Entanglement of Three-Qubit Greenberger-Horne-Zeilinger–Symmetric States

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The first characterization of mixed-state entanglement was achieved for two-qubit states in Werner's seminal work [Phys. Rev. A **40**, 4277 (1989)]. A physically important extension concerns mixtures of a pure entangled state [such as the Greenberger-Horne-Zeilinger (GHZ) state] and the unpolarized state. These mixed states serve as benchmark for the robustness of multipartite entanglement. They share the symmetries of the GHZ state. We call such states GHZ symmetric. Here we give a complete description of the entanglement in the family of three-qubit GHZ-symmetric states and, in particular, of the three-qubit generalized Werner states. Our method relies on the appropriate parametrization of the states and on the invariance of entanglement properties under general local operations. An application is the definition of a symmetrization witness for the entanglement class of arbitrary three-qubit states.

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Introduction.—Entanglement is the essential resource for many tasks in quantum information processing [1,2]. Therefore, it is desirable to precisely characterize the entanglement contained in a quantum state. While our understanding of pure-state entanglement has significantly improved in recent years, entanglement in mixed states has remained a notoriously difficult subject, despite numerous impressive results, e.g., [3–11]. Because of the tremendous experimental progress in producing and controlling multiqubit entanglement (e.g., Refs. [12–16]) this has become also a practical problem, as an accurate assessment of the experimental results is required. The universal tools here are entanglement witnesses [7,17–19]. Despite its flexibility and success in detecting entanglement and distinguishing entanglement classes, characterization by means of witnesses is not always satisfactory. Enhancing the quality of entanglement witnesses requires improvement in the underlying entanglement theory.

An important and in general unsolved question of both practical and theoretical interest is how much noise admixture pure-state entanglement can sustain. Mathematically, this question can be cast as follows. We consider a pure state $|\psi_{\mathrm{ME}}\rangle$ of N qubits that contains a maximum amount of a certain entanglement type. This state gets mixed with the operator $\frac{1}{2^N}\mathbb{1}_{2^N}$, which describes the maximally mixed state of N qubits, serving as a model of unpolarized noise:

$$\rho_{\mathrm{WS}}(p) = p|\psi_{\mathrm{ME}}\rangle\langle\psi_{\mathrm{ME}}| + (1-p)\frac{1}{2^N}\mathbb{1}_{2^N}. \qquad (1)$$

The question then is how small p ($0 \leq p \leq 1$) can be chosen such that $\rho_{\mathrm{WS}}(p)$ still contains a finite amount of the considered entanglement.

For two qubits ($N = 2$), one substitutes $|\psi_{\mathrm{ME}}\rangle$ with the Bell state $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$. Then, $\rho_{\mathrm{WS}}(p)$ represents the so-called Werner states [20]. Although Werner defined them through the symmetry under local unitaries $U \otimes U$, the generalizations of Eq. (1) to three and more qubits are often termed *generalized Werner states* [3,21]. Throughout this article, we shall consider three qubits ($N = 3$) and the maximally entangled GHZ state

$$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \equiv |\mathrm{GHZ}_+\rangle. \qquad (2)$$

We shall give a complete characterization of the entanglement in

$$\rho_{\mathrm{WS}}(p) = p|\mathrm{GHZ}\rangle\langle\mathrm{GHZ}| + \frac{1-p}{8}\mathbb{1}_8 \qquad (3)$$

and the entire family of states with the same symmetry, the GHZ symmetry (see below). After reviewing the known results we present a parametrization for this family that allows to deduce the entanglement type for any given element of the family. Finally we show that our findings can be used as a witness to detect the entanglement type of arbitrary three-qubit states.

For two qubits, Eq. (1) gives the standard Werner state after replacing $|\psi_{\mathrm{ME}}\rangle$ with the Bell state $|\Psi^-\rangle$. In the two-qubit case there is only one type of entanglement, and therefore the problem reduces to finding the maximal value of p such that the state is still not entangled. It can easily be found by computing the concurrence [22].

The three-qubit case, however, is more complex. A state can either be completely separable, biseparable or tripartite entangled. Moreover, there are two inequivalent classes of tripartite entanglement, the GHZ type and the W type [23].

In the space of density matrices, there is a hierarchy of entangled states [4]: the convex hull of the W-type states includes the true W states, the biseparable and the separable ones while the set of GHZ-type states contains all other classes.

The three-qubit generalized Werner states $\rho_{\mathrm{WS}}(p)$ are known to be fully separable if and only if $p \leq p_{\mathrm{sep}} = \frac{1}{5}$ [3] and biseparable if and only if $p \leq p_{\mathrm{bisep}} = \frac{3}{7}$ [9]. One aim of this article is to find the value p_W such that $\rho_{\mathrm{WS}}(p)$ is of W type for $p \leq p_W$ and of GHZ type for $p > p_W$. It turns out that it is advantageous to extend the problem to all mixed states which can be written as affine combinations of $|\mathrm{GHZ}_+\rangle$, $|\mathrm{GHZ}_-\rangle = \frac{1}{\sqrt{2}}(|000\rangle - |111\rangle)$, and the maximally mixed state.

Parametrization of GHZ-symmetric states.—We solve the problem by exploiting its symmetry. The GHZ state, and thus also $\rho_{\mathrm{WS}}(p)$, is invariant under the following transformations (and combinations there of): (i) qubit permutations, (ii) simultaneous three-qubit flips (i.e., application of $\sigma_x \otimes \sigma_x \otimes \sigma_x$), (iii) qubit rotations about the z axis of the form

$$U(\phi_1, \phi_2) = e^{i\phi_1\sigma_z} \otimes e^{i\phi_2\sigma_z} \otimes e^{-i(\phi_1+\phi_2)\sigma_z}. \tag{4}$$

Here, σ_x and σ_z are Pauli operators. We refer to the invariance under the operations (i)–(iii) as *GHZ symmetry.* Except the qubit permutations all those operations are local; therefore (and since qubit permutations always convert GHZ states into GHZ states) GHZ symmetry operations will never turn GHZ-type entanglement into W-type entanglement or vice versa.

An important aspect of this symmetry is that for any decomposition of $\rho_{\mathrm{WS}}(p)$ into pure states there is a GHZ-symmetric decomposition of the same entanglement type. It is generated by replacing each pure state in the decomposition with the equal mixture of all states obtained from that former state by applying the symmetry operations.

In order to identify the set of GHZ-symmetric density matrices we check the action of the symmetry operations on its elements ρ^{S}. First, consider the z rotations (iii). The matrix element $\rho^{\mathrm{S}}_{ijk,lmn}$ is transformed by operations according to Eq. (4) into $\exp[i(i-k-l+n)\phi_1]\exp[i(j-k-m+n)\phi_2]\rho^{\mathrm{S}}_{ijk,lmn}$. Since ϕ_1 and ϕ_2 can take arbitrary values the state remains unchanged only if either the matrix element is zero, or if both $i-k-l+n=0$ and $j-k-m+n=0$. Therefore the only nonzero matrix elements are the diagonal elements, $\rho^{\mathrm{S}}_{000,111}$ and $\rho^{\mathrm{S}}_{111,000}$. Among these elements, permutation invariance forces the diagonal elements to depend only on the number of 1s in the index. Finally, the invariance under collective bit flips implies $\rho^{\mathrm{S}}_{000,000} = \rho^{\mathrm{S}}_{111,111}$ and $\rho^{\mathrm{S}}_{001,001} = \rho^{\mathrm{S}}_{110,110}$. Moreover, we have $\rho^{\mathrm{S}}_{000,111} = \rho^{\mathrm{S}}_{111,000}$ and thus real off-diagonal matrix elements due to Hermiticity. Given the additional constraint $\mathrm{tr}\rho^{\mathrm{S}} = 1$ we find that a state ρ^{S} is fully specified by two independent real parameters. A possible choice is

$$x(\rho^{\mathrm{S}}) = \tfrac{1}{2}[\langle \mathrm{GHZ}_+|\rho^{\mathrm{S}}|\mathrm{GHZ}_+\rangle - \langle \mathrm{GHZ}_-|\rho^{\mathrm{S}}|\mathrm{GHZ}_-\rangle], \tag{5}$$

$$y(\rho^{\mathrm{S}}) = \frac{1}{\sqrt{3}}\left[\langle \mathrm{GHZ}_+|\rho^{\mathrm{S}}|\mathrm{GHZ}_+\rangle + \langle \mathrm{GHZ}_-|\rho^{\mathrm{S}}|\mathrm{GHZ}_-\rangle - \frac{1}{4}\right] \tag{6}$$

such that the Euclidean metric in the (x, y) plane coincides with the Hilbert-Schmidt metric $d(A, B)^2 \equiv \frac{1}{2}\mathrm{tr}(A-B)^\dagger(A-B)$ on the density matrices. The completely mixed state is located at the origin. The set of states ρ^{S} forms a triangle in the (x, y) plane (see Fig. 1). The generalized Werner states (3) are found on the straight line $y = \frac{\sqrt{3}}{2}x$ connecting the origin with the GHZ state. We call it the "Werner line."

For any normalized pure state $|\psi\rangle = (\psi_{000}, \ldots, \psi_{111})$, there exists a corresponding symmetrized state

$$\rho^{\mathrm{S}}(\psi) = \int dU U|\psi\rangle\langle\psi|U^\dagger, \tag{7}$$

where the integral is understood to cover the entire GHZ symmetry group, i.e., unitaries $U(\phi_1, \phi_2)$ as in Eq. (4) and averaging over the discrete symmetries. The coordinates of

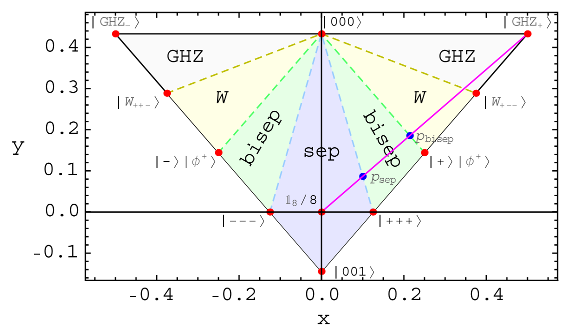

FIG. 1 (color online). The convex set of GHZ-symmetric density matrices ρ^{S}. The upper corners of the triangle are the standard GHZ state $|\mathrm{GHZ}_+\rangle$, and $|\mathrm{GHZ}_-\rangle$. Note that these are the only pure states. Applying σ_z to any one of the qubits changes the sign of x. Therefore for properties invariant under local unitaries, we have a mirror symmetry about the y axis. At the center of the upper horizontal line there is the separable state $\frac{1}{2}(|000\rangle\langle000| + |111\rangle\langle111|)$. The points for the pure states $|001\rangle$, $|+++\rangle$, $|+\rangle|\phi^+\rangle$, $|W_{+--}\rangle$ indicate the positions of the corresponding symmetrized mixed state (here we have used the definitions $|\pm\rangle \equiv \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$, $|\phi^+\rangle \equiv \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, and $|W_{+--}\rangle \equiv \frac{1}{\sqrt{3}}(|+--\rangle + |-+-\rangle + |--+\rangle)$). The solid magenta line (Werner line) represents the generalized Werner states $\rho_{\mathrm{WS}}(p)$ with $p_{\mathrm{sep}} = \frac{1}{5}$ and $p_{\mathrm{bisep}} = \frac{3}{7}$ (see text). The two lower dashed lines are first guesses for the boundaries of fully separable ("sep") and biseparable ("bisep") states from the known values of p_{sep} and p_{bisep}, respectively. The upper dashed line represents a first guess for the boundary between W and GHZ states as $|W_{+--}\rangle$ is the W state with the largest overlap to $|\mathrm{GHZ}_+\rangle$ (cf. Ref. [4]). The intersection with the Werner line occurs at $p = 9/13$.

the symmetrized state can be inferred from the coefficients ψ_{000} and ψ_{111}

$$x(\psi) = \tfrac{1}{2}(\psi_{000}^* \psi_{111} + \psi_{000} \psi_{111}^*), \tag{8}$$

$$y(\psi) = \frac{1}{\sqrt{3}}\left(|\psi_{000}|^2 + |\psi_{111}|^2 - \frac{1}{4}\right). \tag{9}$$

Entanglement properties of GHZ-symmetric states.— After finding and suitably parametrizing the set of GHZ-symmetric states we want to determine the entanglement class of each state (fully separable, biseparable, W, or GHZ). The key idea is that all states in an entanglement class are equivalent under stochastic local operations and classical communication (SLOCC) [23,24]. Mathematically, the corresponding (invertible) local operations are represented by the elements of the group $\mathrm{GL}(2,\mathbb{C})$. That is, applying $\mathrm{GL}(2,\mathbb{C})$ transformations to any qubit does not change the entanglement class of a multiqubit state.

The GHZ-symmetric states of each SLOCC class form a convex set. We characterize each set by finding its boundary starting from the separable states. Our strategy to identify these boundaries is the following. We fix the y coordinate in the interval $-1/(4\sqrt{3}) \le y < \sqrt{3}/4$ and then consider all *pure* states $|\psi\rangle$ of the SLOCC class under consideration whose symmetrized state $\rho^{\mathrm{S}}(\psi)$ has the chosen y according to Eq. (9). States at the boundary are the ones with maximum (or minimum for $x < 0$) x coordinate according to Eq. (8) for a given y, termed $x_{\max}$. Mirror symmetry implies $x_{\min} = -x_{\max}$; therefore, we may restrict our discussion to $x \ge 0$. If $x_{\max}(y)$ does not have the appropriate curvature the boundary is given by the convex hull of $x_{\max}(y)$.

We start with an obvious solution that holds for all SLOCC classes. If, for fixed y, the coefficients of the pure state $|\psi\rangle$ can be chosen equal $|\psi_{000}|^2 = |\psi_{111}|^2 = \frac{1}{2}(\sqrt{3}y + \frac{1}{4})$ the maximum x coordinate is given by $x_{\max} = |\psi_{000}||\psi_{111}| = \frac{1}{2}(\sqrt{3}y + \frac{1}{4})$, i.e., by the lower edge of the triangle of GHZ-symmetric states.

Now consider the separable pure states $|\psi^{\mathrm{sep}}\rangle$. They are equivalent (via local unitaries) to the state $|000\rangle$

$$|\psi^{\mathrm{sep}}\rangle = \left[\bigotimes_{j=1}^{3}\begin{pmatrix} A_j^* & -B_j \\ B_j^* & A_j \end{pmatrix}\right]|000\rangle, \tag{10}$$

where $|A_j|^2 + |B_j|^2 = 1$. For the moduli of the coefficients we find $|\psi_{000}^{\mathrm{sep}}| = |A_1 A_2 A_3|$ and $|\psi_{111}^{\mathrm{sep}}| = \sqrt{(1-|A_1|^2)(1-|A_2|^2)(1-|A_3|^2)}$. Maximizing $x = |\psi_{000}^{\mathrm{sep}}||\psi_{111}^{\mathrm{sep}}|$ subject to the constraint $|\psi_{000}^{\mathrm{sep}}|^2 + |\psi_{111}^{\mathrm{sep}}|^2 = \mathrm{const}$ leads to $x_{\max} = (\frac{1}{4} - \frac{1}{\sqrt{3}}y)^{3/2}$ for $y > 0$. As this function gives a concave boundary (cf. Fig. 2) we use the convex hull

$$x_{\max}^{\mathrm{sep}} = -\frac{\sqrt{3}}{6}y + \frac{1}{8}, \tag{11}$$

which is identical to the first guess from the known result $p_{\mathrm{sep}} = \frac{1}{5}$ (cf. Fig. 1).

For biseparable pure states $|\psi^{\mathrm{bisep}}\rangle$ it suffices (due to the subsequent symmetrization) to consider local equivalence to the state $|0\rangle \otimes |\phi^+\rangle$. That is, we obtain $|\psi^{\mathrm{bisep}}\rangle$ by normalizing the vector $(G_1 \otimes G_2 \otimes G_3)|0\rangle \otimes |\phi^+\rangle$. Here,

$$G_j = \begin{pmatrix} A_j & B_j \\ C_j & D_j \end{pmatrix}, \qquad j = 1, 2, 3$$

denotes an arbitrary $\mathrm{GL}(2,\mathbb{C})$ transformation. The discussion can be restricted to $G_2 = G_3$ because for any $|\psi^{\mathrm{bisep}}\rangle$ the two-qubit part can be made permutation symmetric by a diagonal $\mathrm{GL}(2,\mathbb{C})^{\otimes 2}$ operation without decreasing the coordinates $x(\psi^{\mathrm{bisep}})$, $y(\psi^{\mathrm{bisep}})$ of the corresponding symmetrized state. Maximizing x as before yields

$$x_{\max}^{\mathrm{bisep}} = -\frac{\sqrt{3}}{2}y + \frac{3}{8} \tag{12}$$

for $y > \frac{1}{4\sqrt{3}}$. Again this boundary coincides with the one inferred from $p^{\mathrm{bisep}} = \frac{3}{7}$ (see Fig. 1).

The general pure W state $|\psi^W\rangle$ is found by normalizing $(G_1 \otimes G_2 \otimes G_3)(|001\rangle + |010\rangle + |100\rangle)$. In analogy with the separable states, maximization of $x = |\psi_{000}^W||\psi_{111}^W|$ subject to the constraint $|\psi_{000}^W|^2 + |\psi_{111}^W|^2 = \mathrm{const}$ shows that the maximum is reached for $G_1 = G_2 = G_3$. It leads to polynomial equations whose solutions are given, for convenience, in parametrized form (with $0 \le v \le 1$)

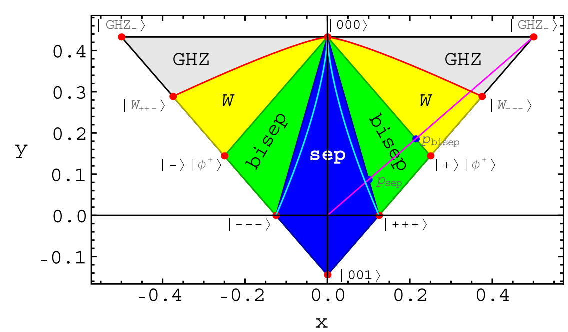

FIG. 2 (color online). The SLOCC classes of three-qubit GHZ-symmetric states ρ^{S}. The dark blue region shows the separable states ("sep") with the light blue lines $x = \pm(\frac{1}{4} - \frac{1}{\sqrt{3}}y)^{3/2}$. Green areas represent the biseparable states ("bisep"). The W states "W"(yellow) and the GHZ states "GHZ" (grey) are separated by the curve Eq. (13) (red line). The Werner line (magenta) crosses that curve at $p_W \approx 0.6955$. Some geometrical aspects are noteworthy. The curve (13) nearly (within a few per cent) describes a circle about the point $\rho^{\mathrm{S}}(001)$. The radius has a minimum in the vicinity of the Werner line. Further, it is intriguing to note that each SLOCC class shares exactly one fourth of the lower edge of the triangle.

$$x_{\max}^{W} = \frac{v^5 + 8v^3}{8(4 - v^2)}, \qquad y = \frac{\sqrt{3}}{4}\frac{4 - v^2 - v^4}{4 - v^2}, \qquad (13)$$

where $y \geq \frac{1}{2\sqrt{3}}$. The second derivative of $x_{\max}^{W}(y)$ shows that the boundary is indeed convex. This completes the characterization of SLOCC classes for GHZ-symmetric three-qubit states.

A particularly interesting point is the intersection of the curve (13) with the Werner line $y_{\mathrm{WS}} = \frac{\sqrt{3}}{2}x$. The corresponding parameter v_W solves the equation

$$1 = 4\frac{4 - v_W^2 - v_W^4}{v_W^3(v_W^2 + 8)}$$

such that $p_W = 0.695\,542\,7\ldots$.

Symmetrization witness.—Although these results might seem of purely theoretical interest they have a surprising application for arbitrary three-qubit states. Suppose ρ is such a state. The twirling operation in Eq. (7) generates the corresponding symmetrized state $\rho^{\mathrm{S}}(\rho)$. The SLOCC class of ρ cannot be lower in the hierarchy described in the introduction than that of $\rho^{\mathrm{S}}(\rho)$. For example, a W state can be projected by the twirling operation Eq. (7) onto a W state, a biseparable state or a fully separable state, but not onto a GHZ state. Hence, the GHZ-symmetrized state $\rho^{\mathrm{S}}(\rho)$ can be used to witness the SLOCC class of the original state ρ, simply by reading off the coordinates of $\rho^{\mathrm{S}}(\rho)$ in Fig. 2. These coordinates $x(\rho)$ and $y(\rho)$ are obtained from the matrix elements of ρ:

$$x(\rho) = \tfrac{1}{2}(\rho_{000,111} + \rho_{111,000}),$$
$$y(\rho) = \frac{1}{\sqrt{3}}\left(\rho_{000,000} + \rho_{111,111} - \frac{1}{4}\right).$$

We will discuss the optimization of this method elsewhere.

Summarizing, we have determined exactly the entanglement properties of an entire family of high-rank (mostly eight) mixed three-qubit states with the same symmetry as the GHZ state. In particular, we have solved the problem for the three-qubit generalized Werner state which is a reference for multiqubit mixed-state entanglement. A practically relevant application of this result is a simple method for detecting the SLOCC class of arbitrary three-qubit states.

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