup \{k\}) = \emptyset$, $T \neq \emptyset$, see Proposition 3.6): let

$$w(T) = 2M|N|^{|T|} \ . \tag{9}$$

A game v is a (strictly) weakly subadditive if and only if $v'_k(Z) \leq v'_k(\emptyset)$ ($v'_k(Z) < v'_k(\emptyset)$), $k \notin Z \subseteq N$, $Z \neq \emptyset$ therefore, by equations (3) and (9) we have that w is also a (strictly) weakly subadditive game.

For the other properties see Proposition 3.6.

Finally, we can apply Theorem 3.4. □

Notice that not all the classes of games (defined in the Preliminaries) are M-closed, the classes of essential, (strictly) superadditive, (strictly) subadditive games are not M-closed. The next example shows this fact.

Example 3.10. (1) Let $v = (0, 0, 10, 50, 0, 0, 20)$, where $S = \{1, 2\}$ is an equivalence set in v. Game v is essential. However, the only game w such that N is an equivalence set in w, and $w'_3 = v'_3$ is $w = (10, 10, 10, 10, 10, 10, -20)$, but w is not essential.

(2) Let $v = (0, 0, 0, 10, 51, 51, 51, 51, 51, 51, 62, 62, 62, 62, 103)$, where $S = \{1, 2,\ 3\}$ is an equivalence set in v. Game v is strictly superadditive. However, the only game w such that N is an equivalence set in w, and $w'_4 = v'_4$, is $w = (10, 10, 10, 10, 61, 61, 61, 61, 61, 61, 72, 72, 72, 72, 113)$, but w is not superadditive. For the subadditive case take $-v$.

Remark 3.11. If $|N| \leq 3$, then the classes of (strictly) superadditive, (strictly) subadditive games coincide with the classes of (strictly) weakly superadditive, (strictly) weakly subadditive games respectively, hence they are M-closed. Furthermore, if $|N| = 2$, then the class of essential games coincides with the class of strictly superadditive games, hence it is M-closed.

Although the above mentioned classes of games are not M-closed, so Theorem 3.4 cannot be applied to them, Young's axiomatization is valid on these classes too, see Young (1985) (p. 71) and the reasoning in Csóka and Pintér (2014).

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