ON SUBJUNCTIVE CLASSES OF GROUPS

Javier Otal

Throughout, the terminology is that of [5]. A class $\underline{X}$ of groups is said to be *subjunctive* if it is $\mathbf{S}_n$ and $\mathbf{N}_0$-closed, that is if subnormal subgroups of $\underline{X}$-groups and groups generated by finitely many normal $\underline{X}$-subgroups are in $\underline{X}$. These classes were introduced in [6] for the study of the subnormal structure of a group and have also been considered under the point of view of the dualization of Gaschütz's formation theory of finite soluble groups ([1],[3]). In the present paper we shall deal with a classification of subjunctive classes in two disjoint types, as the following result shows

<u>Theorem</u> *A subjunctive class either only contains perfect groups having no non-trivial cyclic subnormal subgroups or contains every finite p-group, for some prime p.*

Its proof needs an auxiliary result which we state in a more general form because of its own interest. We recall that an $\underline{N}_1$-group is a group in which every subgroup is subnormal; this class is not $\mathbf{D}_0$-closed and we have

<u>Lemma</u> *Let $\underline{X}$ be a subjunctive class. If $\underline{Y}$ is a $\mathbf{D}_0$-closed subclass of $\underline{N}_1$-groups then every $\underline{Y}$-group is nilpotent and $\mathbf{H}\underline{X} \cap \underline{Y} \leq \underline{X}$.*

<u>Proof</u>: First, let $G \in \underline{Y}$; then $G\times G \in \underline{Y}$ and, in particular, the diagonal subgroup D of $G\times G$ is subnormal. If $D = D_0 \triangleleft D_1 \triangleleft \dots \triangleleft D_r = G\times G$ is a subnormal chain from D to $G\times G$ and Z_i is the i^{th}-term of the upper central series of G we obtain by an easy induction that if $(x,y) \in D_i$ then $xZ_i = yZ_i$. Therefore $G = Z_r$ and then G is nilpotent.

Now, let G be an $\underline{X}$-group with a normal subgroup N such that $G/N \in \underline{Y}$.

Let $\{Z_i/N \mid o \leq i \leq c\}$ be the upper central series of G/N; in order to show that $G/N \in \underline{X}$, we may assume that $Z_i/N \in \underline{X}$ if $i < c$. Let $K = Gx(G/N)$ and denote by G_0 the image isomorphic to G in K under the map that sends x to (x,xN). If $K_i = Z_i x(Z_i/N)$ then $K_i \in \underline{X}$ if $i < c$ and, since $K_{i-1}G_0$ is normal in K_iG_0, we have by an easy induction that $K_iG_0 \in \underline{X}$ if $i < c$. But $K = (Gx1)G_0 = (Gx1)(K_{c-1}G_0) \in \underline{X}$ and then $G/N \in \underline{X}$.

Proof of Theorem: Let $\underline{X}$ be a subjunctive class. Denote by $\underline{U}$ the class of perfect groups having no non-trivial cyclic subnormal subgroups and assume that $\underline{X}$ is not contained in $\underline{U}$. Let $1 \neq G \in \underline{X}$. If G is perfect then it must contain a non-trivial cyclic subnormal subgroup, which necessarily belongs to $\underline{X}$. If G' is proper then $G/G' \in \underline{X}$ by Lemma and each one of its non-trivial cyclic subgroups belongs to $\underline{X}$. Thus $\underline{X}$ contains a non-trivial cyclic group, say H. If H is finite then there exists a prime p such that G contains a cyclic subgroup of order p, which necessarily belongs to $\underline{X}$. If H is infinite then for every prime p the group H/H^p has order p and belongs to $\underline{X}$ by Lemma. Therefore, in any case, $\underline{X}$ contains a copy of a cyclic group of order p, for some prime p. Since any finite p-group P can be subnormally embedded in a finite group Q generated by subnormal subgroups of order p (see [3] p. 204) it follows from the finiteness of Q that $Q \in \underline{X}$ ([1] X.1.d) and then $P \in \underline{X}$, as desired.

Classes of the second type are very common and it is possible to enlarge the stock of finite nilpotent groups contained in a such class obtainning similar results to those of [3] p. 204; sometimes that fact depends on the kind of cyclic groups contained in $\underline{X}$; for example if $\underline{X}$ contains an infinite cyclic group then every finite nilpotent group belongs to $\underline{X}$. On the other hand, P. Hall [2] produced examples of characteristically simple groups with trivial Baer radical; such groups are in $\underline{U}$. It is also clear that a non-abelian simple group is an $\underline{U}$-group;

since $\underline{U}$ is **P** and **D**-closed it can be proved that if $\underline{Y}$ is a class of such groups then $\mathbf{D}_0\underline{Y}$, $\mathbf{D}\underline{Y}$ and the radical class generated by $\underline{Y}$ are subjunctive classes of $\underline{U}$-groups. These groups can be taken to be torsion-free, periodic or mixed (see [4]).

We can apply our result in order to give a negative answer to the $\mathbf{N}_0$-closure of certain classes of groups. For example

Corollary 1 *The following classes are not $\mathbf{N}_0$-closed: (i) The class of groups with trivial Baer radical; (ii) Any $\mathbf{S}$-closed class of torsion-free groups; (iii) A proper and non-trivial quasivariety of groups.*

Proof: (i) Such class contains $\underline{U}$ properly. (ii) Every $\mathbf{S}$-closed subjunctive class must contain a group of order p. (iii) Let $\underline{X}$ be a non-trivial quasivariety, that is $\mathbf{S}$ and $\mathbf{R}$-closed. If $\underline{X} = \mathbf{N}_0\underline{X}$, as above, $\underline{X}$ contains every finite p-group, for some prime p, and then we may deduce that every group is in $\underline{X}$ (Apply [5] 9.11 and 8.19.2).

In contrast with (ii) we have already seen that there are subjunctive classes of $\underline{U}$-groups consisting of torsion-free groups. We also remark that (iii) can be applied to varieties of groups; since a variety is **H**-closed then it is never **P**-closed, except for trivial cases. However there exist proper and non-trivial **P**-closed quasivarieties of groups.

Finally, if $\underline{Y}$ is a class then the class $\mathbf{L}_n\underline{Y}$ is defined to be the class of all groups in which every finitely generated subgroup is contained in some subnormal $\underline{Y}$-subgroup. In [6] it was studied the behaviour of this operator with respect to $\mathbf{N}_0$-closed classes. Joinning [6] Theorem C to our Theorem we readily obtain

Corollary 2 *If $\underline{X}$ is a subjunctive class then $\mathbf{L}_n\underline{X} = \mathbf{N}\underline{X}$ is an $\mathbf{N}$-closed subjunctive class of the same type.*

Remark that, as a consequence, we have just proved that if $\underline{X}$ is contained in $\underline{U}$ then $\underline{U}$ contains $\mathbf{N}\underline{X}$ as well.

Acknowledgement. The author wishes to thank the referee for pointing out the present version of Lemma on the first page of this paper.

REFERENCES

[1] W. Gaschütz. Lectures on subgroups of Sylow type in finite soluble groups. Notes on Pure Mathematics, 11. Australian National University, Canberra 1979.

[2] P. Hall. Wreath products and characteristically simple groups. Proc. Camb. Phil. Soc. 58 (1962), 170-184.

[3] B. Hartley. On Fischer's dualization of Formation Theory. Proc. London Math. Soc. (3) 19 (1969), 193-207.

[4] A. Ju. Ol'sanskii. Infinite groups with cyclic subgroups. Soviet Math. Doklady 20 (1979), 343-346.

[5] D. J. S. Robinson. Finiteness conditions and generalized soluble groups. Springer-Verlag, Berlin 1972.

[6] J. E. Roseblade and S. E. Stonehewer. Subjunctive and Locally Coalescent classes of groups. J. Algebra 8 (1968), 423-435.

Rebut el 15 de febrer del 1985
Facultad de Matemáticas
Departamento de Algebra
Universidad de Zaragoza
50009 Zaragoza
SPAIN