Pub. Mat. UAB
Vol. 30 nº 2-3 Des. 1986

A GEOMETRICAL CHARACTERIZATION OF REFLEXIVITY IN BANACH SPACES

M.J.Chasco and Esteban Indurain

Summary : The main result in this paper is the equivalence, for any Banach space B , between
(i) "Every normalized basic sequence $(a_n)_{n\varepsilon N}$ in B is weakly null" ,
and
(ii) "For every normalized basic sequence $(a_n)_{n\varepsilon N}$ in B ,
$a_1 \varepsilon \overline{\text{span}}\ (a_n-a_{n+1})_{n\varepsilon N}$ " .
Pelczyński proved that (i) characterizes the fact of B being reflexive. So, the same holds for (ii) and we have a "geometrical" characterization of reflexivity.
We finish quoting some equivalent version of the above result.

AMS Subject Class. 1980 : 46 b 10 , 46 b 15
Key words : Reflexive Banach spaces, basic sequences, sequence of differences.

1. Previous Concepts .

-Let B denote a Banach space and K its scalar field, N the set of natural numbers, $[\dots]$ "closed linear span", and $\mathcal{S} = (a_n)_{n\in N}$ be a linearly independent sequence of vectors in B.

Call $K(\mathcal{S}) = \bigcap_{n\in N} [a_n , a_{n+1} , \dots]$ (kernel of $\mathcal{S}$) and

$K_s(\mathcal{S}) = \{K(\mathcal{S}') ; \mathcal{S}'$ is a subsequence (infinite) of $\mathcal{S}\}$ (strict kernel of $\mathcal{S}$)

$\mathcal{S}$ is normalized if $\| a_n \| = 1$ $(n \in N)$

$\mathcal{S}$ is basic if there is a unique sequence of scalars $(\lambda_n)_{n\in N}$ such that

$$x = \sum_1^{\infty} \lambda_n a_n \quad , \text{ for every } x \in [\mathcal{S}] .$$

The sequence $(a_n - a_{n+1})_{n\in N}$ is called sequence of differences of $\mathcal{S}$.

$\mathcal{S}$ is said to be weakly convergent to $x \in B$ if $\lim_n f(a_n) = f(x)$, for every $f \in B^*$ (dual of B) .

$\mathcal{S}$ is said to be minimal if there exists a sequence $(a_n^*)_{n\in N}$ in $[\mathcal{S}]^*$ with $a_n^* (a_m) = \delta_{nm}$ (Kronecker indices) , and uniformly minimal if it also verifies $\sup_n \| a_n \| \cdot \| a_n^* \| < \infty$.

2. The main result .

The result leans on the following two lemmas :

Lemma 1 : Every subsequence $\mathcal{S}'$ of a given sequence $\mathcal{S} = (a_n)_{n\in N}$ has zero strict kernel if and only if the normalized sequence $\mathcal{S}_N = (a_n/\|a_n\|)_n$ has no subsequence weakly convergent to some vector distinct from zero.

Proo. : See |T| , p. 172 .

110

Lemma 2 : Let $\mathcal{S} = (a_n)_{n\in N}$ be a minimal sequence with zero kernel. Let $x \in [\mathcal{S}]$ such that the set $S_x = \{k\in N \ ; \ a_k^*(x) \neq 0\}$ is infinite. We note $S_x = (p_n)_{n\in N}$. Then

$x \in K_s((\sum_{h=1}^{n} a_{p_h}^*(x)\ a_{p_h})_{n\in N})$ if and only if the sequence

$(\sum_{h=1}^{n} a_{p_h}^*(x)\ a_{p_h})_{n\in N}$ is weakly convergent to x .

Proof : (See |I-T|) . It follows from lemma 1 and the third Fréchet's axiom of convergence (see |K|) .

Now, we finally have the

Theorem : Let B be a Banach space. Then the following statements are equivalent :

(i) B is reflexive ,

(ii) Every normalized basic sequence $(a_n)_{n\in N}$ in B is weakly convergent to zero ,

(iii) Every normalized basic sequence $(a_n)_{n\in N}$ in B verifies

$$a_1 \in [a_n - a_{n+1} \ ; \ n \in N] \ .$$

Proof : In |P| has been proved that (i) is equivalent to (ii) .

-(ii) implies (iii) is obvious , considering

$$a_1 - a_n = \sum_{i=1}^{n-1} (a_i - a_{i+1})$$

(iii) implies (ii) :

-Suppose that for every normalized basic sequence $\mathcal{S} = (a_n)_{n\in N}$, $a_1 \in [a_n - a_{n+1} \ ; \ n \in N]$.

Notice that $a_1 \in [a_n - a_{n+1} ; n \in N]$ if and only if $a_1 \in K((a_1 - a_n)_n)$ (see, for instance, |R|, proposition 2.2)

Take $(p_n)_{n\in N}$ a subsequence of N, with $p_1 = 1$. By hypothesis, the sequence $(a_{p_n})_{n\in N}$ also verifies $a_1 \in [a_{p_n} - a_{p_{n+1}} ; n \in N]$, so, it follows that $a_1 \in K_s((a_1 - a_n)_{n\in N})$.

Now, applying lemma 2 to a_1 and $(a_n - a_{n+1})_{n\in N}$, we have that $(a_1 - a_n)_{n\in N}$ is weakly convergent to a_1, and therefore $(a_n)_{n\in N}$ is weakly convergent to zero. ||

3. Equivalent versions.

-In |CH-I| (preprint of this paper) the following equivalent versions of the theorem are given :

(iv) $[a_n ; n \in N] = [a_n - a_{n+1} ; n \in N]$, for every normalized basic sequence $(a_n)_{n\in N}$ in B,

(v) Let $(a_n)_{n\in N}$ be a normalized basic sequence in B. Then, its sequence of differences cannot be uniformly minimal,

(vi) For every normalized basic sequence $(a_n)_{n\in N}$ in B, $[a_n - a_{n+1} ; n \in N]$ cannot be an hyperplane in $[a_n ; n \in N]$.

Acknowledgement. We thank the referee for his valuable suggestions.

4. References .

|CH-I| CHASCO, M.J. - INDURAIN, E. : *Caracterizaciones geométricas de la reflexividad en espacios de Banach. (Preprint) . Pub. S. Mat. García de Galdeano. Serie II. Sección 1 nº 102 . Zaragoza 1986*

|I-T| INDURAIN, E. - TERENZI, P. : *A characterization of basic sequences in Banach spaces . Rend. Acc. Naz. dei XL 104º vol X 1986 (to appear)*

|K| KURATOWSKI, K. : *Topologie, vol I. PWN Warsaw 1952*

|P| PEŁCZYŃSKI, A. : *A note on the paper of I. Singer "Basic sequences and reflexivity of Banach spaces". Studia Math. 21, 371-374 (1962) .*

|R| REYES, A. : *A geometrical characterization of Schauder basis. Arch. Math. 39 , 176-179 (1982).*

|T| TERENZI, P. : *Biorthogonal systems in Banach spaces. Riv. Mat. Univ. Parma 4(4) 165-204 (1978) .*

Rebut el 18 d'Agost de 1986

Departamento de Geometría y Topología
Facultad de Ciencias
50009-ZARAGOZA
ESPAÑA