

## AUTOMORPHISMS OF THE POLYNOMIAL RING IN TWO VARIABLES\*

Warren Dicks

Let  $k$  be a field,  $k[x,y]$  the polynomial ring in two variables, and  $\text{Aut } k[x,y]$  the group of all its  $k$ -algebra automorphisms. Such an automorphism will be denoted by the ordered pair  $(p,q)$  where  $p,q \in k[x,y]$  are the respective images of  $x,y$ .

THEOREM. The group  $\text{Aut } k[x,y]$  is generated by  $(y,x)$ ,  $(x,y-\mu x^n)$   $\mu \in k$ ,  $n \geq 0$ .

Moreover  $\text{Aut } k[x,y] = A *_{\mathcal{C}} B$  where

$$A = \{(\lambda_{11}x + \lambda_{12}y + \lambda_1, \lambda_{21}x + \lambda_{22}y + \lambda_2) \mid \lambda_{11}\lambda_{22} \neq \lambda_{21}\lambda_{12}\},$$

$$B = \{(\lambda_{11}x + \lambda_1, \lambda_{22}y + f(x)) \mid \lambda_{11}\lambda_{22} \neq 0, f(x) \in k[x]\},$$

$$\mathcal{C} = A \cap B = \{(\lambda_{11}x + \lambda_1, \lambda_{21}x + \lambda_{22}y + \lambda_2) \mid \lambda_{11}\lambda_{22} \neq 0\}.$$

The elements of  $A$  are called affine automorphisms, the elements of  $B$  de Jonquières automorphisms, and the elements of the subgroup generated by  $A \cup B$  are called tame automorphisms. The fact that all  $k$ -algebra automorphisms of  $k[x,y]$  are tame was proved by Jung [2] for  $\text{char } k = 0$ , and then by Van der Kulk [8] in the general case. From their work the coproduct decomposition follows fairly easily, but it is not clear who first made the observation. (Kambayashi [3] gives the credit to Shafarevitch [7].)

Rentschler [5] gave a very simple proof of tameness for  $\text{char } k = 0$ , and then along slightly different lines Makar-Limanov [4] gave a fairly simple proof for arbitrary characteristic. (News of Van der Kulk's result seems not to have reached Moscow at that time, for Makar-Limanov refers to the result as

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unpublished work of Shafarevitch.) In the spirit of Serre [6], Roger Alperin [1] gave an explicit example of a tree acted on by  $\text{Aut } k[x,y]$  from which the coproduct decomposition can be read off.

In §1 below we give a modified version of Makar-Limanov's proof, and in §2 recall Alperin's example.

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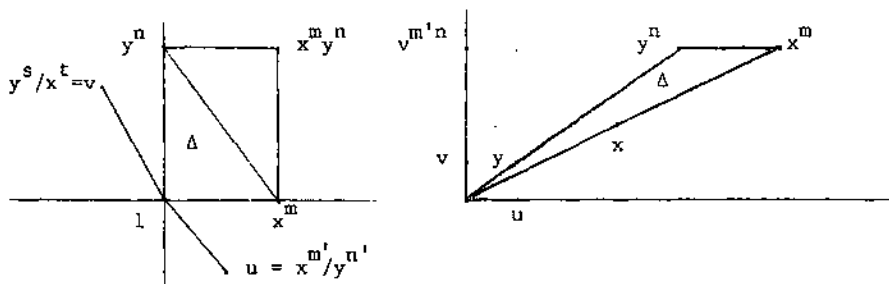
### §1 The support of a primitive element

Let  $(f,g)$  be an automorphism of  $k[x,y]$ . We can write  $f = \sum \lambda_{ij} x^i y^j$ ,  $\lambda_{ij} \in k$  and define  $\text{supp}(f) = \{x^i y^j \mid \lambda_{ij} \neq 0\} \subseteq \langle x,y \rangle$ , where  $\langle x,y \rangle$  is the free abelian group generated by  $x,y$ . Let  $m = x\text{-deg}(f)$ ,  $n = y\text{-deg}(f)$ , that is,  $m$  is the highest exponent of  $x$  occurring in  $\text{supp}(f)$ , and similarly for  $n$ . Set  $\Delta = \{x^i y^j \mid ni + mj \leq mn, i \geq 0, j \geq 0\} \subseteq \langle x,y \rangle$ . Geometrically,  $\text{supp}(f)$  lies in the rectangle determined by  $1, x^m, x^m y^n, y^n$  and  $\Delta$  occupies the triangle determined by  $1, x^m, y^n$ .

The objective of this section is to show  $x^m, y^n \in \text{supp}(f) \subseteq \Delta$  and  $m \mid n$  or  $n \mid m$ .

If  $mn = 0$  this is clear.

Thus we may assume  $mn > 0$ . Let  $m' = m/(m,n)$ ,  $n' = n/(m,n)$ . These are coprime natural numbers, so we can choose natural numbers  $s,t$  such that  $sm' - tn' = 1$ . Let  $u = x^{m'}/y^{n'}$ ,  $v = y^s/x^t$  in  $\langle x,y \rangle$  so  $x = u^s v^{n'}$ ,  $y = u^t v^{m'}$ .



Thus  $k[x, y] \subseteq k[u, v]$  and we can write  $f = \sum \mu_{ij} u^i v^j$  so  $\text{supp}(f) = \{u^i v^j \mid \mu_{ij} \neq 0\}$ . We define the leading  $v$ -component of  $f$  to be  $|f| = (\sum_i \mu_{ij} u^i) v^j \in k[u]^{\times} \times \langle v \rangle$  where  $j = v\text{-deg}(f)$ . If then  $u\text{-deg}(|f|) = i$  we define  $\|f\| = u^i v^j \in \langle u, v \rangle$  called the leading term of  $f$ . This extends to a group homomorphism

$\| \cdot \|: k(u, v)^{\times} \rightarrow \langle u, v \rangle$ . (Notice the superscript  $\times$  is being used to denote the set of nonzero elements.) The following statement indicates the steps in Makar-Limanov's argument.

**THEOREM 1.** (i) There exist  $\alpha, \beta \in k(u)^{\times} \times \langle v \rangle \subseteq k(u, v)^{\times}$  such that

$$|f| = \lambda \alpha^a \quad (\lambda \in k^{\times}, a \in \mathbb{N}^+) \text{ and } x, y \in k[\alpha^{\pm 1}, \beta].$$

(ii) There then exist  $w, z \in \langle u, v \rangle$  such that  $\langle w \rangle = \langle \| \alpha \| \rangle$  or  $\langle \| \alpha \|, \| \beta \| \rangle$  and  $x, y \in \text{semigrp} \langle w^{\pm 1}, z \rangle$ .

(iii) Then  $x^m, y^n \in \text{supp}(f) \subseteq \Delta$  and  $\|f\| = x^m$  and  $\langle w \rangle = \langle x \rangle$ .

(iv) If  $\langle w \rangle = \langle \| \alpha \| \rangle$  then  $m | n$ .

(v) If  $\langle w \rangle = \langle \| \alpha \|, \| \beta \| \rangle$  then  $n | m$ .

**PROOF.** (i) Let  $K = k(u)$  and consider the Laurent series field  $K((v^{-1}))$ . In a natural way  $k(u, v) \subseteq K((v^{-1}))$  and there are maps  $v\text{-deg}: K((v^{-1}))^{\times} \rightarrow \mathbb{Z}$ ,  $| \cdot |: K((v^{-1})) \rightarrow K^{\times} \times \langle v \rangle$  extending the corresponding maps on  $k[u, v]$ . We view  $k^{\times}$  as a subgroup of  $K^{\times} \times \langle v \rangle \subseteq K((v^{-1}))^{\times}$ . Since  $v\text{-deg}(f) > 0$  there exists  $\alpha \in K^{\times} \times \langle v \rangle$  such that the image of  $\alpha$  in  $(K^{\times} \times \langle v \rangle) / k^{\times}$  generates a maximal cyclic subgroup containing the image of  $|f|$ , say  $|f| = \lambda \alpha^a \quad \lambda \in k^{\times}, a \in \mathbb{N}^+$ . By induction on  $a$  we shall show that for any  $f, g \in K((v^{-1}))$  with  $|f| = \lambda \alpha^a \quad \lambda \in k^{\times}, a \in \mathbb{N}^+$  there exists  $\beta \in K^{\times} \times \langle v \rangle$  such that  $|k[f^{\pm 1}, g]| \subseteq k[\alpha^{\pm 1}, \beta]$ .

The case  $a = 0$  is vacuous.

Let us now define a (possibly finite) sequence inductively. Let  $g_1 = g$ . Suppose we have  $g_i$  for some  $i \geq 1$ . If  $|g_i| = \lambda_1 \alpha^{a_1 i}$  for some  $\lambda_1 \in k^{\times}, a_1 \in \mathbb{Z}$  we set  $g_{i+1} = g_i^{-\lambda_1} f^{n_1}$ ; if  $g_i = 0$  or  $g_i \neq 0$  and  $|g_i|$  is not

of this form we let the sequence end at  $g_1$ . Since  $v\text{-deg}(g_1) > v\text{-deg}(g_2) > \dots$  the sequence  $g_1, g_2, \dots$  has a limit  $g_*$  in  $K((v^{-1}))$ ,  $g_* = g^{-\lambda_1} f^{n_1} - \lambda_2 f^{n_2} - \dots$ .

If  $g_* = 0$  then  $k[f^{\pm 1}, g] \subseteq k((f^{-1}))$  so  $|k[f^{\pm 1}, g]^{\times}| \subseteq |k((f^{-1}))^{\times}| \subseteq k[f^{\pm 1}] \subseteq k[\alpha^{\pm 1}]$  and we can take  $\beta$  arbitrary.

Thus we may assume  $g_* \neq 0$  so the sequence is finite and  $k[f^{\pm 1}, g] \subseteq k[f^{\pm 1}, g_*]$ .

If  $|f|, |g_*|$  are algebraically independent over  $k$  then it is easy to see  $|k[f^{\pm 1}, g_*]^{\times}| \subseteq k[|f|^{\pm 1}, |g_*|]$  and we can take  $\beta = |g_*|$ .

This leaves the case where  $|f|, |g_*|$  are algebraically dependent over  $k$ . If  $c = v\text{-deg}(f)$ ,  $d = v\text{-deg}(g_*)$  then  $|f|^d, |g_*|^c$  are algebraically dependent over  $k$  and are  $v$ -homogeneous with the same  $v$ -degree. It follows that  $|f|^d/|g_*|^c$  lies in  $K$  and is algebraic over  $k$  so lies in  $k$ . Thus  $|g_*|^c \equiv |f|^d \equiv \alpha^{\text{ad}} \pmod{k^{\times}}$ . But  $(K^{\times} \times \langle v \rangle)/k^{\times}$  is a torsion-free abelian group, and the image of  $\alpha$  generates a maximal cyclic subgroup, so  $c|ad$  and  $|g_*|^c \equiv \alpha^b \pmod{k^{\times}}$  where  $b = ad/c$ . Say  $|g_*| = \mu \alpha^b$ ,  $\mu \in k^{\times}$ . By the definition of  $g_*$  we know  $a|b$ , say  $b = aq+r$   $0 < r < a$ . Let  $h = g_*/f^q$ . Then  $|h| \equiv \alpha^r \pmod{k^{\times}}$  and the induction hypothesis applies to the pair  $(h, f)$ . Hence there exists  $\beta \in K^{\times} \times \langle v \rangle$  such that  $|k[h^{\pm 1}, f]^{\times}| \subseteq k[\alpha^{\pm 1}, \beta]$ . Now  $|k[f^{\pm 1}, g]^{\times}| \subseteq |k[f^{\pm 1}, h]^{\times}| \subseteq |k[f, h]^{\times}| \langle |f| \rangle \subseteq k[\alpha^{\pm 1}, \beta]$ . By induction  $|k[f^{\pm 1}, g]^{\times}| \subseteq k[\alpha^{\pm 1}, \beta]$  for some  $\beta \in k(u)^{\times} \times \langle v \rangle$ , and (i) is proved since  $x, y \in |k[f, g]^{\times}|$ .

(ii) Recall that two elements of  $\langle u, v \rangle$  are said to be dependent if they generate a cyclic subgroup, and otherwise they are independent, that is, freely generate a free abelian subgroup. If  $\|\alpha\|, \|\beta\|$  are independent then it is clear that  $x, y \in \|k[\alpha^{\pm 1}, \beta]^{\times}\| \subseteq \text{semigrp}\langle \|\alpha\|^{\pm 1}, \|\beta\| \rangle$  and we can take  $w = \|\alpha\|$ ,  $z = \|\beta\|$ . This leaves the case where  $\|\alpha\|, \|\beta\|$  are dependent. Let  $w$  be a generator of  $\langle \|\alpha\|, \|\beta\| \rangle$ , say  $\|\alpha\| = w^i$ ,  $\|\beta\| = w^j$ ,  $w = \|\alpha\|^d \|\beta\|^d$ . Here

$\|\alpha^j\| = \|\beta^i\| = w^{ij}$  so there is a unique  $\mu \in k^x$  such that  $z = \|\alpha^j - \mu\beta^i\| \neq w^{ij}$ .

But  $z$  and  $w^{ij}$  have the same  $v$ -degree so  $w, z$  are independent. Let  $\alpha' = \alpha^c \beta^d$ ,  $\beta' = \alpha^j / \beta^i - \mu$ . Then  $\|k[\alpha^{\pm 1}, \beta^{\pm 1}]^x\| = \|k[\alpha', \beta']^x\| \subseteq \|k[\alpha', \beta']^x\| \langle w \rangle \subseteq \text{semigrp} \langle w^{\pm 1}, z \rangle$ . Thus  $x, y \in \text{semigrp} \langle w^{\pm 1}, z \rangle$  and  $\langle w \rangle = \langle \|\alpha\|, \|\beta\| \rangle$ .

(iii) Geometrically  $x, y \in \text{semigrp} \langle w^{\pm 1}, z \rangle$  means that one of the two half-planes determined by  $w$  contains both  $x$  and  $y$ . Now by (ii)  $\|\alpha\| = w^i$  for some integer  $i$  and on replacing  $w$  with  $w^{-1}$  if necessary we may assume  $i \geq 0$ . By (i),

$\|f\| = \|\alpha\|^a = w^{ia}$  and  $\|f\| \in \text{semigrp} \langle x, y \rangle$  so  $w \in \text{semigrp} \langle x, y \rangle$ . The only way this can happen is for  $w$  to lie along the  $x$  or  $y$  axis, that is,  $w$  is a power of  $x$  or  $y$ . But  $\langle w, z \rangle \supseteq \langle x, y \rangle$  so  $w$  is  $x$  or  $y$ . Thus  $\|f\|$  is a power of  $x$  or  $y$ . But

the only place  $\text{supp}(f)$  meets the  $x$  or  $y$  axes is in  $\Delta$  so  $\|f\| \in \Delta$  and this forces  $\text{supp}(f) \subseteq \Delta$ . The only way  $x\text{-deg}(f)$  can be  $m$  is for  $x^m$  to be in

$\text{supp}(f)$ , and similarly  $y^n \in \text{supp}(f)$ . Thus  $\|f\| = x^m$  or  $y^n$ . But  $u\text{-deg}(x^m) = u\text{-deg}(u^{ms} v^{mn'}) = ms$ ,  $u\text{-deg}(y^n) = u\text{-deg}(u^{nt} v^{m'n}) = nt = ms - (m, n) < ms$  so  $\|f\| = x^m$ . Hence  $\langle w \rangle = \langle x \rangle$ .

(iv) If  $\langle \|\alpha\| \rangle = \langle w \rangle = \langle x \rangle$  then  $\|\alpha\| = x$ . But by (i)  $\|f\| = \|\alpha\|^a = x^a$  and by

(iii)  $\|f\| = x^m$  so  $a = m$ . Thus  $|f| = \lambda \alpha^m$  in  $k(x, y)$  so  $y\text{-deg}(|f|) = m(y\text{-deg}(\alpha))$ .

And  $y\text{-deg}|f| = n$  since  $y^n \in \text{supp}|f|$ , so  $m|n$ .

(v) If  $\langle \|\alpha\|, \|\beta\| \rangle = \langle w \rangle = \langle x \rangle$  then  $n'Z = v\text{-deg}(\langle x \rangle) = v\text{-deg}(\langle \|\alpha\|, \|\beta\| \rangle)$

$= v\text{-deg}(\langle \alpha, \beta \rangle)$ . By (i)  $y \in k[\alpha^{\pm 1}, \beta]$  and this is  $v$ -homogeneous so

$v\text{-deg}(y) \in v\text{-deg}(\langle \alpha, \beta \rangle)$ , that is,  $m'$  is a multiple of  $n'$  so  $n|m$ .

## §2 The Automorphism Group

For any  $p = \sum \mu_{ij} x^i y^j \in k[x, y]^x$  we define  $\text{deg}(p) = \max\{i+j | \mu_{ij} \neq 0\}$ ; if  $\text{deg } p = d$  we define  $p_0 = \sum \mu_{id-i} x^i y^{d-i}$  called the leading component of  $p$ .

THEOREM 2 ([2],[8]). Let  $(p, q)$  be a  $k$ -algebra automorphism of  $k[x, y]$  with  $\text{deg } p \leq \text{deg } q$ . Then either  $(p, q)$  is affine or there is a unique  $\mu \in k^x$  and positive integer  $r$  such that  $\text{deg}(q - \mu p^r) < \text{deg}(q)$ .

PROOF. Let  $(f,g)$  be the inverse of  $(p,q)$  and let  $f$  be as in §1. If  $\deg(p^m) \neq \deg(q^n)$  then  $\deg(f(p,q)) = \max\{\deg(p^m), \deg(q^n)\}$ . But  $f(p,q) = x$  so  $p$  or  $q$  is a polynomial in  $x$  of degree 1 and the desired conclusion follows easily. This leaves the case where  $\deg(p^m) = \deg(q^n)$ . Here  $m \geq n$  so  $n|m$  and  $\deg(p^r) = \deg(q)$  for  $r = \frac{m}{n}$ . We may assume  $(p,q)$  is not affine so  $\deg q > 1$ . Since  $f(p,q) = x$  it follows that  $p_0, q_0$  are algebraically dependent over  $k$ . Hence  $q_0/p_0^r$  is algebraic over  $k$  so lies in  $k$ , say  $q_0/p_0^r = \mu$ . Then  $\deg(q - \mu p^r) < \deg(q)$  as desired.

By induction on  $\deg(q)$  it follows easily from Theorem 2 that all  $k$ -algebra automorphisms of  $k[x,y]$  are tame. It is even a simple matter to obtain the decomposition.

THEOREM 3.  $\text{Aut } k[x,y] = A *_C B$ .

PROOF. Let  $\Gamma$  be the oriented graph whose vertices are the  $k$ -subspaces of  $k[x,y]$  and whose edges are the inclusion maps. Then  $\text{Aut } k[x,y]$  acts in a natural way on the graph  $\Gamma$ . Let  $T$  be the orbit of  $k+kx \rightarrow k+kx+ky$ . We claim that  $T$  is a tree.

Any vertex of  $T$  is of the form  $k+kp$  or  $k+kp+kq$  where  $(p,q)$  is some automorphism. We define  $\deg(k+kp) = \deg(p)$  and  $\deg(k+kp+kq) = \max\{\deg(p), \deg(q)\} - \frac{1}{2}$ . It is easy to see these are well-defined.

Consider a vertex of the form  $k+kp$ . We can find an automorphism  $(p,q)$  with  $\deg(q)$  minimal, so  $\deg(q) < \deg(p)$  or  $(p,q)$  is affine. All the neighbours of  $k+kp$  are of the form  $k+kp+k(q+h)$  where  $h \in k[p]$ . The only neighbour of  $k+kp$  with smaller degree is  $k+kp+kq$ ; all the others have greater degree.

Consider a vertex of the form  $k+kp+kq$  where  $\deg(q) < \deg(p)$ . The neighbours are of the form  $k+k(\alpha p + \beta q)$  where  $\alpha, \beta \in k^x$  are not both zero; only  $k+kq$  has smaller degree, all the others have greater degree.

Finally, the vertex  $k+kx+ky$  has smaller degree than all its neighbours.

Thus every path from  $k+kx+ky$  is strictly increasing (so  $T$  has no circuits) and from each vertex there is a strictly decreasing path which must necessarily arrive at  $k+kx+ky$  (so  $T$  is connected). Hence  $T$  is a tree.

Now  $k+kx \rightarrow k+kx+ky$  is a transversal in  $T$  for the action of  $\text{Aut } k[x,y]$  and the stabilizer of  $k+kx$  is  $B$  while the stabilizer of  $k+kx+ky$  is  $A$ . This implies  $G = A *_C B$ . cf [6].

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Bedford College  
Regent's Park  
London, NW1 4NS