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BEST APPROXIMATION IN METRIC SPACES

T. D. Narang *

1. Introduction. The notion of strict convexity and uniform convexity in normed linear spaces was extended to metric spaces in [1] and certain existence and uniqueness theorems on best approximation were proved in these spaces in [1] and [2]. In this note we shall give a relationship between the two types of convexities in metric spaces and further discuss some results on best approximation in metric spaces. We shall also extend the notion of sun introduced in normed linear spaces by Efimov and Steckin [3] to metric spaces.

2. Strictly Convex And Uniformly Convex Metric Spaces.

If x,y,z are any three points in a metric space (X,d) then z is said to be a *point between* x and y if

$$d(x,z) + d(z,y) = d(x,y).$$

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Further, z is said to be a *mid-point* of x and y if

$$d(x,z) = d(z,y) = \frac{1}{2} d(x,y).$$

A metric d defined on X is said to be a *convex metric* if for each pair x,y in X, d determines at least one mid-point and d is said to be a *strongly convex metric* if for every pair it determines a unique mid-point.

A metric space (X,d) is said to be a *convex metric space* or a *strongly convex metric space* according as the metric d is convex or strongly convex.

A strongly convex metric space (X,d) is said to be *strictly convex* if $d(x,x_o) \leqslant r$, $d(y,x_o) \leqslant r$ imply $d(z,x_o) < r$, unless $x = y$, where x_o is arbitrary but fixed point of X, z is the mid-point of x and y, and r is any finite real number.

A strongly convex metric space (X,d) is said to be *uniformly convex* if there corresponds to each pair of positive numbers (ϵ, r) a positive number δ such that $d(x,y) < \epsilon$ whenever $d(x,x_o) \leqslant r$, $d(y,x\ ) \leqslant r$, $d(z,x_o) > r - \delta$, z being the mid-point of x and y and the other points being arbitrary.

A metric space (X,d) is said to be *totally complete* if every bounded closed subset of X is compact.

As is easy to see, a compact space is totally complete but a totally complete space need not be compact e.g. the real line with the usual metric.

It was shown in [1] that every uniformly convex metric space is strictly convex and a compact strictly convex metric space is uniformly convex. However, we have:

Theorem 1. Every totally complete strictly convex metric space is uniformly convex.

The proof given in [1] works in this situation too as the set

$$S = \{ \langle x,y \rangle : d(x,x_o) \leqslant r,\ d(y,x_o) \leqslant r,\ d(x,y) \geqslant \epsilon \}$$

is closed as well as bounded in the totally complete space $X \times X$ and so compact.

3. Best Approximation Map In a Metric Space.

Given a subset K of a metric space (X,d) and $x \in X$, a point $y_o \in K$ such that $d(x,y_o) = d(x,K)$ is called a *point of best approximation* to x in K.

The mapping π_k which takes each point of the space to those points of K which are nearest to it, is called the *metric projection*.

We shall denote by $\pi_K(x)$ the set of best approximation elements of x in K i.e.

$$\pi_K(x) = \{k \in K : d(x,k) = d(x,K)\}.$$

The set K is said to be *proximinal* if each point of X has a best approximation in K and it is said to be *Chebyshev* if each point of X has a unique best approximation in K.

For Chebyshev sets the metric projection is single-valued.

Theorem 2. If G is a Chebyshev subset of a metric space (X,d) then $\pi_G(z) = \pi_G(x)$ where $z \in X$ is any element between x and $\pi_G(x)$.

Proof. By the definition of z,

$$d(x,z) + d(z,\pi_G(x)) = d(x,\pi_G(x)).$$

Let $g \in G$. Then

$$d(x,z) + d(z,g) \geqslant d(x,g)$$

implies

$$\begin{aligned} d(z,g) &\geqslant d(x,g) - d(x,z) \\ &\geqslant d(x,\pi_G(x)) - d(x,z) \\ &= d(z,\pi_G(x)) \end{aligned}$$

i.e. $d(z,\pi_G(x)) \leqslant d(z,g)$ for all $g \in G$ and so $\pi_G(x) \in G$ is a best approximation to $z \in X$. Since G is Chebyshev, $\pi_G(x) = \pi_G(z)$.

Remark 1. This result is analogous to the following result proved in normed linear spaces by M. Nicolescu (see Lemma 2.1[4], p. 364):

Let E be a normed linear space and G a Chebyshev set in E, then $\pi_G[\alpha x + (1-\alpha)\pi_G(x)] = \pi_G(x)$, $x \in E$, $0 \leqslant \alpha \leqslant 1$.

Remark 2. The concept of a 'sun' was introduced in approximation theory by Efimov and Steekin [3] as:

A Chebyshev subset G of a normed linear space E is called a *sun* if we have

$$\pi_G[\alpha x + (1-\alpha)\pi_G(x)] = \pi_G(x),\ x \in E,\ \alpha \geqslant 0$$

i.e. if $\pi_G(x) \in G$ is best approximation to $x \in E$ then $\pi_G(x)$ is also a best approximation to all points on the ray $\overrightarrow{\pi_G(x)X}$ As is easy to see, this concept is meaningful in any linear metric space.

Motivated by Theorem 2, we now extend the notion of sun to any metric space (X,d).

A point $z \in X$ is said to be on the *ray* $\overrightarrow{xy}$ if either

z is between x and y or y is between x and z i.e. either $d(x,y) = d(x,z) + d(z,y)$ or $d(x,z) = d(x,y) + d(y,z)$.

A Chebyshev set G in a metric space (X,d) is called a *sun* if for each $x \in X$, $\pi_G(z) = \pi_G(x)$ for every z on the ray $\overrightarrow{\pi_G(x)X}$.

It will be interesting to study suns in metric spaces.

<u>Theorem 3</u>. If (X,d) is a metric space, G a subset of X and $g_o \in G$, then $\pi_G^{-1}(g_o) = \{x \in X : d(x,g_o) = d(x,G)\}$ is closed and $x_o \in \pi_G^{-1}(g_o) \Rightarrow z \in \pi_G^{-1}(g_o)$ for every z between x_o and g_o.

<u>Proof</u>. $\pi_G^{-1}(g_o) = \{x \in X : d(x,g_o) = d(x,G)\}$

$$= \{x \in X : d(x,g_o) \leq d(x,g) \text{ for all } g \in G\}$$

$$= \bigcap_{g \in G} \{x \in X : d(x,g_o) \leq d(x,g)\}.$$

The closedness of $\pi_G^{-1}(g_o)$ now follows from the continuity of d.

Now $x_o \in \pi_G^{-1}(g_o) \Rightarrow d(x_o,g_o) \leq d(x_o,g)$ for all $g \in G$. Since z is between x_o and g_o,

$$d(x_o,z) + d(z,g_o) = d(x_o,g_o).$$

Write

$$d(z,g) \geqslant d(g,x_o) - d(x_o,z) \quad \text{for all} \quad g \in G$$

$$\geqslant d(x_o,g_o) - d(x_o,z)$$

$$= d(z,g_o)$$

i.e. $d(z,g_o) \leqslant d(z,g)$ for all $g \in G$

i.e. $z \in \pi_G^{-1}(g_o)$.

Remark 1. This result is analogous to the following result proved in normed linear spaces (see [4]. p.143 and p.354):

Let X be a normed linear space, G a linear subspace of X and $g_o \in G$. Then the set $\pi_G^{-1}(g_o)$ is closed and

$$x \in \pi_G^{-1}(g_o) \Rightarrow \alpha x + (1-\alpha)\, g_o \in \pi_G^{-1}(g_o),\ 0 \leqslant \alpha \leqslant 1.$$

Remark 2. If G is a Chebyshev set in X then Theorem 3 gives that $\pi_G^{-1}(\pi_G(x))$ is closed for every $x \in X$.

If f is a mapping from a non-empty set X into a non-empty set Y then the *graph* of f is the set

$$G(f) = \{(x, f(x)) : x \in X\}.$$

It is well known that for continuous mappings in metric spaces the graph is closed. It is also well known that for Chebyshev sets the metric projection need not necessarily be continuous. However, we have:

<u>Theorem 4</u>. If K is a Chebyshev set in a metric space (X,d) then the graph of the metric projection π_K is closed.

<u>Proof</u>. $G(\pi_K) = \{(x, \pi_K(x)) : x \in X\}$.

Let (y,z) be a limit point of $G(\pi_K)$. Then there exists a sequence $\langle (y_n,\pi_K(y_n)) \rangle$ in $G(\pi_K)$ such that

$$(y_n,\pi_K(y_n)) \longrightarrow (y,z).$$

This implies, $y_n \longrightarrow y$ and $\pi_K(y_n) \longrightarrow z$.

Consider

$$\begin{aligned} |d(y_n,\pi_K(y_n)) - d(y,\pi_K(y))| &= |d(y_n,K) - d(y,K)| \\ &\leq d(y_n,y). \end{aligned}$$

So, $d(y_n,\pi_K(y_n)) \leq d(y_n,y) + d(y,\pi_K(y))$ implies

$$\begin{aligned} d(y,z) &\leq d(y,y_n) + d(y_n,\pi_K(y_n)) + d(\pi_K(y_n),z) \\ &\leq 2d(y,y_n) + d(y,\pi_K(y)) + d(\pi_K(y_n),z). \end{aligned}$$

This implies

$$d(y,z) \leqslant d(y,\pi_K(y)).$$

Since K is closed, $z \in K$ and so $d(y,z) \geqslant d(y,\pi_K(y))$. Thus $d(y,z) = d(y,\pi_K(y))$. Since K is Chebyshev, $z = \pi_K(y)$ i.e. $(y,z) \in G(\pi_K)$ and so $G(\pi_K)$ is closed.

REFERENCES

[1] G.C. Ahuja, T.D. Narang and Swaran Thehan; Best Approximation on Convex Sets in a Metric Space, J. Approximation Theory 12(1974), 94-97.

[2] G.C. Ahuja, T.D. Narang and Swaran Thehan; Certain Results on Best Approximation and Continuity of Metric Projections, Nanta Mathematica 11(1977), 103-107.

[3] N.V. Efimov and S.B. Steckin; Some Properties of Chebyshev Sets, Doklady Akad. Naud SSSR 118 (1958), 17-19.

[4] Ivan Singer; Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces, Springer-Verlag, 1970.

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Department of Mathematics
Guru Nanak Dev University,
Amritsar - 143005 (India)