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CENTRAL EXTENSION AND COVERINGS

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The theory of central extensions has a lot of analogy with the theory of covering spaces. It is mentioned for example in [1]. In this paper we show that the category of central extensions of a perfect group and a certain category of covering spaces of a certain space are equivalent (see Theorem 1). Then the facts about central extensions will follow from the corresponding facts about coverings (see Corollaries 1-3).

We start with some definitions to make this work selfcontained.

Definition 1. (see [2] § 5) A pair $(X;\mathfrak{C})$ is called a central extension of a group G if $\mathfrak{C} : X \to G$ is an epimorphism and $\mathrm{kernel}(\mathfrak{C}) \subset \mathrm{center}\, X$

Definition 2. (see [2] § 5) The central extension $(X;\mathfrak{C})$ of a group G is called universal if for every central extension $(Y;\psi)$ of G there is one and only one homomorphism $h : X \to Y$ such that $\psi \circ h = \mathfrak{C}$.

It follows from [2] Theorem 5.3 that if a group G has universal central extension $(X;\mathfrak{C})$ then G and X are perfect.

We shall denote by $E(G)$ the category of central extensions of G. Morphisms in this category are homomorphisms over G.

Now we describe a category $\mathrm{Cov}_{\cdot}^{ab}(X)$ of pointed abelian coverings over a connected space X with a base point. Objects of $\mathrm{Cov}_{\cdot}^{ab}(X)$ are principal G-fibrations over X with a base point in the fibre over the base point of X. G is a discrete abelian group. Such principal G-fibrations are regular coverings and they are induced from the universal covering of BG by a map $f : X \to BG$. If E_1 and E_2 are

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two coverings induced respectively by $f_1 : X \to BG_1$ and $f_2 : X \to BG_2$ then morphisms of E_1 in E_2 in the category. $\mathrm{Cov.}^{ab}(X)$ are those pointed maps from E_1 in E_2 over X which are induced by maps $h : BG_1 \to BG_2$ such that $h \circ f_1$ is homotopic to f_2. The category $\mathrm{Cov.}^{ab}(X)$ has an initial object. It is the universal, pointed covering.

Let us suppose now that G is a perfect group. Then the fundamental group of BG is perfect and we can apply the "+" construction to get BG^+. BG^+ is simply-connected and therefore $\Omega(BG^+)$ is connected.

<u>Theorem 1</u>. Let G be a perfect group. Then the categories $\mathrm{Cov.}^{ab}(\Omega(BG^+))$ and $\mathcal{E}(G)$ are equivalent. The full subcategory of $\mathrm{Cov.}^{ab}(\Omega(BG^+))$ which objects are connected coverings and the category of central extensions (X,φ) of G such that X's are perfect, are also equivalent.

<u>Proof</u>. We shall define two functors $F : \mathcal{E}(G) \to \mathrm{Cov.}^{ab}(\Omega BG^+)$ and $J : \mathrm{Cov.}^{ab}(\Omega BG^+) \to \mathcal{E}(G)$ such that the compositions $F \circ J$ and $J \circ F$ are natural isomorphic to the identity functors.

Let $1 \to H \to X \xrightarrow{\varphi} G \to 1$ be a central extension. Then $BH \to BX \to BG$ is a fibration. Let $tr : H_2(BG) \to H_1(BH)$ be a transgression homomorphism in the Serre spectral sequence of this fibration. The homomorphism tr we can consider as an element $t \in H^2(BG,H) = H^2(BG^+;H)$. We have the following long sequence of fibrations

$$(*_X) \quad \to \quad \Omega\mathfrak{F}(X) \to \Omega BG^+ \xrightarrow{\delta=\Omega t} K(H,1) \to \mathfrak{F}(X) \to BG^+ \xrightarrow{t} K(H,2) \quad ,$$

where $\mathfrak{F}(X)$ is a homotopy fibre of t.

We set $F(X:\varphi) = (\delta^!(EH) \to \Omega(BG^+))$ where $\delta^!(EH) \to \Omega(BG^+)$ is a covering induced by δ from the universal covering over BH. The base point of $\delta ! EH$ we choose in the fibre over the base point of $\Omega(BG^+)$. The homomorphism $f : (X_1, \varphi_1) \to (X_2, \varphi_2)$ of central extensions induces a map between sequences of fibrations $(*_{X_1})$ and $(*_{X_2})$. As a part of this map we get a commutative diagram

$$\begin{array}{ccc} & \overset{\delta_1}{\nearrow} & BH_1 \\ \Omega(BG^+) & & \big\downarrow f_* \\ & \underset{\delta_2}{\searrow} & BH_2 \end{array} .$$

This diagram induces a morphism between coverings $\delta_1 !(EH_1) \to \Omega(BG^+)$ and $\delta_2 !(EH_2) \to \Omega(BG^+)$ in the category $\mathrm{Cov.}^{ab}(\Omega(BG^+))$.

Now we shall define a functor $J : \mathrm{Cov.}^{ab}(\Omega BG^+) \to E(G)$. Let $(E \overset{p}{\to} \Omega BG^+) \in \mathrm{Cov.}^{ab}(\Omega BG^+)$ and let us suppose that $p : E \to \Omega BG^+$ is a principal K fibration. $(p : E \to \Omega BG^+)$ is induced from the universal covering over BK by a map $x : \Omega(BG^+) \to BK$. We have the following isomorphisms

$$H^1(\Omega(BG^+);K) \approx \mathrm{Hom}(\pi_1(\Omega BG^+);K) \approx \mathrm{Hom}(\pi_2(BG^+);K) \approx H^2(BG^+;K) .$$

Therefore there is $y \in H^2(BG^+;K)$ which corresponds to x by these isomorphisms. Let us form the following sequence of fibrations

$$(**) \qquad \to \Omega BG^+ \xrightarrow{\Omega y = x} K(H;1) \to Y = \mathrm{Fibre}(y) \to BG^+ \xrightarrow{y} K(H;2) .$$

Let $i : BG \to BG^+$ be a natural map in the "+" construction. Let

$$(***) \qquad K(H;1) \to S = i!Y \to BG$$

be a fibration induced by i from the fibration

$$K(H;1) \to Y \to BG^+ .$$

After applying functor π_1 to the fibration (***) we get an exact sequence

$$(****) \quad 1 \to H \to \pi_1(S) = T \to G \to 1 .$$

The action of $\pi_1(BG)$ on the fibre in the fibration (***) is trivial because this fibration is induced from the fibration over the simply-connected space BG^+. Therefore the extension (****) is central.

A map in the category $Cov.^{ab}(\Omega BG^+)$ induces a homotopy commutative diagram

$$\begin{array}{ccc} \Omega BG^+ & \longrightarrow & BH_1 \\ \| & & \downarrow \\ \Omega BG^+ & \longrightarrow & BH_2 \end{array} .$$

Hence we get a homotopy commutative diagram

$$\begin{array}{ccccc} K(H_1,1) & \longrightarrow & Y_1 & \longrightarrow & BG^+ \\ \downarrow & & \downarrow & & \| \\ K(H_2,1) & \longrightarrow & Y_2 & \longrightarrow & BG^+ \end{array}$$

and consequently a map between central extensions

$$\begin{array}{ccccc} H_1 & \to & T_1 & \to & G \\ \downarrow & & \downarrow & & \| \\ H_2 & \to & T_2 & \to & G \end{array} .$$

The proof that the compositions $F \circ J$ and $J \circ F$ are natural isomorphic to the identities follows immediately from definitions of F and J and I omit it.

If a principal H-fibration $E \to \Omega BG^+$ is connected then $\pi_1(\Omega BG^+) \to \pi_1(BH)$ is an epimorphism. This implies that $\pi_1(Y) = 0$ and therefore $H_1(T) = 0$. Consequently $J(E \to \Omega BG^+) = (1 \to H \to T \to G \to 1)$ is an extension of G such that T is perfect.

If $1 \to H \to X \to G \to 1$ is a central extension with X perfect then $\pi_1(\Omega BG^+) \to \pi_1(BH)$ is an epimorphism and consequently the induced covering over ΩBG^+ is connected.

The following corollaries, usually proved in an algebraic way, follow immediately from Theorem 1 .

Corollary 1. There exists a universal central extension of a perfect group G .

Proof. The universal central extension is an initial object in the category $E(G)$. The category $Cov.^{ab}(\Omega BG^+)$ has an initial object. It is a universal covering. Therefore there is an initial object in $E(G)$.

Corollary 2. $(X;\varphi)$ is a universal extension iff $H_1(X) = 0$ and $H_2(X) = 0$. Then we have $\ker \varphi = H_2(G)$.

Proof. The principal fibration corresponding to $(X;\varphi)$ is $\Omega BX^+ \to \Omega BG^+$. This covering is universal if and only if $\pi_0(\Omega BX^+) = 0$ and $\pi_1(\Omega BX^+) = 0$. Hence we have that $(X;\varphi)$ is universal if and only if $H_1(BX^+) = H_1(X) = 0$ and $H_2(BX^+) = H_2(X) = 0$. The fibration $\Omega BX^+ \to \Omega BG^+$ is induced from the universal covering over $B(\ker\varphi)$ by a map $\Omega BG^+ \to B(\ker\varphi)$. If it is universal then
$\ker\varphi = \pi_1(\Omega BG^+) = \pi_2(BG^+) = H_2(BG^+) = H_2(G)$.

Corollary 3. The isomorphism classes of central extensions (X,φ) of G such that X's are perfect,are in one to one correspondence with subgroups of $H_2(G)$.

Proof. The isomorphism classes of connected coverings over ΩBG^+ are in one to one correspondence with subgroups of $\pi_1(\Omega BG^+) = H_2(G)$.

Some steps in the proofs given below can be shown using the following proposition which itself seems to be interesting.

Proposition 1. Let us suppose that $0 \to H \to X \to G \to 1$ is a central extension of a perfect group G by a group H. Then $BH \to BX^+ \to BG^+$ is a fibration. (The "+" construction is done with respect to a maximal perfect subgroup of X.)

Proof. Let us assume first that X is perfect. Let F be a fibre of $BX^+ \to BG^+$. There is a map of a fibration $BH \to BX \to BG$ into a fibration $F \to BX^+ \to BG^+$. This map induces a map of Serre spectral sequences. This map is an isomorphism on $E^2_{*,0}$ and on $E^{\infty}_{*,*}$-terms. Therefore it is isomorphism on $E^2_{0,*}$-terms. This means that a map $H_*(BH;Z) \to H_*(F;Z)$ is an isomorphism. F is a fibre of a map between nilpotent spaces therefore it is nilpotent. It implies that $BH \to F$ is a homotopy equivalence.

Let now X be arbitrary and let X' be a maximal, perfect subgroup of X. The extension $0 \to H' = \mathrm{Ker}(i) \to X' \overset{i}{\to} G \to 1$ is also central. Moreover BX'^+ is a universal cover of BX^+. If F is a fibre of $BX^+ \to BG^+$ then only $\pi_1(F)$ is non-zero and it appears in the following exact sequence

$$0 \to \pi_2(BG^+) \to \pi_1(F) \to \pi_1(BX^+) \to 1$$

$\pi_1(BX^+)$ is abelian. This implies that $\pi_1(F)$ is nilpotent. Repeating once more arguments with the Serre spectral sequence we get that F is homotopically equivalent to $K(H,1)$.

In [3] we have introduced "$+_P$" construction in the case if $H_1(X;Z_P) = 0$. (Z_P is a ring of integers localized outside P.)

Definition 3. We say that G is P-perfect if $H_1(G,Z_p) = 0$.

We shall study central extensions of a P-perfect group G by finitely generated Z_p-modules. We shall denote this category by $E_p(G)$. We have the following proposition.

Proposition 2. Let $0 \to H \to X \to G \to 1$ be a central extension of a P-perfect group G by a finitely generated Z_p-module H . Then $BH \to BX^{+}P \to BG^{+}P$ is a fibration. (The "$+_p$"-construction is done with respect to a maximal P-perfect subgroup of X).

The proof of Proposition 2 is exactly the same as the proof of Proposition 1.

Let X be a P-local space. We define a category $Cov._{,P}^{ab}(X)$. Objects of $Cov._{,P}^{ab}(X)$ are principal M-fibrations over X with a fixed base point in the fibre over the base point of X . M is a Z_p-module.

Theorem 2. Let G be a P-perfect group. Then the categories $Cov._P^{ab}(\Omega BG^{+}P)$ and $E_p(G)$ are equivalent. The full subcategory of $Cov._{,P}^{ab}(\Omega BG^{+}P)$ which objects are connected coverings and the category of central extensions (X,φ) of G such that X's are P-perfect ,are also equivalent.

Corollary 4. i) There exists a universal central extension of a P-perfect group G in the category $E_p(G)$.

ii) (X,φ) is a universal central extension of a P-perfect group G in the category $E_p(G)$ if and only if $H_1(X;Z_p) = 0$ and $H_2(X;Z_p) = 0$. Then we have that $\ker \varphi = H_2(G;Z_p)$.

iii) The isomorphism classes of central extensions of G by P-perfect groups in the category $E_p(G)$ are in one to one correspondence with

Z_p-submodules of $H_2(G,Z_p)$.

The proofs are the same as before.

Proposition 3. Let $H \to X \to G$ be a central extension of G . Then there is a central extension of G by $H \otimes Z_p$ together with a natural map

$$\begin{array}{ccccccccc} 0 & \to & H & \to & X & \to & G & \to & 1 \\ & & i\downarrow & & \downarrow & & \| & & \\ 0 & \to & H\otimes Z_p & \to & X_p & \to & G & \to & 1 \end{array}$$

where i is Z_p-localization ($i(a) = a\otimes 1$) .

Proof. We have a fibration

(*) $BH \to BX \to BG$.

Bousfield and Kan have introduced the fibrewise localization functor. After applying it to a fibration (*) we obtain a fibration

(**) $(BH)_p \to (BX)_p^+ \to BG$

and a fibre map of (*) into (**).
From the fibration (**) we get the following exact sequence

$$0 \to H\otimes Z_p \to \pi_1(BX_p^f) := X_p \to G \to 1 .$$

The action of $\pi_1(BG) = G$ on fibres of (*) and (**) are compatible therefore (**)is a central extension. ■
Proposition 3 is of course a special case of a more general result proved by algebraic method in [0].Proposition 1 is of course well known.The related results about "+" construction are also in A.J. Berrick "An Approach to Algebraic K-theory",Pitman research notes in Math. 56 (London,1982).

References

[0] P.Hilton, Relative nilpotent groups, Lecture Notes in Math. 915, Springer-Verlag 1982.

[1] M.A. Kervaire, Multiplicateurs de Schur et K-théorie, in Essays on Topology and Related Topics, Springer-Verlag 1970.

[2] J. Milnor, Introduction to algebraic K-theory, Princeton University Press, 1971.

[3] Z. Wojtkowiak, On fibrations which are also cofibrations, Quart. J. Math. Oxford (2), 30 (1979), 505-512.

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