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TRIPLES $\mathbb{H}$-AZUMAYA. GRUPO DE BRAUER

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El grupo de Brauer para álgebras que son simultáneamente H-módulos y H-comódulos, siendo H un álgebra de Hopf conmutativa y coconmutativa en la categoría de R-módulos (R anillo conmutativo), fue estudiado por F.W. Long en |6|.

En este trabajo, para un álgebra de Hopf $\mathbb{H}$ en una categoría cerrada simétrica $\mathcal{C}$, se estudia el grupo de Brauer de los $\mathbb{H}$-dimódulos triple que son $\mathbb{H}$-Azumaya. Este grupo generaliza el de Long y, si se consideran $\mathbb{H}$-dimódulos triple con coestructura (respectivamente estructura) trivial, se obtiene el grupo de Brauer de $\mathbb{H}$-módulos (respectivamente $\mathbb{H}$-comódulos) triple de Azumaya definidos por J.M. Fernández Vilaboa en |1|. El grupo de Brauer de la categoría cerrada $\mathcal{C}$ se tiene cuando la estructura y la coestructura son las triviales.

Teniendo en cuenta las técnicas utilizadas y el contexto en que se desarrollan queda de manifiesto el carácter esencialmente no-aditivo de esta teoría haciendo innecesaria la utilización de argumentos de dimensión y localización usados por Long en su estudio.

Sea $\mathcal{C}$ una categoría cerrada simétrica con igualadores, coiguladores, objeto base K e isomorfismo natural de conmutatividad τ . Denotaremos con $\otimes$ el "producto tensor" en $\mathcal{C}$ y para cada objeto M de C (profinito en C (|8| (1.6.3))) α_M y β_M ($\bar{a}_M$ y $\bar{b}_M$) son la unidad y la counidad, respectivamente, de la C-adjunción $M \otimes - \dashv [M,-] : C \longrightarrow C$ ($M\otimes - \dashv \hat{M}\otimes - : C \longrightarrow C$ con $\hat{M} = [M,K]$ |8| (2.3.9)).

Con $\mathbb{H} = (\mathbb{C} = (C, \varepsilon_C, \delta_C)$, $\mathbb{T} = (C, \eta_C, \mu_C)$, $\tau^C, \lambda)$ se denotará un álgebra de Hopf en C conmutativa y coconmutativa respecto al cotriple $\mathbb{C}$ (|8| (1.1.9), (1.1.11), (1.1.12)).

1. $\mathbb{H}$-DIMODULOS TRIPLE. PROPIEDADES

(1.1) __Definición__. Una terna (M, ϕ_M, ρ_M) se dirá que es un $\mathbb{H}$-dimódulo si:

i) (M, ϕ_M) es un $\mathbb{H}$-módulo por la izquierda (|7| (3.2.1))
ii) (M, ρ_M) es un $\mathbb{H}$-comódulo por la derecha (|7| (3.2.2))
iii) $\rho_M \circ \phi_M = (\phi_M \otimes C) \circ (C \otimes \rho_M)$

(1.2) __Definición__. La terna $(\mathbb{A}, \phi_A, \rho_A) = ((A, \eta_A, \mu_A); \phi_A; \rho_A)$ es un $\mathbb{H}$-dimódulo triple por la izquierda si,

i) $\mathbb{A} = (A, \eta_A, \mu_A)$ es un triple en C
ii) $(\mathbb{A}, \phi_A)$ es un $\mathbb{H}$-módulo triple por la izquierda (|1| (2.1.1))
iii) $(\mathbb{A}, \rho_A)$ es un $\mathbb{H}$-comódulo triple por la derecha (|1|(2.2.1))
iv) $\rho_A \circ \phi_A = (\phi_A \otimes C) \circ (C \otimes \rho_A)$.

Un morfismo de $\mathbb{H}$-dimódulos triple será un morfismo de $\mathbb{H}$-dimódulos (|2| (1.2)) que lo es también de triples.

(1.3) __Proposición__. Si $(\mathbb{A}, \phi_A, \rho_A)$ y $(\mathbb{B}, \phi_B, \rho_B)$ son $\mathbb{H}$-dimódulos triple entonces:

i) $(\mathbb{A}\,\mathbb{B}, \phi_{A\otimes B}, \rho_{A\otimes B}) := ((A\otimes B, \eta_A \otimes \eta_B, (\mu_A \otimes \mu_B) \circ (A \otimes \tau^B_A \otimes B);$
$\phi_{A\otimes B} = (\phi_A \otimes \phi_B) \circ (C \otimes \tau^C_A \otimes B) \circ (\delta_C \otimes A \otimes B)$; $\rho_{A\otimes B} = (A \otimes B \otimes \mu_C) \circ$

$\circ (A \otimes \tau_B^C \otimes C) \circ (\rho_A \otimes \rho_B))$ es $\mathbb{H}$-dimódulo triple.

ii) $(\mathbb{A}^\circ, \phi_A, \rho_A) := ((A, \eta_A, \mu_A \circ \tau_A^A) ; \phi_A ; \rho_A)$ y

$(\bar{\mathbb{A}}, \phi_A, \rho_A) := ((A, \eta_A, \mu_{\bar{A}} := \mu_A \circ \tau_A^A \circ (A \otimes \phi_A) \circ (\rho_A \otimes A)); \phi_A ; \rho_A)$

son $\mathbb{H}$-dimódulos triple.

iii) Definiendo el producto smash de $\mathbb{A}$ y $\mathbb{B}$ como

$\mathbb{A} \# \mathbb{B} = (A \otimes B, \eta_{A\#B}, \mu_{A\#B})$ con $\eta_{A\#B} = \eta_{A\otimes B} = \eta_A \otimes \eta_B$ y

$\mu_{A\#B} = (\mu_A \otimes \mu_B) \circ (A \otimes \tau_A^B \otimes B) \circ (A \otimes B \otimes \phi_A \otimes B) \circ (A \otimes \rho_B \otimes A \otimes B)$

entonces, $(\mathbb{A} \# \mathbb{B}, \phi_{A\otimes B}, \rho_{A\otimes B})$ es un $\mathbb{H}$-dimódulo triple. Además, el producto smash, así como el producto tensor, de $\mathbb{H}$-dimódulos triple es asociativo.

<u>Demostración</u>.

i) (|2| (1.5)i))

ii) $\phi_A \circ (C \otimes \mu_{\bar{A}}) = \mu_A \circ (\phi_A \otimes \phi_A) \circ (C \otimes \tau_A^C \otimes A) \circ (\delta_C \otimes \tau_A^A) \circ (C \otimes A \otimes \phi_A) \circ$

$\circ (C \otimes \rho_A \otimes A) = \mu_A \circ (\phi_A \otimes A) \circ (C \otimes \tau_A^A) \circ (C \otimes \phi_A \otimes \phi_A) \circ (\delta_C \otimes \rho_A \otimes A) = \mu_{\bar{A}} \circ \phi_{A \otimes A}$

($\mathbb{H}$ es conmutativa y coconmutativa).

Análogamente se prueban las restantes condiciones.

iii) De la conmutatividad y coconmutatividad de $\mathbb{H}$, junto con la naturalidad del isomorfismo de conmutatividad τ, se obtiene que $(\mathbb{A} \# \mathbb{B}, \phi_{A\otimes B})$ es un $\mathbb{H}$-módulo triple por la izquierda y $(\mathbb{A} \# \mathbb{B}, \rho_{A\otimes B})$ es un $\mathbb{H}$-comódulo triple por la derecha. La relación de compatibilidad entre ambas estructuras es la misma que la del $\mathbb{H}$-dimódulo triple $(\mathbb{A}\mathbb{B}, \phi_{A\otimes B}, \rho_{A\otimes B})$.

(1.4) **<u>Proposición</u>**. Si $(\mathbb{A}, \phi_A, \rho_A)$ y $(\mathbb{B}, \phi_B, \rho_B)$ son $\mathbb{H}$-dimódulos triple, entonces se tienen los siguientes isomorfismos de $\mathbb{H}$-dimódulos triple:

i) $(\mathbb{A},\phi_A,\rho_A) \simeq (\bar{\bar{\mathbb{A}}},\phi_A,\rho_A)$

ii) $(\mathbb{A}\ \mathbb{B},\phi_{A\otimes B},\rho_{A\otimes B}) \simeq (\mathbb{B}\ \mathbb{A},\phi_{B\otimes A},\rho_{B\otimes A})$

iii) $(\bar{\mathbb{B}}\#\bar{\mathbb{A}},\phi_{B\otimes A},\rho_{B\otimes A}) \simeq (\overline{\mathbb{A}\#\mathbb{B}},\phi_{B\otimes A},\rho_{A\otimes B})$

<u>Demostración</u>.

i) $q_A := \phi_A \circ \tau_C^A \circ \rho_A : A \longrightarrow A$ es un isomorfismo de $\mathbb{H}$-dimódulos triple cuyo inverso es: $\phi_A \circ \tau_C^A \circ (A\otimes\lambda)\circ\rho_A$.

$(\phi_A \circ \tau_C^A \circ (A\otimes\lambda)\circ\rho_A)\circ q_A = \phi_A \circ \tau_C^A \circ (\phi_A\otimes\lambda)\circ(C\otimes\rho_A)\circ\tau_C^A\circ\rho_A =$

$= \phi_A \circ \tau_C^A \circ (A\otimes\mu_C)\circ(A\otimes\lambda\otimes C)\circ(A\otimes\delta_C)\circ\rho_A = \phi_A \circ \tau_C^A \circ$

$\circ (A\otimes(\eta_C\circ\varepsilon_C))\circ\rho_A = 1_A$

ii) (|2| (1.5)i))

iii) $S_{B,A} := \tau_A^B \circ (B\otimes\phi_A)\circ(\rho_B\otimes A) : B\otimes A \longrightarrow A\otimes B$ es un isomorfismo de $\mathbb{H}$-dimódulos triple cuyo inverso es:

$(B\otimes\phi_A)\circ(B\otimes\lambda\otimes A)\circ(\rho_B\otimes A)\circ\tau_B^A$ (prueba análoga a i)).

(1.5) <u>Proposición</u>. Si (M,ϕ_M,ρ_M) y (N,ϕ_N,ρ_N) son $\mathbb{H}$-dimódulos y además M es profinito en C, entonces

i) $([M,N],\ \phi_{[M,N]} := [M,\phi_N \circ (C\otimes\beta_M(N))\circ(C\otimes\phi_M\otimes[M,N])\circ$
$\circ(C\otimes\lambda\otimes[M,N])\circ(\delta_C\otimes M\otimes[M,N])\circ(\tau_C^M\otimes[M,N])]\circ$
$\circ(\alpha_M(C\otimes[M,N]))$, $\rho_{[M,N]} := (\nabla_{MKN}\otimes\mu_C)\circ(\hat{M}\otimes\rho_N\otimes C)\circ$
$\circ(\hat{M}\otimes\beta_M(N)\otimes\lambda)\circ(\hat{M}\otimes M\otimes\tau_{[M,N]}^C)\circ(\hat{M}\otimes\rho_M\otimes[M,N])\circ(\bar{a}_M\otimes[M,N])$
es un $\mathbb{H}$-dimódulo (|2| (1.3)ii)).

ii) $(\mathbb{E}(M),\phi_{E(M)},\rho_{E(M)}) := ((E(M) = [M,M],\ \eta_{E(M)} = \alpha_M^*,$
$\mu_{E(M)} = d_{MMM})\ ;\ \phi_{E(M)} := \phi_{[M,M]};\ \rho_{E(M)} := \rho_{[M,M]})$ es un $\mathbb{H}$-dimódulo triple. Además si N es también profinito en C, entonces los $\mathbb{H}$-dimódulos triple $\mathbb{E}(M\otimes N)$ y $\mathbb{E}(M)\ \mathbb{E}(N)$ son isomorfos (|2| (1.5)ii)).

(1.6) __Lema__. Si (M,ϕ_M,ρ_M) es un $\mathbb{H}$-dimódulo con M profinito en $\mathcal{C}$, entonces el morfismo $g_M := [M,\phi_M \circ \tau_C^M] \circ \alpha_M(C)$ verifica:

i) $g_M:(\mathbb{T}, C\otimes\eta_C) \longrightarrow (\mathbb{E}(M),\rho_{E(M)})$ es un morfismo de $\mathbb{H}$-comódulos triple.

ii) $\phi_{E(M)} = \mu_{E(M)} \circ (\mu_{E(M)} \otimes E(M)) \circ (E(M) \otimes \tau_{E(M)}^{E(M)}) \circ (g_M \otimes g_M \otimes E(M)) \circ$
$\circ (\tau_C^C \otimes E(M)) \circ (C\otimes\lambda\otimes E(M)) \circ (\delta_C \otimes E(M))$

iii) $(E(M), \phi_{E(M)} \circ (C \otimes \mu_{E(M)}) \circ (C \otimes \tau_{E(M)}^{E(M)}) \circ (C \otimes E(M) \otimes g_M) \circ$
$\circ (C \otimes \tau_{E(M)}^{C}))$ es un $\mathbb{H}$ $\mathbb{H}$-módulo por la izquierda (|7| (3.1.16)).

__Demostración__.

i) $\mu_{E(M)} \circ (g_M \otimes g_M) = [M, \beta_M(M) \circ (\beta_M(M) \otimes E(M)) \circ (M \otimes g_M \otimes g_M)] \circ$
$\circ (\alpha_M(C\otimes C)) = [M, \phi_M \circ \tau_C^M \circ (\phi_M \otimes C) \circ (\tau_C^M \otimes C)] \circ$
$\circ (\alpha_M(C\otimes C)) = g_M \circ \mu_C$ ($\mathbb{H}$ conmutativa).

$g_M \circ \eta_C = \alpha_M = \eta_{E(M)}$

$\rho_{E(M)} \circ g_M = (\nabla_{MKM} \otimes \mu_C) \circ (\hat{M} \otimes \rho_M \otimes \lambda) \circ (\hat{M} \otimes \phi_M \otimes C) \circ (\hat{M}\otimes\tau_{C\otimes M}^{C}) \circ$
$\circ (\hat{M} \otimes \tau_{C\otimes C}^{M}) \circ (\hat{M}\otimes\rho_M\otimes C) \circ (\bar{a}_M\otimes C) = (\nabla_{MKM} \otimes \mu_C) \circ (\hat{M}\otimes\phi_M\otimes C\otimes C) \circ$
$\circ (\hat{M}\otimes C\otimes\rho_M\otimes\lambda) \circ (\hat{M} \otimes \tau_C^{M\otimes C}) \circ (\hat{M} \otimes\rho_M \otimes C) \circ (\bar{a}_M \otimes C) =$
$= (\nabla_{MKM} \otimes C) \circ (\hat{M} \otimes \phi_M \otimes C) \circ (\hat{M} \otimes \tau_C^{M\otimes C}) \circ (\hat{M}\otimes M\otimes\mu_C \otimes C) \circ$
$\circ (\hat{M}\otimes M\otimes C\otimes\lambda\otimes C) \circ (\hat{M} \otimes M \otimes \delta_C \otimes C) \circ (\hat{M} \otimes\rho_M \otimes C) \circ (\bar{a}_M \otimes C) =$
$= (\nabla_{MKM} \otimes C) \circ (\hat{M} \otimes\phi_M \otimes C) \circ (\hat{M} \otimes\tau_C^{M\otimes C}) \circ (\hat{M} \otimes M \otimes (\eta_C \circ \varepsilon_C) \otimes C) \circ$
$\circ (\hat{M}\otimes\rho_M\otimes C) \circ (\bar{a}_M \otimes C) = g_M \otimes \eta_C; (\mu_C \circ (C \otimes \lambda) \circ \delta_C = \eta_C \circ \varepsilon_C)$

ii) $\mu_{E(M)} \circ (\mu_{E(M)} \circ E(M)) \circ (E(M) \otimes \tau_{E(M)}^{E(M)}) \circ (g_M \otimes g_M \otimes E(M)) \circ$
$\circ (\tau_C^C \otimes E(M)) \circ (C \otimes \lambda \otimes E(M)) \circ (\delta_C \otimes E(M)) = [M, \beta_M(M) \circ (M \otimes g_M) \circ$
$\circ (\beta_M(M) \otimes C) \circ (\beta_M(M) \otimes E(M) \otimes C) \circ (M \otimes g_M \otimes E(M) \otimes C) \circ (M\otimes C\otimes\tau_{E(M)}^{C}) \circ$
$\circ (M \otimes \tau_C^C \otimes E(M)) \circ (M \otimes C \otimes\lambda \otimes E(M)) \circ (M \otimes \delta_C \otimes E(M))] \circ$
$\circ (\alpha_M(C \otimes E(M))) = \phi_{E(M)}$

iii) $(E(M),\phi_{E(M)})$ es un $\mathbb{H}$-módulo por la izquierda. Además, $(E(M);\mu_{E(M)}\circ\tau^{E(M)}_{E(M)}\circ(E(M)\otimes g_M))$ es un $\mathbb{H}$-módulo por la derecha ($g_M:\mathbb{T}\longrightarrow\mathbb{E}(M)^{\circ}$ es un morfismo de triples al ser $\mathbb{H}$ conmutativa) y $\mu_{E(M)}\circ\tau^{E(M)}_{E(M)}\circ(\phi_{E(M)}\otimes g_M)=$

$=\mu_{E(M)}\circ\tau^{E(M)}_{E(M)}\circ(\mu_{E(M)}\otimes g_M)\circ(\mu_{E(M)}\otimes E(M)\otimes C)\circ$

$\circ(E(M)\otimes\tau^{E(M)}_{E(M)}\otimes C)\circ(g_M\otimes g_M\otimes E(M)\otimes C)\circ(\lambda\otimes C\otimes E(M)\otimes C)\circ$

$\circ(\delta_C\otimes E(M)\otimes C)=\mu_{E(M)}\circ(g_M\otimes E(M))\circ(\mu_C\otimes\mu_{E(M)})\circ$

$\circ(C\otimes\tau^{E(M)\otimes E(M)}_{C})\circ(C\otimes\tau^{E(M)}_{E(M)}\otimes C)\circ(\lambda\otimes g_M\otimes E(M)\otimes C)\circ$

$\circ(\delta_C\otimes E(M)\otimes C)=\mu_{E(M)}\circ(\mu_{E(M)}\otimes E(M))\circ(E(M)\otimes\mu_{E(M)}\otimes E(M))\circ$

$\circ(E(M)\otimes\tau^{E(M)\otimes E(M)}_{E(M)})\circ(g_M\otimes g_M\otimes E(M)\otimes g_M)\circ(\lambda\otimes C\otimes E(M)\otimes C)\circ$

$\circ(\delta_C\otimes E(M)\otimes C)=\phi_{E(M)}\circ(C\otimes\mu_{E(M)})\circ(C\otimes\tau^{E(M)}_{E(M)})\circ(C\otimes E(M)\otimes g_M).$

(1.7) <u>Lema</u>. Si (M,ϕ_M,ρ_M) es un $\mathbb{H}$-dimódulo con M profinito en C, entonces el morfismo $g'_M:=(\nabla_{MKM}\otimes\lambda)\circ(\hat{M}\otimes\rho_M)\circ\nabla^{-1}_{MKM}$ verifica:

i) $\tau^{E(M)}_{C}\circ g'_M:(E(M),\rho_{E(M)})\longrightarrow(C\otimes E(M),\ C\otimes\rho_{E(M)})$ es un morfismo de $\mathbb{H}$-comódulos por la derecha.

ii) $(E(M),g'_M)$ es un $\mathbb{H}$-comódulo por la derecha.

iii) $(E(M),\ \phi_{E(M)},g'_M)$ es un $\mathbb{H}$-dimódulo.

<u>Demostración</u>.

i) $(C\otimes\rho_{E(M)})\circ\tau^{E(M)}_{C}\circ g'_M=(C\otimes\nabla_{MKM}\otimes\mu_C)\circ(C\otimes\hat{M}\otimes\rho_M\otimes C)\circ$

$\circ(C\otimes\hat{M}\otimes\beta_M(M)\otimes\lambda)\circ(C\otimes\hat{M}\otimes M\otimes\tau^{C}_{E(M)})\circ(\tau^{\hat{M}\otimes M\otimes C}_{C}\otimes E(M))\circ$

$\circ(\hat{M}\otimes\rho_M\otimes\tau^{E(M)}_{C})\circ(\hat{M}\otimes M\otimes g'_M)\circ(\bar{a}_M\otimes E(M))=(C\otimes\nabla_{MKM}\otimes\mu_C)\circ$

$\circ(C\otimes\hat{M}\otimes\rho_M\otimes C)\circ(\tau^{\hat{M}\otimes M}_{C}\otimes C)\circ(\hat{M}\otimes M\otimes\lambda\otimes\lambda)\circ(\hat{M}\otimes\rho_M\otimes C)\circ$

$\circ(\hat{M}\otimes\beta_M(M)\otimes C)\circ(\hat{M}\otimes M\otimes\tau^{C}_{E(M)})\circ(\hat{M}\otimes\rho_M\otimes E(M))\circ(\bar{a}_M\otimes E(M))=$

$=(\tau^{E(M)}_{C}\otimes C)\circ(\nabla_{MKM}\otimes\lambda\otimes C)\circ(\hat{M}\otimes\rho_M\otimes C)\circ(\hat{M}\otimes M\otimes\mu_C)\circ$

$\circ(\hat{M}\otimes\rho_M\otimes\lambda)\otimes(\hat{M}\otimes\beta_M(M)\otimes C)\circ(\hat{M}\otimes M\otimes\tau^{C}_{E(M)})\circ$

$\circ(\hat{M}\otimes\rho_M\otimes E(M))\circ(\bar{a}_M\otimes E(M)) = ((\tau^{E(M)}_{C}\circ g'_M)\otimes C)\circ\rho_{E(M)}$

ii) $(g'_M\otimes C)\circ g'_M = (\nabla_{MKM}\otimes\lambda\otimes\lambda)\circ(\hat{M}\otimes\rho_M\otimes C)\circ(\bar{M}\otimes\rho_M)\circ$

$\circ\nabla^{-1}_{MKM} = (\nabla_{MKM}\otimes\lambda\otimes\lambda)\circ(\bar{M}\otimes M\otimes\delta_C)\circ(\hat{M}\otimes\rho_M)\circ\nabla^{-1}_{MKM} =$

$=(E(M)\otimes\delta_C)\circ g'_M$

$(E(M)\otimes\varepsilon_C)\circ g'_M = 1_{E(M)}$

iii) Por (1.5) $(E(M),\phi_{E(M)})$ es $\mathbb{H}$-módulo por la izquierda y por ii) $(E(M), g'_M)$ es $\mathbb{H}$-comódulo por la derecha. Además,

$(\nabla^{-1}_{MKM}\otimes C)\circ g'_M\circ\phi_{E(M)} = (\hat{M}\otimes M\otimes\lambda)\circ(\hat{M}\otimes\rho_M)\circ(\bar{M}\otimes\beta_M(M))\circ$

$\circ(\bar{a}_M\otimes\phi_{E(M)}) = (\hat{M}\otimes\phi_M\otimes\lambda)\circ(\hat{M}\otimes C\otimes\rho_M)\circ(\bar{M}\otimes C\otimes\beta_M(M))\circ$

$\circ(\hat{M}\otimes\tau^{M}_{C}\otimes E(M))\circ(\bar{M}\otimes\phi_M\otimes C\otimes E(M))\circ(\hat{M}\otimes\tau^{M}_{C}\otimes C\otimes E(M))\circ$

$\circ(\hat{M}\otimes M\otimes\tau^{C}_{C}\otimes E(M))\circ(\hat{M}\otimes M\otimes C\otimes\lambda\otimes E(M))\circ(\hat{M}\otimes M\otimes\delta_C\otimes E(M))\circ$

$\circ(\bar{a}_M\otimes C\otimes E(M)) = (\bar{M}\otimes\phi_M\otimes C)\circ(\hat{M}\otimes C\otimes\beta_M(M)\otimes C)\circ$

$\circ(\hat{M}\otimes\tau^{M}_{C}\otimes E(M)\otimes C)\circ(\hat{M}\otimes\phi_M\otimes C\otimes E(M)\otimes C)\circ(\bar{M}\otimes\tau^{M}_{C}\otimes$

$\otimes C\otimes E(M)\otimes C)\circ(\bar{M}\otimes M\otimes\tau^{C}_{C}\otimes E(M)\otimes C)\circ(\bar{M}\otimes M\otimes C\otimes\lambda\otimes E(M)\otimes C)\circ$

$\circ(\bar{a}_M\otimes\delta_C\otimes E(M)\otimes C)\circ(C\otimes g'_M) = (\nabla^{-1}_{MKM}\otimes C)\circ(\phi_{E(M)}\otimes C)\circ(C\otimes g'_M)$

Por lo tanto, ya que $\nabla^{-1}_{MKM}\otimes C$ es un isomorfismo al ser M profinito en C, ambas estructuras son compatibles.

(1.8) **Proposición**. Si (M,ϕ_M,ρ_M) es un $\mathbb{H}$-dimódulo con M profinito en C y $(\mathbb{A},\phi_A,\rho_A)$ es un $\mathbb{H}$-dimódulo triple, se tienen los siguientes isomorfismos de $\mathbb{H}$-dimódulos triple:

i) $\mathbb{E}(M)^{\circ}\#\mathbb{A} \simeq \mathbb{E}(M)^{\circ}\mathbb{A}$

ii) $\mathbb{A}\#\mathbb{E}(M)^{\circ} \simeq \mathbb{A}\,\mathbb{E}(M)^{\circ}$

Demostración.

i)

$h_{M,A} := (\mu_{E(M)} \otimes A) \circ (\tau_{E(M)}^{E(M)} \otimes A) \circ (E(M) \otimes g_M \otimes A) \circ (E(M) \otimes \tau_C^A) \circ$
$\circ (E(M) \otimes \rho_A) : E(M) \otimes A \longrightarrow E(M) \otimes A$ en donde $g_M : C \longrightarrow E(M)$ es el morfismo definido en (1.6).

$h_{M,A}$ es un morfismo de $\mathbb{H}$-comódulos. En efecto:

$$\rho_{E(M)\otimes A} \circ h_{M,A} = (\mu_{E(M)} \otimes A \otimes \mu_C) \circ (\tau_{E(M)}^{E(M)} \otimes \tau_A^C \otimes C) \circ$$
$$\circ (E(M) \otimes E(M) \otimes \mu_C \otimes \rho_A) \circ (E(M) \otimes \tau_{E(M)}^C \otimes C \otimes A) \circ (\rho_{E(M)} \otimes \rho_{E(M)} \otimes A) \circ$$
$$\circ (E(M) \otimes g_M \otimes A) \circ (E(M) \otimes \tau_C^A) \circ (E(M) \otimes \rho_A) = (\mu_{E(M)} \otimes A \otimes \mu_C) \circ$$
$$\circ (\tau_{E(M)}^{E(M)} \otimes \tau_A^C \otimes C) \circ (E(M) \otimes \tau_{E(M)}^C \otimes A \otimes C) \circ (\rho_{E(M)} \otimes \tau_{E(M)}^A \otimes C) \circ$$
$$\circ (E(M) \otimes A \otimes g_M \otimes C) \circ (E(M) \otimes A \otimes \delta_C) \circ (E(M) \otimes \rho_A) = (h_{M,A} \otimes C) \circ \rho_{E(M)\otimes A}$$

(g_M morfismo de $\mathbb{H}$-comódulos (1.6)i), $\mathbb{H}$ conmutativa y coconmutativa).

$h_{M,A}$ es morfismo de $\mathbb{H}$-módulos:

$$\phi_{E(M)\otimes A} \circ (C \otimes h_{M,A}) = (\mu_{E(M)} \otimes A) \circ (\tau_{E(M)}^{E(M)} \otimes A) \circ (\phi_{E(M)} \otimes g_M \otimes \phi_A) \circ$$
$$\circ (C \otimes E(M) \otimes \tau_C^{C\otimes A}) \circ (C \otimes \tau_{E(M)}^C \otimes \rho_A) \circ (\delta_C \otimes E(M) \otimes A) =$$
$$= h_{MA} \circ \phi_{E(M)\otimes A} \quad ((1.6)\text{iii})).$$

Además, por la propiedad ii) de (1.6), por la naturalidad del isomorfismo de simetria τ y por ser $\mathbb{H}$ conmutativa y coconmutativa se obtiene que $h_{M,A}$ es un morfismo de triples, es decir,

$$h_{M,A} \circ \mu_{E(M)\circ\# A} = \mu_{E(M)\circ\otimes A} \circ (h_{M,A} \otimes h_{M,A}) \quad y \quad h_{M,A} \circ (\eta_{E(M)} \otimes \eta_A) =$$
$$= \eta_{E(M)} \otimes \eta_A \,.$$

El morfismo $(\mu_{E(M)} \otimes A) \circ (\tau^{E(M)}_{E(M)} \otimes A) \circ$
$\circ (E(M) \otimes g_M \otimes A) \circ (E(M) \otimes \tau^{A}_{C}) \circ (E(M) \otimes A \otimes \lambda) \circ (E(M) \otimes \rho_A)$ resulta inverso de $h_{M,A}$ como consecuencia de ser g_M un morfismo de triples (1.6) y por las propiedades del antípodo.

Para la demostración de ii) se utiliza el morfismo $h'_{A,M} := (\phi_A \otimes E(M)) \circ \tau^{A\otimes E(M)}_{C} \circ (A \otimes g'_M)$ en donde $g'_M : E(M) \longrightarrow E(M) \otimes C$ es el morfismo del lema (1.7). Con cálculos análogos a los anteriores y teniendo en cuenta las propiedades de (1.7), resulta que $h'_{A,M}$ es un isomorfismo de $\mathbb{H}$-dimódulos triple cuyo inverso está definido por:

$$(h'_{A,M})^{-1} := (\phi_A \otimes E(M)) \circ (\lambda \otimes A \otimes E(M)) \circ \tau^{A\otimes E(M)}_{C} \circ (A \otimes g'_M)$$

(1.9) **Corolario**. Si (M, ϕ_M, ρ_M) y (N, ϕ_N, ρ_N) son $\mathbb{H}$-dimódulos con M y N profinitos en $\mathcal{C}$ y $(\mathbb{A}, \phi_A, \rho_A)$ es un $\mathbb{H}$-dimódulo triple, entonces:

i) $\mathbb{E}(M)^\circ \# \mathbb{E}(N)^\circ$ y $\mathbb{E}(M\otimes N)^\circ$ son $\mathbb{H}$-dimódulos triple isomorfos.

ii) $\mathbb{A} \# \mathbb{E}(M)^\circ$ y $\mathbb{E}(M)^\circ \# \mathbb{A}$ son $\mathbb{H}$-dimódulos triple isomorfos.

Demostración.

i) $\mathbb{E}(M)^\circ \# \mathbb{E}(N)^\circ \simeq \mathbb{E}(M)^\circ \mathbb{E}(N)^\circ$ por (1.8)i).
$\mathbb{E}(M)^\circ \mathbb{E}(N)^\circ \simeq \mathbb{E}(M\otimes N)^\circ$ por (1.5)ii).

ii) Se sigue de la proposición anterior (1.8) y de (1.4)ii)

(1.10) **Proposición**. Si (M, ϕ_M, ρ_M) es un $\mathbb{H}$-dimódulo con M profinito en $\mathcal{C}$, entonces se tienen los siguientes isomorfismos de $\mathbb{H}$-dimódulos triple:

i) $\overline{\mathbb{E}(M)^\circ} = \mathbb{E}(M)$

ii) $\mathbb{E}(M) \simeq \mathbb{E}(\hat{M})^\circ$

Demostración.

i) Se define $j_M := [M, \beta_M(M) \circ (M \otimes \phi_{E(M)}) \circ (\rho_M \otimes E(M)] \circ$
$\circ (\alpha_M(E(M)) = \nabla_{MKM} \circ (\hat{M} \otimes (\beta_M(M) \circ (M \otimes \phi_{E(M)}) \circ (\rho_M \otimes E(M)))) \circ$
$\circ (\hat{a}_M \otimes E(M))$;$(j_M^{-1} = \nabla_{MKM} \circ (\hat{M} \otimes (\beta_M(M) \circ (M \otimes \phi_{E(M)}) \circ$
$\circ (M \otimes \lambda \otimes E(M)) \circ (\rho_M \otimes E(M)))) \circ (\bar{a}_M \otimes E(M))$

j_M es morfismo de triples:

$j_M \circ \mu_{\overline{E(M)^\circ}} = [M, \beta_M(M) \circ (M \otimes \phi_{E(M)}) \circ (\rho_M \otimes \mu_{E(M)}) \circ (M \otimes E(M) \otimes \phi_{E(M)}) \circ$
$\circ (M \otimes \rho_{E(M)} \otimes E(M)] \circ (\alpha_M(E(M) \otimes E(M))) = [M, \beta_M(M) \circ$
$\circ (M \otimes \mu_{E(M)}) \circ (M \otimes \phi_{E(M) \otimes E(M)}) \circ (\rho_M \otimes E(M) \otimes \phi_{E(M)}) \circ$
$\circ (M \otimes \rho_{E(M)} \otimes E(M))] \circ (\alpha_M(E(M) \otimes E(M))) = [M, \beta_M(M) \circ (M \otimes \mu_{E(M)}) \circ$
$\circ (M \otimes E(M) \otimes \phi_{E(M)}) \circ (\rho_{M \otimes E(M)} \otimes E(M)) \circ (M \otimes \phi_{E(M)} \otimes E(M)) \circ$
$\circ (\rho_M \otimes E(M) \otimes E(M))] \circ (\alpha_M(E(M) \otimes E(M))) = [M, \beta_M(M) \circ (M \otimes \phi_{E(M)}) \circ$
$\circ (\rho_M \otimes E(M)) \circ (\beta_M(M) \otimes E(M)) \circ (M \otimes \phi_{E(M)} \otimes E(M)) \circ (\rho_M \otimes E(M) \otimes E(M))] \circ$
$\circ (\alpha_M(E(M) \otimes E(M))) = \mu_{E(M)} \circ (j_M \otimes j_M)$;$((\mathbb{E}(M), \phi_{E(M)}, \rho_{E(M)})$
es $\mathbb{H}$-dimódulo triple, $\mathbb{H}$ es conmutativa y coconmutativa y
$\beta_M(M)$ es un morfismo de $\mathbb{H}$-comódulos por la derecha por la
conmutatividad de $\mathbb{H}$).

Además, $j_M \circ \eta_{\overline{E(M)^\circ}} = \eta_{E(M)}$.

j_M es morfismo de $\mathbb{H}$-módulos:

$\phi_{E(M)} \circ (C \otimes j_M) = [M, \phi_M \circ (C \otimes \phi_M) \circ (C \otimes C \otimes \beta_M(M)) \circ$
$\circ (C \otimes C \otimes \phi_M \otimes E(M)) \circ (C \otimes C \otimes \lambda \otimes M \otimes E(M)) \circ (C \otimes \delta_C \otimes M \otimes E(M)) \circ$

$\circ(C\otimes\tau_C^M\otimes E(M))\circ(C\otimes\phi_M\otimes C\otimes E(M))\circ(C\otimes\lambda\otimes\rho_M\otimes E(M))\circ$

$\circ(\delta_C\otimes M\otimes E(M))\circ(\tau_C^M\otimes E(M))]\circ(\alpha_M(C\otimes E(M)))=$

$=[M,\phi_M\circ(C\otimes\beta_M(M))\circ(C\otimes\phi_M\otimes E(M))\circ(\mu_C\otimes\mu_C\otimes M\otimes E(M))\circ$

$\circ(C\otimes C\otimes\tau_C^C\otimes M\otimes E(M))\circ(C\otimes\tau_C^C\otimes\lambda\otimes M\otimes E(M))\circ(C\otimes\lambda\otimes\delta_C\otimes M\otimes E(M))\circ$

$\circ(\delta_C\otimes\tau_C^M\otimes E(M))\circ(C\otimes\rho_M\otimes E(M))\circ(\tau_C^M\otimes E(M))]\circ(\alpha_M(C\otimes E(M)))=$

$=[M,\phi_M\circ(C\otimes\beta_M(M))\circ(C\otimes\phi_M\otimes E(M))\circ(C\otimes\lambda\otimes M\otimes E(M))\circ$

$\circ(\delta_C\otimes M\otimes E(M))\circ(\tau_C^M\otimes E(M))\circ(M\otimes\mu_C\otimes E(M))\circ(\rho_M\otimes C\otimes E(M))]\circ$

$\circ(\alpha_M(C\otimes E(M)))=[M,\beta_M(M)\circ(M\otimes\phi_{E(M)})\circ(M\otimes\mu_C\otimes E(M))\circ$

$\circ(\rho_M\otimes C\otimes E(M))]\circ(\alpha_M(C\otimes E(M)))=j_M\circ\phi_{E(M)}.$

j_M es morfismo de $\mathbb{H}$-comódulos:

$\rho_{E(M)}\circ j_M=(\nabla_{MKM}\otimes\mu_C)\circ(\hat{M}\otimes\rho_M\otimes\lambda)\circ(\hat{M}\otimes\beta_M(M)\otimes C)\circ$

$\circ(\hat{M}\otimes M\otimes\tau_{E(M)}^C)\circ(\hat{M}\otimes\rho_M\otimes\phi_{E(M)})\circ(\hat{M}\otimes\rho_M\otimes E(M))\circ$

$\circ(\bar{a}_M\otimes E(M))=(\nabla_{MKM}\otimes C)\circ(\hat{M}\otimes\beta_M(M)\otimes C)\circ(\hat{M}\otimes M\otimes\rho_{E(M)})\circ$

$\circ(\hat{M}\otimes M\otimes\phi_{E(M)})\circ(\hat{M}\otimes\rho_M\otimes E(M))\circ(\bar{a}_M\otimes E(M))=$

$=(j_M\otimes C)\circ\rho_{E(M)}$ ($\mathbb{H}$ coconmutativa).

Análogamente, $j_M':=\nabla_{MKM}\circ(\hat{M}\otimes(\phi_M\circ\tau_C^M\circ(\beta_M(M)\otimes C)\circ$

$\circ(M\otimes\rho_{E(M)})))\circ(\bar{a}_M\otimes E(M))$ es otro isomorfismo de $\mathbb{H}$-dimódulos triple entre $\overline{\mathbb{E}(M)^\circ}$ y $\mathbb{E}(M)$ cuyo inverso es

$(j_M')^{-1}:=\nabla_{MKM}\circ(\hat{M}\otimes(\phi_M\circ(\lambda\otimes M)\circ\tau_C^M\circ(\beta_M(M)\otimes C)\circ$

$\circ(M\otimes\rho_{E(M)})))\circ(\bar{a}_M\otimes E(M)).$

ii) $m_M:=[\hat{M},(\hat{M}\otimes\bar{\delta}_C)\circ(\nabla_{MKM}^{-1}\otimes\hat{M})\circ\tau_{E(M)}^{\hat{M}}]\circ$

$\circ(\alpha_{\hat{M}}(E(M))):\mathbb{E}(M)\longrightarrow E(\hat{M})$ es un morfismo de $\mathbb{H}$-dimódulos triple cuyo inverso es: $\nabla_{MKM}\circ(\beta_{\hat{M}}(\hat{M})\otimes M)\circ(\hat{M}\otimes\tau_{E(\hat{M})}^M)\circ$

$\circ(\bar{a}_M \otimes E(\hat{M}))$.

2. $\mathbb{H}$-DIMODULOS TRIPLE $\mathbb{H}$-AZUMAYA

(2.1) **Proposición**. Si $(\mathbb{A},\phi_A,\rho_A)$ es un $\mathbb{H}$-dimódulo triple con A progenerador en C (|8| (1.6.3)), los morfismos:

$$\chi_{\bar{A},A} := [A,\mu_A \circ (\mu_{\bar{A}} \otimes A)] \circ (\alpha_A(A\otimes A)) : \bar{\mathbb{A}}\#\mathbb{A} \longrightarrow \mathbb{E}(A)$$

$$\chi_{A,\bar{A}} := [A,\mu_A \circ (A\otimes\mu_{\bar{A}}) \circ \tau^{A}_{A\otimes A}] \circ (\alpha_A(A\otimes A)) : \mathbb{A}\#\bar{\mathbb{A}} \longrightarrow \mathbb{E}(A)^{\circ} \quad (1.3)ii)$$

son de $\mathbb{H}$-dimódulos triple

Demostración.

$\chi_{\bar{A},A}$ es morfismo de triples:

$$\mu_{E(A)} \circ (\chi_{\bar{A},A} \otimes \chi_{\bar{A},A}) = [A,\mu_A \circ (\mu_{\bar{A}} \otimes A) \circ (\mu_A \otimes A\otimes A) \circ (\mu_{\bar{A}} \otimes A \otimes A \otimes A)] \circ$$
$$\circ(\alpha_A(A\otimes A\otimes A\otimes A)) = [A,\mu_A \circ (\mu_A \otimes A) \circ (\tau^A_A \otimes A) \circ (\mu_A \otimes \phi_A \otimes A) \circ$$
$$\circ(\mu_A \otimes \tau^C_A \otimes \phi_A \otimes A) \circ (\tau^A_A \otimes \mu_C \otimes \rho_A \otimes A \otimes A) \circ (A \otimes \tau^C_A \otimes C \otimes A \otimes A \otimes A) \circ$$
$$\circ(\rho_A \otimes \phi_A \otimes C \otimes A \otimes A \otimes A) \circ (\rho_A \otimes \rho_A \otimes A \otimes A \otimes A)] \circ (\alpha_A(A\otimes A\otimes A\otimes A)) =$$
$$=[A,\mu_A \circ (\mu_A \otimes \mu_A) \circ (\tau^A_A \otimes A \otimes A) \circ (A \otimes \mu_A \otimes A \otimes A) \circ (A \otimes \phi_A \otimes A \otimes A \otimes A) \circ$$
$$\circ(A\otimes C\otimes\phi_A\otimes A\otimes A\otimes A) \circ (A\otimes C\otimes\tau^A_{C\otimes A} \otimes A \otimes A) \circ (A \otimes C \otimes\phi_A \otimes C \otimes \tau^A_A \otimes A) \circ$$
$$\circ(A \otimes \delta_C \otimes A \otimes C \otimes A \otimes \phi_A \otimes A) \circ (\rho_A \otimes \rho_A \otimes \rho_A \otimes A \otimes A)] \circ$$
$$\circ(\alpha_A(A \otimes A \otimes A \otimes A)) = \chi_{\bar{A},A} \circ \mu_{\bar{A}\#A} \quad (\mathbb{H} \text{ conmutativa}).$$

Además, $\chi_{\bar{A},A} \circ (\eta_A \otimes \eta_A) = \eta_{E(A)}$.

$\chi_{\bar{A},A}$ es morfismo de $\mathbb{H}$-módulos:

$$\phi_{E(A)} \circ (C \otimes \chi_{\bar{A},A}) = [A,\mu_A \circ \phi_{A\otimes A} \circ (C \otimes \mu_A \otimes A) \circ (\tau^A_{C\otimes A} \otimes A) \circ$$
$$\circ(\phi_A \otimes C \otimes \phi_A \otimes A) \circ (C \otimes A \otimes \tau^C_C \otimes A \otimes A) \circ (\tau^{A\otimes C\otimes C}_C \otimes A \otimes A) \circ$$

$\circ(\rho_A \otimes C \otimes \lambda \otimes A \otimes A) \circ (A \otimes \delta_C \otimes A \otimes A)] \circ (\alpha_A(C \otimes A \otimes A)) =$

$= [A, \mu_A \circ (\mu_A \otimes \phi_A) \circ (\phi_{A\otimes A} \otimes C \otimes A) \circ (C \otimes \tau^{C}_{A\otimes A} \otimes A) \circ (\delta_C \otimes \tau^{A}_{A} \otimes A) \circ$

$\circ(\tau^{A}_{C} \otimes \phi_A \otimes A) \circ (\phi_A \otimes C \otimes C \otimes A \otimes A) \circ (\tau^{A}_{C} \otimes \tau^{C}_{C} \otimes A \otimes A) \circ$

$\circ(A \otimes \tau^{C\otimes C}_{C} \otimes A \otimes A) \circ (A \otimes C \otimes C \otimes \lambda \otimes A \otimes A) \circ (\rho_A \otimes \delta_C \otimes A \otimes A)] \circ$

$\circ(\alpha_A(C \otimes A \otimes A)) = [A, \mu_A \circ (\mu_A \otimes A) \circ (\phi_A \otimes \phi_A \otimes A) \circ (C \otimes \phi_A \otimes C \otimes A \otimes \phi_A) \circ$

$\circ(C \otimes C \otimes \tau^{C}_{A} \otimes A \otimes C \otimes A) \circ (\tau^{C}_{C} \otimes \tau^{A}_{C\otimes A} \otimes C \otimes A) \circ$

$\circ(\tau^{A}_{C\otimes C} \otimes C \otimes \tau^{C}_{A} \otimes A) \circ (\rho_A \otimes C \otimes \mu_C \otimes C \otimes A \otimes A) \circ (A \otimes \delta_C \otimes \tau^{C}_{C} \otimes A \otimes$

$\otimes A) \circ (A \otimes \delta_C \otimes \lambda \otimes A \otimes A) \circ (A \otimes \delta_C \otimes A \otimes A)] \circ (\alpha_A(C \otimes A \otimes A)) =$

$= [A, \mu_A \circ (\mu_A \otimes A) \circ (\phi_A \otimes \phi_A \otimes A) \circ (C \otimes \phi_A \otimes C \otimes A \otimes \phi_A) \circ$

$\circ(C \otimes C \otimes \tau^{C}_{A} \otimes A \otimes C \otimes A) \circ (\tau^{C}_{C} \otimes \tau^{A}_{C\otimes A} \otimes C \otimes A) \circ (\tau^{A}_{C\otimes C} \otimes C \otimes \tau^{C}_{A} \otimes A) \circ$

$\circ(\rho_A \otimes C \otimes (\eta_C \circ \varepsilon_C) \otimes C \otimes A \otimes A) \circ (A \otimes C \otimes \delta_C \otimes A \otimes A) \circ (A \otimes \delta_C \otimes A \otimes A)] \circ$

$\circ(\alpha_A(C \otimes A \otimes A)) = [A, \mu_A \circ (\mu_{\bar{A}} \otimes A) \circ (A \otimes \phi_{A\otimes A})] \circ (\alpha_A(C \otimes A \otimes A)) =$

$= \chi_{\bar{A},A} \circ \phi_{\bar{A}\# A}$ ($\mathbb{H}$ conmutativa y coconmutativa).

$\chi_{\bar{A},A}$ es morfismo de $\mathbb{H}$-comódulos:

$\rho_{E(A)} \circ \chi_{\bar{A},A} = (\nabla_{AKA} \otimes \mu_C) \circ (\hat{A} \otimes \rho_A \otimes \lambda) \circ (\hat{A} \otimes \mu_A \otimes C) \circ$

$\circ(\hat{A} \otimes \mu_{\bar{A}} \otimes A \otimes C) \circ (\hat{A} \otimes A \otimes \tau^{C}_{A\otimes A}) \circ (\hat{A} \otimes \rho_A \otimes A \otimes A) \circ (\bar{a}_A \otimes A \otimes A) =$

$= (\nabla_{AKA} \otimes \mu_C) \circ (\hat{A} \otimes \mu_A \otimes \mu_C \otimes \lambda) \circ (\hat{A} \otimes \mu_A \otimes \tau^{C}_{A} \otimes C \otimes C) \circ (\hat{A} \otimes A \otimes A \otimes$

$\otimes \mu_C \otimes \rho_A \otimes C) \circ (\hat{A} \otimes \tau^{A\otimes C}_{A} \otimes C \otimes A \otimes C) \circ (\hat{A} \otimes \rho_A \otimes \phi_A \otimes C \otimes A \otimes C) \circ (\hat{A} \otimes \rho_A \otimes$

$\otimes \rho_A \otimes A \otimes C) \circ (\hat{A} \otimes A \otimes \tau^{C}_{A\otimes A}) \circ (\hat{A} \otimes \rho_A \otimes A \otimes A) \circ (\bar{a}_A \otimes A \otimes A) = (\nabla_{AKA} \otimes C) \circ$

$\circ(\hat{A} \otimes \mu_A \otimes C) \circ (\hat{A} \otimes \mu_A \otimes A \otimes C) \circ (\hat{A} \otimes \tau^{A}_{A} \otimes A \otimes C) \circ (\hat{A} \otimes \bar{A} \otimes \phi_A \otimes A \otimes C) \circ$

$\circ(\hat{A} \otimes A \otimes C \otimes \tau^{C}_{A\otimes A}) \circ (\hat{A} \otimes A \otimes C \otimes \mu_C \otimes A \otimes A) \circ (\hat{A} \otimes A \otimes C \otimes \mu_C \otimes C \otimes A \otimes A) \circ$

$\circ(\hat{A} \otimes A \otimes C \otimes \mu_C \otimes C \otimes \tau^{A\otimes A}_{C}) \circ (\hat{A} \otimes A \otimes \delta_C \otimes \lambda \otimes \tau^{A}_{C} \otimes \rho_A) \circ (\hat{A} \otimes A \otimes \delta_C \otimes$

$\otimes \rho_A \otimes A) \circ (\hat{A} \otimes \rho_A \otimes A \otimes A) \circ (\bar{a}_A \otimes A \otimes A) = (\nabla_{AKA} \otimes \mu_C) \circ (\hat{A} \otimes \mu_A \otimes$

$\otimes C \otimes C) \circ (\bar{A} \otimes \mu_{\bar{A}} \otimes \tau_A^C \otimes C) \circ (\bar{a}_A \otimes \rho_A \otimes \rho_A) = (x_{\bar{A},A} \otimes C) \circ \rho_{A \otimes A}$

Análogamente se prueba que $x_{A,\bar{A}} : \mathbb{A}\#\bar{\mathbb{A}} \longrightarrow \mathbb{E}(A)^\circ$ es un morfismo de $\mathbb{H}$-dimódulos triple.

(2.2) **Definición**. Si $(\mathbb{A}, \phi_A, \rho_A)$ es un $\mathbb{H}$-dimódulo triple con A progenerador en $\mathcal{C}$, se dirá que $\mathbb{A}$ es $\mathbb{H}$-Azumaya si $x_{\bar{A},A}$ y $x_{A,\bar{A}}$ son isomorfismos.

(2.3) **Proposición**.

i) Si $\mathbb{A}$ es $\mathbb{H}$-Azumaya, también lo es $\bar{\mathbb{A}}$.

ii) Si (M, ϕ_M, ρ_M) es un $\mathbb{H}$-dimódulo con M progenerador en $\mathcal{C}$, entonces $\mathbb{E}(M)^\circ$ es $\mathbb{H}$-Azumaya.

iii) Si $\mathbb{A}$ y $\mathbb{B}$ son $\mathbb{H}$-Azumaya, $\mathbb{A}\#\mathbb{B}$ es $\mathbb{H}$-Azumaya.

Demostración.

i) $x_{\bar{\bar{A}},\bar{A}} \circ (S_{\bar{A},A})^{-1} = j_A \circ x_{A,\bar{A}}$ siendo $S_{\bar{A},A}$ y j_A los isomorfismos de $\mathbb{H}$-dimódulos triple obtenidos en (1.4)iii) y (1.10)i) respectivamente.

En efecto:

$x_{\bar{\bar{A}},A} \circ (S_{\bar{A},A})^{-1} = [A, \mu_A \circ \tau_A^A \circ (\mu_A \otimes \phi_A) \circ (A \otimes A \otimes \mu_C \otimes \phi_A) \circ$

$\circ (\tau_A^{A \otimes C} \otimes C \otimes \lambda \otimes A) \circ (\rho_A \otimes \phi_A \otimes C \otimes C \otimes A) \circ (\rho_A \otimes \rho_A \otimes C \otimes A) \circ$

$\circ (\tau_A^A \otimes C \otimes A) \circ (A \otimes \phi_A \otimes C \otimes A) \circ (\rho_A \otimes \rho_A \otimes A) \circ (A \otimes \tau_A^A)] \circ$

$\circ (a_A(A \otimes A)) = [A, \mu_A \circ (\phi_A \otimes A) \circ (\mu_C \otimes A \otimes A) \circ$

$\circ (C \otimes \tau_{C \otimes A}^A) \circ (C \otimes \mu_A \otimes \mu_C \otimes A) \circ (C \otimes \tau_A^A \otimes C \otimes \lambda \otimes A) \circ (C \otimes A \otimes \phi_A \otimes$

$\otimes C \otimes C \otimes A) \circ (C \otimes \tau_{A \otimes C}^A \otimes C \otimes C \otimes A) \circ (\tau_C^A \otimes A \otimes \delta_C \otimes C \otimes A) \circ$

$\circ (A \otimes C \otimes A \otimes \delta_C \otimes A) \circ (A \otimes C \otimes \rho_A \otimes A) \circ (A \otimes C \otimes \phi_A \otimes A) \circ (A \otimes \delta_C \otimes A \otimes A) \circ$

$\circ(\rho_A \otimes \tau_A^A)] \circ (\alpha_A(A\otimes A)) = [A, \mu_A \circ (\phi_A \otimes A) \circ (\mu_C \otimes A \otimes A) \circ$

$\circ(C\otimes \tau_{C\otimes A}^A) \circ (C \otimes \mu_A \otimes C \otimes A) \circ (C \otimes \tau_A^A \otimes C \otimes A) \circ$

$\circ(C \otimes A \otimes \phi_A \otimes C \otimes A) \circ (C \otimes \tau_{A\otimes C}^A \otimes (\eta_C \circ \varepsilon_C) \otimes A) \circ$

$\circ(\tau_C^A \otimes A \otimes \delta_C \otimes A) \circ (A \otimes C \otimes \rho_A \otimes A) \circ (A \otimes C \otimes \phi_A \otimes A) \circ (A\otimes \delta_C \otimes A \otimes A) \circ$

$\circ(\rho_A \otimes \tau_A^A)] \circ (\alpha_A(A\otimes A)) = [A, \beta_A(A) \circ (A \otimes \chi_{A,\bar{A}}) \circ (A \otimes \phi_{A\otimes A}) \circ$

$\circ(\rho_A \otimes A \otimes A)] \circ (\alpha_A(A\otimes A)) = [A, \beta_A(A) \circ (A \otimes \phi_{E(A)}) \circ (\rho_A \otimes \chi_{A,\bar{A}})] \circ$

$\circ(\alpha_A(A\otimes A)) = j_A \circ \chi_{A,\bar{A}}$ ($\chi_{A,\bar{A}}$ es un morfismo de $\mathbb{H}$-módulos por la izquierda y $\mathbb{H}$ es conmutativa y coconmutativa). De donde $\chi_{\bar{A},\bar{A}} = j_A \circ \chi_{A,\bar{A}} \circ S_{\bar{A},A}$ es una composición de isomorfirmos.

Además, $\chi_{\bar{A},\bar{\bar{A}}} = Z_A \circ \chi_{\bar{A},A} \circ S_{A,\bar{A}}$ en donde

$Z_A := [A, \beta_A(A) \circ (\phi_A \otimes E(A)) \circ (\tau_C^A \otimes E(A)) \circ (A \otimes \tau_C^{E(A)}) \circ (A\otimes \rho_{E(A)})] \circ$

$\circ(\alpha_A(E(A))) : E(A) \longrightarrow E(A)$ es un isomorfismo de $\mathbb{H}$-dimódulos triple ($(Z_A)^{-1} := [A, \beta_A(A) \circ (\phi_A \otimes E(A)) \circ (\lambda \otimes A \otimes E(A)) \circ$

$\circ(\tau_C^A \otimes E(A)) \circ (A \otimes \tau_C^{E(A)}) \circ (A \otimes \rho_{E(A)})] \circ (\alpha_A(E(A))))$ y por lo tanto $\chi_{\bar{A},\bar{\bar{A}}}$ es también una composición de isomorfismos.

ii) $\mu_{E(M)} \circ (\mu_{E(M)} \otimes E(M)) \circ \tau_{E(M)}^{E(M)\otimes E(M)} \circ (E(M) \otimes j_M \otimes E(M)) \circ$

$\circ(E(M) \otimes \phi_{E(M)} \otimes E(M)) \circ (E(M) \otimes \tau_C^{E(M)\otimes E(M)}) \circ (E(M) \otimes E(M) \otimes$

$\otimes \nabla_{MKM} \otimes \lambda) \circ (E(M) \otimes E(M) \otimes \hat{M} \otimes \rho_M) \circ (E(M) \otimes E(M) \otimes \nabla_{MKM}^{-1}) =$

$= [M, \beta_M(M) \circ (M \otimes \phi_{E(M)}) \circ (M \otimes \mu_C \otimes E(M)) \circ (\rho_M \otimes C \otimes E(M)) \circ$

$\circ(\beta_M(M) \otimes C \otimes E(M)) \circ (M \otimes \mu_{E(M)} \otimes C \otimes E(M)) \circ (M \otimes \tau_{E(M)}^{E(M)} \otimes C \otimes E(M)) \circ$

$\circ(M \otimes E(M) \otimes \tau_{E(M)\otimes C}^{E(M)}) \circ (M \otimes E(M) \otimes E(M) \otimes \nabla_{MKM} \otimes \lambda) \circ$

$\circ(M \otimes E(M) \otimes E(M) \otimes \hat{M} \otimes \rho_M) \circ (M \otimes E(M) \otimes E(M) \otimes \nabla_{MKM}^{-1})] \circ$

$\circ(\alpha_M(E(M) \otimes E(M) \otimes E(M))) = [M, \beta_M(M) \circ (M \otimes \phi_{E(M)}) \circ$

$\circ(M \otimes \mu_C \otimes E(M)) \circ (\rho_M \otimes C \otimes E(M) \circ (\beta_M(M) \otimes C \otimes E(M)) \circ$

$\circ(M \otimes \tau^C_{E(M)} \otimes E(M)) \circ (M \otimes \lambda \otimes E(M) \otimes E(M)) \circ (\rho_M \otimes E(M) \otimes E(M)) \circ$

$\circ \tau_M^{E(M)\otimes E(M)} \circ (E(M) \otimes E(M) \otimes \beta_M(M)) \circ (\tau^M_{E(M)\otimes E(M)} \otimes E(M))] \circ$

$\circ(\alpha_M(E(M) \otimes E(M) \otimes E(M))) = [M, \beta_M(M) \circ (\beta_M(M) \otimes E(M)) \circ$

$\circ \tau_M^{E(M)\otimes E(M)} \circ (E(M) \otimes \phi_{E(M)} \otimes \beta_M(M)) \circ (\rho_{E(M)} \otimes E(M) \otimes M \otimes$

$\otimes E(M)) \circ (\tau^M_{E(M)\otimes E(M)} \otimes E(M))] \circ (\alpha_M(E(M) \otimes E(M) \otimes E(M)) =$

$= \mu_{E(M)} \circ \tau^{E(M)}_{E(M)} \circ (\mu_{E(M)} \otimes E(M)) \circ (E(M) \otimes \phi_{E(M)} \otimes E(M)) \circ$

$\circ(\rho_{E(M)} \otimes E(M) \otimes E(M))$. Por lo tanto $\chi_{\overline{E(M)^\circ}, E(M)^\circ} =$

$= \chi_{E(M)^\circ} \circ (j_M \otimes E(M)) \circ h'_{\overline{E(M)^\circ}, M}$ siendo $h'_{\overline{E(M)^\circ}, M}$ y j_M los isomorfismos de $\mathbb{H}$-dimódulos triple obtenidos en (1.8)ii) y (1.10)i) respectivamente y $\chi_{E(M)^\circ} = [E(M), \mu_{E(M)} \circ \tau^{E(M)}_{E(M)} \circ (\mu_{E(M)} \otimes E(M))] \circ$ $\circ(\alpha_{E(M)}(E(M) \otimes E(M))) : (\mathbb{E}(M)^\circ)^e \longrightarrow \mathbb{E}(E(M)^\circ)$ el isomorfirmo de $\mathbb{H}$-dimódulos triple deducido del caracter 1-Azumaya del triple $\mathbb{E}(M)^\circ$ al ser M progenerador en $\mathcal{C}$ (|2| (1.8)).

Análogamente se comprueba que $\chi_{E(M)^\circ, \overline{E(M)^\circ}} =$ $= \chi_{E(M)^\circ} \circ (E(M) \otimes j'^{-1}_M) \circ h_{M, \overline{E(M)^\circ}}$ en donde $h_{M, \overline{E(M)^\circ}}$ y j'_M son los isomorfismos de $\mathbb{H}$-dimódulos triple obtenidos en (1.8)i) y (1.10)i) respectivamente.

iii) Con cálculos similares a los utilizados hasta aquí, se obtiene:

$\chi_{\overline{A\#B}, A\#B} = f \circ (E(A) \otimes (m_B)^{-1}) \circ h'_{E(A), \hat{B}} \circ$
$\circ((m_A)^{-1} \otimes m_B) \circ (E(\hat{A}) \otimes \chi_{\bar{B}, B}) \circ ((h_{\hat{A}, B})^{-1} \otimes B) \circ$

$\circ(\tau^{B}_{E(\bar{A})} \otimes B) \circ (h'_{\bar{B},\bar{A}} \otimes B) \circ (B \otimes m_A \otimes B) \circ$

$\circ(B \otimes \chi_{\bar{A},A} \otimes B) \circ ((S_{B,A})^{-1} \otimes A \otimes B)$ en donde S, m, h y h' son los isomorfismos de (1.4)iii), (1.10)ii), (1.8)i) y (1.8)ii) respectivamente y

$f = [A \otimes B, (\beta_A(A) \otimes \beta_B(B)) \circ (A \otimes \tau^{B}_{E(A)} \otimes E(B))] \circ$

$\circ (\alpha_{A\otimes B}(E(A) \otimes E(B))) : E(A) \otimes E(B) \longrightarrow E(A \otimes B)$

es el isomorfismo de (1.5)ii). $\chi_{\overline{A\#B},A\#B}$ es composición de isomorfismos por ser $\mathbb{A}$ y $\mathbb{B}$ $\mathbb{H}$-Azumayas.

Además, $\chi_{A\#B,\overline{A\#B}} = f \circ h_{A,E(B)} \circ \circ (\chi_{A,\bar{A}} \otimes E(B)) \circ$

$\circ(A \otimes (h'_{\bar{A},B})^{-1}) \circ (A \otimes \tau^{E(B)}_{A}) \circ (A \otimes h_{B,\bar{A}}) \circ (A \otimes \chi_{B,\bar{B}} \otimes A) \circ$

$\circ(A \otimes B \otimes (S_{B,A})^{-1})$ también es un isomorfismo, de donde, $\mathbb{A}\#\mathbb{B}$ es un triple $\mathbb{H}$-Azumaya.

3. $BD^{\mathbb{H}}(C)$, GRUPO DE BRAUER DE $\mathbb{H}$-AZUMAYAS

(3.1) **Definición**. Dos $\mathbb{H}$-dimódulos triple $(\mathbb{A}, \phi_A, \rho_A)$ y $(\mathbb{B}, \phi_B, \rho_B)$ que son $\mathbb{H}$-Azumayas están relacionados si, y sólo si, existen $\mathbb{H}$-dimódulos (M, ϕ_M, ρ_M) y (N, ϕ_N, ρ_N) con M y N progeneradores en C, tales que $\mathbb{A}\#\mathbb{E}(M)^{\circ}$ y $\mathbb{B}\#\mathbb{E}(N)^{\circ}$ son isomorfos como $\mathbb{H}$-dimódulos triple. Esta relación es de equivalencia, y el conjunto cociente se denotará por $BD^{\mathbb{H}}(C)$.

(3.2) **Proposición**. $BD^{\mathbb{H}}(C)$ es un grupo que se denominará el grupo de Brauer de $\mathbb{H}$-Azumayas.

Demostración.

Para $[(\mathbb{A}, \phi_A, \rho_A)]$ y $[(\mathbb{B}, \phi_B, \rho_B)]$ elementos de $BD^{\mathbb{H}}(C)$

se define la operación: $[(\mathbb{A},\phi_A,\rho_A)]\,[(\mathbb{B},\phi_B,\rho_B)] :=$ $=[(\mathbb{A}\#\mathbb{B}, \phi_{A\otimes B}, \rho_{A\otimes B})]$ (1.3)iii). El elemento neutro es la clase de $(\mathbb{E}(M)^{\circ},\phi_{E(M)}, \rho_{E(M)})$ siendo (M,ϕ_M,ρ_M) cualquier $\mathbb{H}$-dimódulo con M progenerador en C (2.3)ii). El inverso de $[(\mathbb{A},\phi_A,\rho_A)]$ es $[(\bar{\mathbb{A}},\phi_A,\rho_A)]$ (2.2) y (2.3)i).

(3.3) **Proposición**. El siguiente diagrama de monomorfismos de grupos:

$$\begin{tikzcd}
 & BM(C,\mathbb{H}) \arrow[dr, "P_3"] & \\
B(C) \arrow[ur, "P_1"] \arrow[dr, "P_2"'] & & BD^{\mathbb{H}}(C) \\
 & BC(C,\mathbb{H}) \arrow[ur, "P_4"'] &
\end{tikzcd}$$

es conmutativo, siendo $BM(C,\mathbb{H})$ (respectivamente $BC(C,\mathbb{H})$) el grupo de Brauer de $\mathbb{H}$-módulos triple (respectivamente $\mathbb{H}$-comódulos triple) 1-Azumaya (|1| (2.1.8), (2.2.7)) y $B(C)$ el grupo de Brauer de triples 1-Azumaya en C (|1| (2.1.11)).

Demostración.

Si $(\mathbb{A},\phi_A)$ es un $\mathbb{H}$-módulo triple por la izquierda entonces $(\mathbb{A},\phi_A,\rho_A = A\otimes \eta_C)$ es un $\mathbb{H}$-dimódulo triple. Además, como en este caso los morfismos $x_{A,\bar{A}}$ y $x_{\bar{A},A}$ (2.1) coinciden con $x_{\mathbb{A}} = [A,\mu_A\circ(\mu_A\otimes A)\circ(\tau_A^A\otimes A)]\circ\alpha_A(A\otimes A)$ (|2| (1.6)) se define de forma canónica el monomorfismo $P_3 : BM(C,\mathbb{H}) \longrightarrow BD^{\mathbb{H}}(C)$, $[(\mathbb{A},\phi_A)] \longrightarrow [(\mathbb{A},\phi_A,\rho_A = A\otimes\eta_C)]$. (Nótese que si $(\mathbb{B},\phi_B,\rho_B) =$ $= B\otimes\eta_C)$ es un $\mathbb{H}$-dimódulo triple definido a partir del $\mathbb{H}$-módulo triple $(\mathbb{B},\phi_B)$, entonces los $\mathbb{H}$-dimódulos triple $(\mathbb{A}\#\mathbb{B}, \phi_{A\otimes B}, \rho_{A\otimes B})$ y $(\mathbb{A}\,\mathbb{B}, \phi_{A\otimes B}, \rho_{A\otimes B})$ coinciden).

Análogamente se obtiene el monomorfismo

$$P_4 : BC(C,\mathbb{H}) \longrightarrow BD^{\mathbb{H}}(C) \qquad [(\mathbb{A},\rho_A)] \longrightarrow [(\mathbb{A},\phi_A = \varepsilon_C \otimes A, \rho_A)].$$

Los morfismos P_1 y P_2 que completan el diagrama son los definidos en ([1], (2.1.12), (2.2.7)):

$$P_1([\mathbb{A}]) := [(\mathbb{A}, \varepsilon_C \otimes A)]$$

$$P_2([\mathbb{A}]) := [(\mathbb{A}, A \otimes \eta_C)] .$$

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