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A NOTE ON PRIMITIVE GROUPS WITH SMALL MAXIMAL SUBGROUPS

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Julio Lafuente of the Universidad de Zaragoza has raised the following problem, which is of some significance to the theory of Schunck classes and their projectors in finite groups (cf. [1]):

PROBLEM. If G is a primitive group with non-abelian minimal normal subgroup $S(G)$ and maximal subgroup H such that $G = HS(G)$, does it necessarily follow that $H \cap S(G) \neq 1$?

Here a primitive (finite) group is one which possesses a maximal subgroup with trivial core (i.e., a maximal subgroup H of G satisfying $\mathrm{Core}_G(H) = 1$, where $\mathrm{Core}_G(H) = \bigcap_{g\in G} H^g$ is the most comprehensive normal subgroup of G contained in H); further, $S(G)$ denotes the socle of G, the product of all minimal normal subgroups.

The purpose of this note is to provide a rather large class of examples to answer Lafuente's question in the negative.

PROPOSITION. Let H be a finite group with a non-abelian simple subgroup E such that $\mathrm{Core}_H(E) = 1$ and $E \leq K$ whenever $K \neq 1$ is a subgroup of H normalised by E.

Put $N = N_H(E)$ and define a group

$$G = E \wr_N H$$

to be the twisted wreath product with respect to the action of N on E given by viewing N to be a subgroup of $\mathrm{Aut}(E)$ in the obvious way.

Then G is a primitive group with $S(G) = E^*$, the base group, being non-abelian and minimal normal in G, and $S(G)$ is complemented in G by a maximal subgroup isomorphic with H.

P r o o f. The reader is referred to [3], I.15.10, for a definition of twisted wreath product. However, we shall find it more convenient to adopt the notation from [2]. All of the information on twisted wreath products we shall need in the sequel is included in the following statements:

G is a semi-direct product of H and $E^* = E_1 \times \ldots \times E_n$, where $n = |H:N|$ and $E_i \cong E$ $(i = 1,\ldots,n)$; the action of H upon E^* is such that

(1) E_1 is normalised by N (in fact, $N_H(E_1) = N$), and NE_1 is isomorphic with a subgroup of Aut(E)E containing Inn(E)E, where N is supposed to be embedded in Aut(E); observe that this is possible, as $C_H(E) = 1$ follows from the hypothesis that E is contained in every non-trivial subgroup of H which is normalised by E;

(2) if $\{1,\ldots,n\}$ represents the set of cosets of N in H, with 1 corresponding to N, then any $h \in H$ permutes the elements of $\{E_1,\ldots,E_n\}$ in the same way as it acts on $\{1,\ldots,n\}$ (or, by means of right multiplication, on the set of cosets of N in H); in particular, the components E_i of E^* are permuted transitively by H.

To begin with the proof of our assertions on G, we observe that from transitivity of the action of H upon $\{E_1,\ldots,E_n\}$ (see (2)) together with the fact that this set comprises all minimal normal subgroups of E^*, we get immediately that E^* is minimal normal in G. Moreover, from $\mathrm{Core}_H(N) = 1$ (this being a consequence of $\mathrm{Core}_H(E) = 1$, $N = N_H(E)$, and $S(N) = E$) we have that the permutation representation of H on $\{E_1,\ldots,E_n\}$ is faithful. Hence we may derive from (2) and $Z(E) = 1$ that $E^* = S(G)$ is the unique minimal normal subgroup of G; cf. [2] for a similar argument.

It remains to show that H is a maximal subgroup of G. To this end we assume that

$$H \leq X < G.$$

Then $X = G \cap X = HE^* \cap X = HY$, where $Y = E^* \cap X \trianglelefteq X$. Thus we have to prove that $Y = 1$. Aiming for a contradiction we assume that $Y \neq 1$. Then we let

$$\pi_i : E^* \longrightarrow E_i \quad (i = 1,\ldots,n)$$

be the projection of $E_1 \times \ldots \times E_n$ onto E_i; and we consider the subgroup $Y_i = Y^{\pi_i}$ of E_i. Since $Y \trianglelefteq X = HY$, it is inferred from (2) that $Y_i \cong Y_j$ for all $i,j \in \{1,\ldots,n\}$; moreover, Y is a subdirect subgroup of $Y_1 \times \ldots \times Y_n$ and

$$(Y \cap E_1) \times \ldots \times (Y \cap E_n) \leq Y \leq Y_1 \times \ldots \times Y_n.$$

Recall from (1) that E_1 is invariant under the action of $N \leq H$. It is obvious that π_1 commutes with this action, and as E normalises Y, we deduce that E normalises $Y_1 = Y^{\pi_1} \leq E_1$. Since the automorphisms induced in E_1 by E are precisely the inner automorphisms of E_1, we see that either $Y_1 = 1$ or $Y_1 = E_1$. Hence we are left to exclude the latter possibility. In this case our observations in the preceding paragraph show that

$$Y_i = E_i \text{ for } i = 1,\ldots,n.$$

Now, $Y_1 = E_1$ says that for each $e_1 \in E_1$ there exist $e_2 \in E_2, \ldots, e_n \in E_n$ such that $e_1e_2 \ldots e_n \in Y$. Therefore we may consider two cases as follows.

C a s e 1. For each $e_1 \in E_1$ there is exactly one $(n-1)$-tuple $(e_2, \ldots, e_n) \in E_2 \times \ldots \times E_n$ such that $e_1e_2 \ldots e_n \in Y$.

Clearly, $|Y| = |E|$; indeed, we see that π_1 induces an isomorphism from Y to E_1. Consequently,

$$X = HY \text{ with } E \cong Y \trianglelefteq X,$$

and so there is a homomorphism

$$\tau: H \longrightarrow \mathrm{Aut}(E),$$

where by construction of the twisted wreath product $G = E \,\wr_N\, H$ the subgroup N of H is faithfully represented (by means of τ) as a group of automorphisms of E: note that $Y_1 = E_1$. If K denotes the kernel of τ, then $E \not\leq K$, whence the hypothesis on the embedding of E in H forces $K = 1$. Then $E^{\tau} = \mathrm{Inn}(E) \trianglelefteq \mathrm{Aut}(E) \geq H^{\tau}$ gives $E \trianglelefteq H$, contradicting the hypothesis that $\mathrm{Core}_H(E) = 1$.

In what follows we shall not actually make use of the hypothesis of Case 2, but shall need the conclusion $N \trianglelefteq H$, which emerged at the end of Case 1, in a slightly more general setting. (We have preferred to consider these two cases for the convenience of the reader only.)

C a s e 2. There is some $e_1 \in E_1$ such that there exist $e_2, e_2' \in E_2, \ldots, e_n, e_n' \in E_n$ with $e_1e_2 \ldots e_n, e_1e_2' \ldots e_n' \in Y$ and $(e_2, \ldots, e_n) \neq (e_2', \ldots, e_n')$.

In the sequel we shall tacitly assume whenever an equation such as $y = f_1 \ldots f_n$ for some $y \in Y$ is written that the f_i are elements of E_i $(i = 1, \ldots, n)$.

Multiplying $e_1e_2 \ldots e_n$ by $(e_1e_2' \ldots e_n')^{-1}$ yields an element

$$y^* = 1 \cdot e_2(e_2')^{-1} \cdot \ldots \cdot e_n(e_n')^{-1} \in Y.$$

Taking into account that Y is normalised by N and that H acts transitively upon $\{E_1, \ldots, E_n\}$, we see that firstly, there is some $y' = f_1' \ldots f_n' \in Y$ such that $f_1' \neq 1$ and $f_k' = 1$ for at least one $k \in \{2, \ldots, n\}$ and secondly, given that y' is as above $f_j' \neq 1$ for $j = 1$ only cannot possibly hold: in fact, in this case $Y \cap E_1 \neq 1$ would lead to $Y \cap E_1 = E_1$ and $E^* \leq Y$ (contradicting $HY = X < G = HE^*$), since $Y \cap E_1$ is normalised by a subgroup of $\mathrm{Aut}(E_1)$ containing all inner automorphisms (namely, by $N \geq E$).

With each $y = f_1 \ldots f_n \in Y$ there is associated the support of y

$$\sigma(y) = \sigma(f_1 \ldots f_n) = \{k \in \{1, \ldots, n\} \mid f_k \neq 1\}.$$

We choose $y = f_1 \ldots f_n \in Y \setminus \{1\}$ such that $|\sigma(y)|$ is minimal. W.l.o.g. $f_1 \neq 1$. By what has been said above, $\Delta = \sigma(y)$ is a proper subset of $\{1, \ldots, n\}$. Put $d = |\Delta|$ and let

$$H_\Delta = \{h \in H \mid \delta^h \in \Delta \text{ for each } \delta \in \Delta\} \leq H$$

be the stabiliser in H of the set Δ.

We claim that $N \leq H_\Delta$. By way of contradiction, suppose that $N_\Delta = N \cap H_\Delta <$ N. Then for any $x \in N \setminus N_\Delta$ and for any $y^* = g_1^* \dots g_n^* \in Y$ such that $\sigma(y^*) = \Delta$ we calculate

$$[y^x, y^*] = [f_1{}^x f_2^* \dots f_n^*, g_1^* \dots g_n^*]$$
$$= [f_1{}^x, g_1^*][f_2^*, g_2^*] \dots [f_n^*, g_n^*],$$

where $f_1{}^x \in E_1$, $f_i^* \in E_i$ $(i = 2, \dots, n)$ and $f_i^* \neq 1$ implies $i \in (\Delta \setminus \{1\})^x$. Since y was chosen with $|\Delta| = |\sigma(y)|$ being minimal, $\Delta \cap \Delta^x \subset \Delta$ forces the above commutator to be 1: its non-trivial j-th entries can occur only in places j from $\Delta \cap \Delta^x$. Then, in particular, we have

$$[f_1{}^x, g_1^*] = 1.$$

Elements $y^* = g_1^* \dots g_n^* \in Y$ with the above-mentioned property can be found as conjugates of y by elements of N_Δ. Thus we get

$$[f_1{}^x, f_1{}^z] = 1 \text{ whenever } x \in N \setminus N_\Delta \text{ and } z \in N_\Delta,$$

and this may be reformulated in terms of subgroups A and B of E_1 as follows:

$$[A,B] = 1, \text{ where}$$
$$A = \langle f_1{}^x \mid x \in N \setminus N_\Delta \rangle \leq E_1 \text{ and } B = \langle f_1{}^z \mid z \in N_\Delta \rangle \leq E_1.$$

The group of automorphisms induced in E_1 by N, however, contains all inner automorphisms of E_1, and so $E_1 = \langle f_1{}^m \mid m \in N \rangle$ follows from simplicity of E_1 together with $f_1 \neq 1$. From the definitions of A and B it is obvious that these groups are non-trivial subgroups of E_1, generating the whole of E_1, and centralising each other. This is clearly impossible with a non-abelian simple group E_1, and our assertion that $N = N_\Delta \leq H_\Delta$ holds.

Now we contemplate the group

$$H_\Delta[H_\Delta, y],$$

where $H_\Delta \leq H$ and $[H_\Delta, y] \leq Y$ is invariant under the action of H_Δ: note that

$$\sigma(y^\#) = \sigma(y) \text{ for each non-trivial } y^\# \in [H_\Delta, y].$$

Knowing that $N \leq H_\Delta$, we may repeat the arguments from the first part of this proof to conclude that $[H_\Delta, y] \cong E$ by means of (an obvious modification of the arguments in) Case 1: observe that $E \leq N \leq H_\Delta$ and that Δ was chosen with minimal cardinality. Then from $N_H(E) = N \leq H_\Delta \leq N_H(E)$ (see Case 1) we get

$$H_\Delta = N.$$

We aim to show that d = 1, that is to say, $\Delta = \{1\}$, which already has been seen to yield a contradiction. Assume that $d > 1$. As before, the N-invariant subgroup $D = [H_\Delta, y]$ is N-isomorphic with E and may be written as

$$D = \{(e_1, e_2, \dots, e_n) \mid e_1 \in E_1\};$$

here $(e_2, \dots, e_n) \in E_2 \times \dots \times E_n$ is determined uniquely by $e_1 \in E_1$, and is such that $e_i \neq 1$ iff $i \in \Delta$ for any $(e_1,\ldots,e_n) \in D\setminus\{1\}$; moreover, the map given by

$$E_1 \ni e_1 \mapsto e_i \in E_i,\ (e_1,\ldots,e_n) \in D,$$

is an isomorphism. Therefore, by calculating commutators $[a,b^x]$ ($a,b \in D$, $x \in H$) in much the same way as previously, we obtain that for no element $x \in H$ we can have $\emptyset \subset \Delta \cap \Delta^x \subset \Delta$; here one has to appeal to the choice of Δ once more. Since $|H:H_\Delta| = |H:N| = n$, we find that

$$\{1,\ldots,n\} = \dot{\bigcup}_{i=1}^{n} \Delta^{h_i}$$

is a disjoint union, when a decomposition of $H = \dot{\bigcup}_{i=1}^{n} H_\Delta h_i$ with $h_i \in H$ as a disjoint union of right cosets is given. Hence $d = 1$, as desired, and our proof is complete. □

EXAMPLES of pairs E,H satisfying the hypothesis of our Proposition may be found by considering, e.g.,

$E = A_n$, $H = A_{n+1}$ ($n \geq 5$) [where E is maximal in H], or

$E = PSL(2,p^n)$, $H = PSL(2,p^{2n})$ ($p^n \geq 3$) [where $N_H(E) \cong PGL(2,p^n)$ is maximal in H].

Apparently one can find a group $H \gneq E$ such that the pair E,H satisfies the above-mentioned properties for many, if not all, non-abelian finite simple groups E. Nevertheless we have not been able to prove a statement of this sort; of course, one would be interested in a proof not relying on the classification of all finite simple groups. More specifically, one might wonder whether the embedding of E into an alternating group, which arises from the permutation representation of E on the cosets of a suitable maximal subgroup of E, yields a pair E,H to which the Proposition can be applied.

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