Pub. Mat. UAB
Vol. 26 Nº 3 Des. 1982

POLYNOMIAL AND RELATED ALGEBRAS AS COHOMOLOGY RINGS (REPORT ON RECENT PROGRESS)

Larry Smith

§ 1. Rings of Invariants as Algebras over the Steenrod Algebra

Let p be a prime and

$$P^* = P[x_1,\ldots,x_n]$$

a polynomial algebra over $\mathbb{Z}/p$. A question of basic importance is to decide if P^* can occur as the $\mathbb{Z}/p$ cohomology of a topological space.

Of course a necessary condition for P^* to be a cohomology ring is that it be an unstable algebra over the Steenrod algebra (for $p \neq 2$ the generators x_j all have even degree, so the Bockstein is identically zero and only $\mathcal{P}^*$, the algebra of Steenrod reduced powers is relevant). By assuming this extra structure we can then try to either construct a space X with $P^* \simeq H^*(X; \mathbb{Z}/p)$, or try to use higher order cohomology operations, and or, operations in extraordinary cohomology to prove no such space X can exist. Hindsight now shows that in fact another approach, using ideas from Galois theory, and invariant theory, provides a complete answer to the realization problem for <u>non-modular</u> polynomial algebras (We say that

$$P^* \simeq P[x_1, \ldots, x_n]$$

is non-modular if

$$\deg x_i \not\equiv 0 \bmod 2p : i = 1, \ldots, n.)$$

This hindsight suggests a natural division of the realization problem; namely, first construct a class of examples of unstable polynomial algebras over the mod p Steenrod algebra, and then worry about which of these can occur as cohomology rings. One elegant way of constructing unstable algebras over the Steenrod algebra is provided by invariant theory, and was exploited to good advantage by Clark and Ewing [6]. We start with

G : a finite group

ρ : $G \hookrightarrow GL(n; \mathbb{Z}/p)$ a faithfull representation.

Let $V := V\rho \simeq \underset{n}{\oplus} \mathbb{Z}/p$ be the representation space for ρ and form

$$P(V) = P[V^*]$$

the graded polynomial algebra on the dual vector space V^* of V, where the grading results from the requirement:

$$\deg v = 2 : \forall v \in V.$$

The action of G can be extended to $P(V)$ in the obvious way and so we can form the ring of invariants

$$H^* := P(V)^G = \{f \in P(V) \mid gf = f \ \forall g \in G\}.$$

(The study of rings of invariants was in fact an important local industry in Göttingen around the turn of the century, so it is only natural that I should spend a certain apprenticeship in this area.)

The Steenrod algebra acts on $P(V)$ in a unique way compatable with the Cartan formula and the unstability condition, namely, via the condition

$$P^k v = \begin{cases} v & : k = 0 \\ v^p & : k = 1 \\ 0 & : \text{otherwise} \end{cases} \quad : p \neq 2$$

and $\beta v = 0$

or

$$Sq^k v = \begin{cases} v & : \ k = 0 \\ v^2 & : \ k = 2 \\ 0 & : \ \text{otherwise (in particular } \beta v = Sq^1 v = 0) \end{cases} \quad p = 2$$

Moreover since G is acting by linear transformations on V and raising to the p^{th} power is linear in characteristic p, it follows that the action of G commutes with the action of the Steenrod algebra, and hence $H^* = P(V)^G$ inherits from P(V) the structure of an unstable algebra over the Steenrod algebra.

Example 1: $D^*(n) := P(V)^{GL(V)}$

This algebra was originally studied by Dickson who showed

$$D^*(n) \simeq P[y_1, \ldots, y_n]$$

$$\deg y_i = 2(p^n - p^{n-i}) \ : \ i = 1, \ldots, n$$

Later on we will see that D*(n), which we refer to as the Dickson algebra, plays a crucial role in the classification of unstable polynomial algebras over the Steenrod algebra. For now let me just mention that the action of the Steenrod algebra on D*(n) is completely determined by the formulae [17]:

$$P^{p^j} y_k = \left\{ \begin{array}{lcl} y_{k+1} & : & j-k = n-1 \\ -y_1 y_k & : & j = n-1 \\ 0 & : & \text{otherwise} \end{array} \right\} \qquad p > 2$$

with an analogous formula for $p=2$, and the fact that the P^{p^j} generate $\mathcal{P}^*$.

Example 2 (Steenrod-Wilkerson):

The polynomial algebra in question is $A^* := P[x_4, x_{2p+2}]$ where

$$P^1 x_4 = x_{2p+2}$$

is the crucial formula. In [18] Steenrod verified by tedious calculation that A^* admits an unstable $\mathcal{P}^*$-algebra structure. (N. B. When $p = 3$ $A^* \cong H^*(BSp(2); \mathbb{Z}/3)$.) In [20] Wilkerson observed that

$$A^* \simeq P(V)^G \quad : \quad \dim_{\mathbb{Z}/p} V = 2$$

where

$$G \hookrightarrow GL(n; \mathbb{Z}_p)$$

is the subgroup generated by (N. B. $\mathbb{Z}_p$:= p-adic integers)

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} -1 , & 0 \\ \theta+\theta^{-1}, & 1 \end{pmatrix}$$

where $\theta = \exp\left(\frac{2\pi i}{p+1}\right)$ (N. B. one needs to check that $\theta + \theta^{-1} \in \mathbb{Z}_p$) and the action of G on V is via mod p reduction from $\mathbb{Z}_p$. (The group G comes from the Shepard and Todd list [11].)

Our primary interest in introducing this construction is however to construct a *large* class of unstable polynomial algebras over the Steenrod algebra. Polynomial rings of invariants, *however in characteristic zero*, were long known to arise from the canonical representation of the Weyl group of a compact connected Lie group on the universal covering space of a maximal torus. These representations are generated by real reflections. In [11] Shepard and Todd introduced a complex analog of reflections, classified all the finite groups that admit complex reflection representations and showed by explicit calculation that the resulting rings of invariants were polynomial algebras (over $\mathbb{C}$!) Ewing and Clark exploited the work of Shepard and Todd by carrying the Shepard and Todd classification kicking and screaming down to characteristic p. To be more specific one introduces a characteristic free definition of complex reflections, namely:

Definition: An automorphism

$$\rho : V \to V$$

is called a pseudo reflection if $1-\rho$ has rank one.

The motivation for this is clear. If you are going to have a reflection across a complex hyperplane, then the orthogonal complement of the hyperplane is a complex line = real 2-plane, so we can also make a "funny house mirror" by also rotating the image in the orthogonal complement of the mirror.

One then proves [4] [5].

Theorem (Chevalley-Bourbaki): Let $\rho : G \hookrightarrow GL(V)$ be a finite dimensional representation of the finite group G which is generated by pseudo reflections. If $|G| \not\equiv 0 \bmod p$, where p is the characteristic of the ground field, then

$$P(V)^G \cong P[x_1, \ldots, x_n]$$

where: if

$$\deg x_i = 2d_i \quad i = 1, \ldots, n,$$

then

$$|G| = d_1 \ldots d_n \quad .$$

Thus in the non-modular situation one can construct lots of example of unstable polynomial algebras over the Steenrod algebra. By utelizing their mod p reduction of Shepard and Todd, Clark and Ewing can provide the following complete list of irreduciable examples

Number	Rank	Order	Type	Primes
1	n	$(n+1)!$	$[4,6,\cdots,2(n+1)]$	$p \nmid (n+1)!$
2a*	n	$qm^{n-1}n!$	$[2m,4m,\cdots,2(n-1)m,2qn]$	$p \nmid n!, p \equiv 1 \bmod m$
2b	2	$2m$	$[4,2m]$	$m > 2, p \equiv \pm 1 \bmod m$
3	1	m	$[2m]$	$p \equiv 1 \bmod m$
4	2	24	[8,12]	$p \equiv 1 \bmod 3$
5	2	72	[12,24]	$p \equiv 1 \bmod 3$
6	2	48	[8,24]	$p \equiv 1 \bmod 12$
7	2	144	[24,24]	$p \equiv 1 \bmod 12$
8	2	96	[16,24]	$p \equiv 1 \bmod 4$
9	2	192	[16,48]	$p \equiv 1 \bmod 8$
10	2	288	[24,48]	$p \equiv 1 \bmod 12$
11	2	576	[48,48]	$p \equiv 1 \bmod 24$
12	2	48	[12,16]	$p \equiv 1,3 \bmod 8,\ p \neq 3$
13	2	96	[16,24]	$p \equiv 1 \bmod 8$
14	2	144	[12,48]	$p \equiv 1,19 \bmod 24$
15	2	288	[24,48]	$p \equiv 1 \bmod 24$
16	2	600	[40,60]	$p \equiv 1 \bmod 5$
17	2	1200	[40,120]	$p \equiv 1 \bmod 20$
18	2	1800	[60,120]	$p \equiv 1 \bmod 15$
19	2	3600	[120,120]	$p \equiv 1 \bmod 60$
20	2	360	[24,60]	$p \equiv 1,4 \bmod 15$
21	2	720	[24,120]	$p \equiv 1,49 \bmod 60$
22	2	240	[24,40]	$p \equiv 1,9 \bmod 20$
23	3	120	[4,12,20]	$p \equiv 1,4 \bmod 5$
24	3	336	[8,12,28]	$p \equiv 1,2,4 \bmod 7$
25	3	648	[12,18,24]	$p \equiv 1 \bmod 3$
26	3	1296	[12,24,36]	$p \equiv 1 \bmod 3$
27	3	2160	[12,24,60]	$p \equiv 1,4 \bmod 15$
28	4	1152	[4,12,16,24]	$p \neq 2$ or 3
29	4	7680	[8,16,24,40]	$p \equiv 1 \bmod 4,\ p \neq 5$
30	4	14,400	[4,24,40,60]	$p \equiv 1,4 \bmod 5$
31	4	64·6!	[16,24,40,48]	$p \equiv 1 \bmod 4,\ p \neq 5$
32	4	216·6!	[24,36,48,60]	$p \equiv 1 \bmod 3$
33	5	72·6!	[8,12,20,24,36]	$p \equiv 1 \bmod 3$
34	6	108·9!	[12,24,36,48,60,84]	$p \equiv 1 \bmod 3,\ p \neq 7$
35	6	72·6!	[4,10,12,16,18,24]	$p \neq 2,3,$ or 5
36	7	8·9!	[4,12,16,20,24,28,36]	$p \neq 2,3,5,$ or 7
37	8	192·10!	[4,16,24,28,36,40,48,60]	$p \neq 2,3,5,$ or 7

* where $m > 1$ and $m = qr$.

If one drops the non-modular restriction, viz. $|G| \not\equiv 0$, then simple examples, e.g.

$$P[\sigma_1, \ldots, \sigma_n] = P(V)^{\Sigma_n}$$

$$\dim V = n, \qquad \Sigma_n \text{ symmetric group}$$

show that there are rings of invariants that are polynomial. In fact in the strictly modular case, namely when G is a p-group, there is a characterization of the groups and representations [10] for which $P(V)^G$ is polynomial. For example one has long known:

<u>Example 3:</u> Let

$$Up(V) = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \in GL(V) \right\}$$

be the subgroup of upper triangular matrices. Then

$$P(V)^{Up(V)} = P[z_1, \ldots, z_n]$$

$$\deg z \;.. = 2i(p;n)$$

For example when n = 2 we find

$$P[t_1, t_2]^{Up(2)} = P[\rho_1, \rho_2]$$

$$\rho_1 = t_1 \ , \quad \rho_2 = t_2^p - t_2^{p-1} t_1$$

and the Steenrod algebra action is determined by

$$P^1\rho_2 = \rho_1^{p-1}\rho_2$$

$$P^1\rho_1 = \rho_1^p$$

Thus we see that invariant theory can provide us with examples of "nice" algebras, in particular polynomial algebras, that are unstable algebras over the Steenrod algebra. We should now ask which of these occur as cohomology rings.

§ 2. Realizing Unstable Algebras over $\mathcal{P}^*$ as Cohomology Rings

In the preceeding section we saw how invariant theory provided a large class of unstable algebras over the Steenrod algebra, in particular an extensive class of non-modular unstable polynomial algebras over the Steenrod algebra. (We say that a graded $\mathbb{Z}/p$ algebra A^* is non-modular if $(QA^*)^d = 0 \ \forall \ d \equiv 0 \bmod p$, where $QA^* := \mathbb{Z}/p \otimes_{A^*} IA^*$ is the module of indecomposables.) The basic construction of spaces realizing rings of invariants as cohomology rings is due to Clark and Ewing [6] building on work of Holzsager [9] and Sullivan [19]

Theorem 1: Let $\rho : G \hookrightarrow GL(n; \mathbb{Z}/p)$ be a faithfull representation of the group G. Assume (NMC) $|G| \not\equiv 0 \bmod p$ (non-modularity condition). Then there exists a space $X(\rho)$ such that

$$H^*(X(\rho); \mathbb{Z}/p) \simeq P(V)^G$$

where $V := V_\rho \simeq \bigoplus_n \mathbb{Z}/p$

is the representation space of ρ.

Sketch of Proof: Let $\mathbb{Z}_p$ denote the p-adic integers. Note that the representation ρ lifts to a representation

$$\tilde{\rho} : G \hookrightarrow GL(n; \mathbb{Z}_p) .$$

This is because the kernals of the successive reduction maps

$$q : GL(n; \mathbb{Z}/p^k) \to GL(n; \mathbb{Z}/p^{k-1}) : k > 1$$

are all p-groups, so the obstruction to lifting a $\mathbb{Z}/p^{k-1}$ representation of G over q to a $\mathbb{Z}/p^k$ representation all lie in

$$H^1(G; \ker q) = 0$$

because ker q is a p-group. Let

$$\tilde{V} = \tilde{V}_{\tilde{\rho}} \simeq \oplus_n \mathbb{Z}_p$$

be the representation space of $\tilde{\rho}$. The operation of G on $\tilde{V}$ induces by functorality an operation of G on the Eilenberg-MacLane space $K(\tilde{V},2)$, and WOLOG we may suppose this action is free. Set

$$X(V;G) := K(\tilde{V},2)/G ,$$

so that we have a regular covering

$$\pi : K(\tilde{V},2) \downarrow X(V;G).$$

Now

$$K(\tilde{V},2) \simeq [\underset{n}{X}\, \mathbb{C}P(\infty)]_{\hat{p}}$$

where $[\ 19\ ]_{\hat{p}}$ denotes the p-adic completion. Thus

$$H^*(K(\tilde{V};2)) \simeq H^*([\underset{n}{X}\, \mathbb{C}P(\infty)]_p) \simeq H^*(\underset{n}{X}\, \mathbb{C}P(\infty))$$

$$\simeq P(V)$$

where

$$H^*(\ ) := H^*(\ ;\mathbb{Z}/p)\ .$$

Moreover one easily sees that the action of G on $H^2(K(\tilde{V};2))$ induced by that on $K(\tilde{V};2)$ coincides with the contragradient representation V^* of ρ. Since $p \nmid |G|$ a transfer / spectral sequence argument shows

$$H^*(X(V;G)) \simeq P(V)^G$$

as was to be shown. □

<u>N.B.</u> Here, and throughout, $H^*(\ ) = H^*(\ ;\mathbb{Z}/p)$ denotes cohomology with coefficients in the field of p elements unless explicitely stated to the contrary.

__Remark:__ It is worthwhile noting that there is a quite different construction of a space $\bar{X}(V;G)$ with $H^*(\bar{X}(V;G)) \simeq P(V)^G$. The advantages gained with this construction is that the resulting space $\bar{X}(V;G)$ is simply connected, and the construction is applicable in a more general context. The construction runs as follows. Begin as before with $K(\tilde{V};2)$. Since $K(\tilde{V};2)$ is a p-complete loop space it follows by a remark of George Cooke [7] that the p-adic integers $\mathbb{Z}_p$ act on $K(\tilde{V};2)$. Thus in fact the p-adic group ring $\mathbb{Z}_p(G)$ acts on $K(\tilde{V};2)$. In $\mathbb{Z}_p(G)$ there is the idempotent

$$e := \sum_{g \in G} g$$

and moreover since $p \nmid G$ one has

$$\mathrm{Im}\{e : P(V) \circlearrowleft\} = P(V)^G .$$

Now given a p-complete loop space and an idempotent one can apply the Eckmann-Hilton dual of an old construction of George Cooke and myself [8] to construct a space $K(\tilde{V};2)^e =: \bar{X}(V;G)$. Using the facts that

$$\pi_2(K(\tilde{V};2)) \xrightarrow{\sim} H_2(K(\tilde{V};2); \mathbb{Z})$$

and

$$H^*(K(\tilde{V};2); \mathbb{Z}/p) \simeq P(V)$$

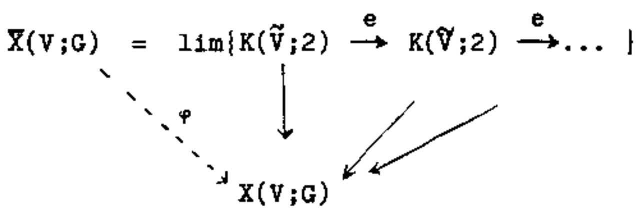
one readily shows in this case that

$$H^*(\bar{X}(V;G);\mathbb{Z}/p) \simeq \mathrm{Im}\{e : P(V) \circlearrowleft\} = P(V)^G .$$

At the Workshop in Barcelong G. Mislin, A. Zabrodsky and I constructed a proof of:

<u>Prop.</u> (MSZ): $\bar{X}(V;G)$ is the homotopy equivalent to the Bousfield - Kan p-completion of $X(V;G)$.

<u>Proof:</u> By construction one has a diagram of solid arrows



inducing the dotted map φ. Because $\bar{X}(V,G)$ is p-complete φ factors through the p-completion of $X(V;G)$. Moreover since φ_* is a $\mathbb{Z}/p$ homology isomorphism the same is true of the induced map

$$\bar{X}(V;G) \to [X(V,G)]_{\hat{p}}$$

and the result follows. □

In any case if we start with:

$$\rho : G \hookrightarrow GL(n; \mathbb{Z}/p)$$

then $H^* := P(V)^G$ is a polynomial algebra over the Steenrod algebra to be found in the list compiled by Clark and Ewing and moreover there is a space X such that $H^*(X; \mathbb{Z}/p) \cong H^*$.

Let me summarize the preceeding discussion in the following result:

<u>Theorem 2</u> (Clark-Ewing): Let $\rho : G \hookrightarrow GL(n; \mathbb{Z}/p)$ be generated by pseudo reflections and assume

(NMC) $\quad p \nmid |G|$

Then:

<u>Algebraic Part:</u> There is a representation $G \hookrightarrow GL(n;\mathbb{C})$ of G as a complex pseudo reflection group, such that the polynomial algebras

$$P(\oplus_n \mathbb{C})^G \qquad P(\oplus_n \mathbb{Z}/p)^G$$

have the same type.

Topological Part: There exists a space $X(V;G)$ such that $H^*(X(V;G)) \simeq P(V)^G$; $V = \bigoplus_n \mathbb{Z}/p$ the representation space of ρ.

Remarks:

(1) The existence of a complex "lifting" of a given mod p representation can be explained as follows. We have already seen that (NMC) allows us to construct a p-adic lifting $G < GL(n; \mathbb{Z}_p)$. But G being a finite group means that this representation is already defined in a finite extension of $\mathbb{Q}$ (simply adjoin enough roots of unity), and hence over $\mathbb{C}$.

(2) In addition Clark and Ewing determine the character fields (N.B. Since they prove that the Schur index is always 1 it doesn't matter which definition of "character field" one is using) of the complex hyperplane groups in the Shepard-Todd list. Thus starting from a complex hyperplane group one can read off over which finite fields $\mathbb{Z}/p$ it admits (pseudo reflection) representations.

Along with the question of realizing $P(V)^G$ as a cohomology ring, we should also look at the homotopy classification of maps between such spaces. The construction offered by Clark and Ewing delivers a rather explicit space $X(V;G)$ and one can prove: (see [14])

(2) the Steenrod algebra action on A* lifts to an unstable action on $P[x_1,\ldots,x_n]$, and

(3) $P[y_1,\ldots,y_n]$ is closed under this lifted action.

Then there exists a topological space A such that

$$H^*(A;\mathbb{Z}/p) \simeq A^* .$$

The construction of Clark and Ewing provides many examples of spaces whose $\mathbb{Z}/p$ cohomology is a polynomial algebra. For an odd prime p another very natural question to study is that of realizing symmetric algebras i.e., free commutative algebras as cohomology rings. Armed with a good realization theorem for symmetric algebras, and a corresponding classification of maps, one could try to mimic with these spaces as building blocks the upside down Postnikov tower (Sullivan's minimal model construction) to get more complete information about "$\mathrm{Im}\{H^* : \mathrm{Top} \to \mathrm{Un\,Al}/\mathcal{P}^*\}$".

The minute one starts to talk about symmetric algebras, the Bockstein behaviour becomes important. Even in the simpelest case viz

$$P[x] \otimes E[y] \;:\; \beta y \neq 0$$

and

$$P[u] \otimes E[v] \;:\; \beta : u = 0$$

Prop. 3: With the notations preceeding

$$[X(V',G'),X(V'',G'')] \simeq \mathrm{Morpf}((V',G'),(V'',G''))$$

where

$$\mathrm{Morpf}((V',G'),(V'',G'')) := \left\{ (\varphi,\psi) \;\middle|\; \begin{array}{l} \varphi : V' \to V'' \\ \psi : G' \to G'' \\ \text{and} \\ \varphi(g'v') = \psi(g')\varphi(v') \\ \forall g' \in G',\ v' \in V' \end{array} \right\}$$

This classification of maps comes in handy when one trys to use the spaces $X(X;G)$ as "building blocks" to construct spaces realizing other interesting unstable algebras over the Steenrod algebra as cohomology rings. Here [12] for example is a sample result in this direction. (This result has also been obtained independently by Howard Hiller.)

Prop. 4: Let

$$A^* = P[x_1,\ldots,x_n]/(y_1,\ldots,y_n)$$

be a graded complete intersection, that is an unstable algebra over the Steenrod algebra. Assume that:

(1) $$\left(\prod_{i=1}^{n} \deg x_i\right)\left(\prod_{i=1}^{n} \deg y_i\right) \not\equiv 0(p);$$

the two examples behave very differently, as Aguadé has shown. As a sample of his results one has [3].

<u>Prop. 5:</u> (J. Aguadé) Suppose p an odd prime, and

$$S^* \simeq E[y] \otimes P[x]$$

is an unstable algebra over the Steenrod algebra. Let 2d =: deg x then $d|p-1$ and all such S* occur as cohomology rings.

<u>Sketch of Proof:</u> To see that $d|p-1$ we write $\beta y = x^r$. There is the Adam relation

$$P^1 \beta P^{dr-1} = (dr - 1)\beta P^{dr} + P^{dr}\beta$$

so applying this to y gives

$$P^1 \beta P^{dr-1} y = (dr - 1)\beta P^{dr} y + P^{dr} \beta y$$

$$= 0 + P^{dr}(x^r) = x^{pr}$$

where $P^{dr} y = 0$ by unstability. But this says P^1 acts nontrivially on S* so $d|p-1$.

To construct the examples where $\beta y = x$ Aguadé proceeds as follows. Let $\zeta \in \mathbb{Z}/p^{\times} \simeq \mathbb{Z}/p-1$ be a generator, and set $\xi := \zeta^{p-1/d}$. Then ξ induces an action of $\mathbb{Z}/d$ on $K(\mathbb{Z}/p;1) = B\,\mathbb{Z}/p$. Let $Y := K(\mathbb{Z}/p;1)/\xi$ be the orbit space. One then has (where deg u = 1)

$$H^*(Y;\mathbb{Z}/p) \simeq H^*(B\,\mathbb{Z}/p)^{\xi} = (E[u] \otimes P[\beta u])^{\xi}$$

$$\simeq \quad E(u(\beta u)^{d-1}) \otimes P((\beta u)^d)$$

as required. □

Recalling how the construction of Clark and Ewing is the many variable generalization of the one variable construction of Holzsager [9] and Sullivan [19] one is tempted to try to proceed analogously starting with Aguadé's construction to prove:

<u>Prop. 6:</u> Suppose

$$S^* \simeq E(y_1,\dots,y_n) \otimes P(\beta y_1,\dots,\beta y_n)$$

is an unstable algebra over the Steenrod algebra where

$$\text{(NMC)} \qquad \prod_{i=1}^{n} \deg \beta y_i \not\equiv 0(p).$$

Then there exists a space Y such that $H^*(Y;\mathbb{Z}/p) \simeq S^*$.

The idea of the construction would be to start with $\rho : G \hookrightarrow GL(n;\mathbb{Z}/p)$. Let $V := \oplus_n \mathbb{Z}/p$ be the representation space of ρ. The action of G on V induces a free action on $BV = K(V;1)$ so we can form the orbit space $Y := BV/_G \simeq B(G \times_\rho V)$. As in Proposition (1), if $p \nmid |G|$ one obtains

$$H^*(Y;\mathbb{Z}/p) \simeq H^*(BV;\mathbb{Z}/p)^G \simeq [E(V) \otimes P(\beta V)]^G .$$

<u>However</u> it is almost never the case that

$$[E(V) \otimes P(\beta V)]^G \simeq E(Y) \otimes P(\beta Y)$$

where $P(\beta Y) \simeq P(\beta V)^G$, (see for example, [4; Ex]) even in the nicest cases. For example, pick an enormous prime p. Let Σ_1 act on $V := \oplus_n \mathbb{Z}/p$ via the adjoint representation. Then one sees the obvious map

$$\varphi : E(\sigma_1,\ldots,\sigma_n) \otimes P(\beta\sigma_1,\ldots,\beta\sigma_n) \to P(V)^G$$

is simply <u>not even monic.</u> To see this choose a basis $v_1,\ldots,v_n$ for V and recall

$$v_1^2 + \ldots + v_n^2 = p(\sigma_1,\ldots,\sigma_n) .$$

But

$$v_1^2 + \ldots + v_n^2 = 0 \in E(V)$$

so $p(\sigma_1,\ldots,\sigma_n) \in \ker \varphi$. Thus a proof of Prop. 6 must proceed along other lines. The proof in [13] runs more or less as follows:

Begin as before with $G < GL(V)$. Let

$$\psi_k : X(V;G) \to X(V;G)$$

be the map induced by the morphism (recall Prop. 3)

$$\lambda_k : (V,G) \to (V,G) \mid \lambda_k(v) = kv,\ \lambda_k g = g.$$

Form the fiber square (Δ := diagonal map)

$$\begin{array}{ccc} Y_k & \longrightarrow & X \\ \downarrow & & \downarrow \Delta \\ X & \xrightarrow{(1,\psi_k)} & X \times X \end{array} \qquad X := (V,G)$$

defining Y_k. Then for $k = p+1$ one finds

$$H^*(Y_{p+1}, \mathbb{Z}/p) \simeq E(Y) \otimes P(\beta Y)$$

where

$$P(\beta Y) = P(\beta V)^G .$$

So for all the examples of rings of invariants, and related rings, which we have shown to occur as cohomology rings have satisfied the non modularity condition.

We have however, at least as algebra over the Steenrod algebra, the other extreme case of $P(V)^{Up(V)}$, etc, namely $P(V)^G$ where G is a p-group. The following result settles the realization question for these modular rings of invariants in the negative [15].

<u>Prop. 7:</u> Let p be an odd prime and $G < GL(n; \mathbb{Z}/p)$ a p-group such that

$$R^* := P(V)^G; \; V = \bigoplus_n \mathbb{Z}/p$$

is a polynomial algebra. Then R* cannot arise as the $\mathbb{Z}/p$ cohomology of a space.

Thus a polynomial algebra

$$P^* := P(x_1, \ldots, x_n)$$

where $\deg x_i = 2p^{a_i}$, $i = 1, \ldots, n$, and at least one a_i positive, that occurs as a ring of invariants can never be the cohomology algebra of a space. Therefore we cannot separate the realization question into a non-modular theory, but must move from a non-modular theory to a "mixed" theory. A prototype example here is the Dickson algebra

$$D^*(n) := P(V)^{GL(V)} \simeq P[y_1, \ldots, y_n]$$
$$\deg y_i = 2p^n - 2p^{n-i}; \; i = 1, \ldots, n.$$

One reason for singling out $D^*(n)$ for special study is the following result proved jointly with Bob Switzer, in an attempt to clarify the work of Adams and Wilkerson to be discussed in the next section.

Prop. 8 (joint with R.M. Switzer): Let $H^* \in \mathrm{UnId}/\mathcal{P}^*$ (:= Unstable Integral Domein over the Steenrod algebra.) ANASC hat $H^* \simeq P(V)^G$ for some $G < GL(V)$, where $\deg v = 2$ $\forall\, v \in V$, is that H^* be a finite algebraic extension in $\mathrm{UnId}/\mathcal{A}^*$ of $D^*(n)$.

Finally the realization question for $D^*(n)$ is settled by [17]

Prop. 9 (joint with R.M. Switzer): ANASC that $D^*(n)$ occur as a cohomology algebra is:

$$n = 1$$

or

$$n = 2 \quad \text{and} \quad p \leq 3.$$

N.B. For $n = 1$ the example are due to Holszager and Sullivan. For $n = 2$ they are all classical, viz., $\mathbb{C}P(\infty)$, $BSU(3)$ when $p = 2$

The case $P(V)^{GL(2;\,\mathbb{Z}/3)}$ has recently been realized by A. Zabrodsky [23].

§ 4. Classification; the Theorem of Adams and Wilkerson

So for we have seen how invariant theory can help us to construct nice examples of algebras $H^* \in \text{UnAl}/\mathcal{P}^*$, which in turn can often be realized as cohomology rings. Invariant theory can also be used to classify objects in $H^* \in \text{UnAl}/\mathcal{P}^*$. Here is the most exciting result on the realization problem for polynomial algebras since J.F. Adams settled the one variable case mod 2.

<u>Theorem 10:</u> (J.F. Adams and C.W. Wilkerson): Let $H^* \simeq P[x_1,\ldots,x_n] \in \text{UnAl}/\mathcal{P}^*$ and suppose

$$p \nmid d_1 \ldots d_n$$

where

$$2d_j := \deg x_j \quad : \quad j = 1,\ldots,n .$$

Then there exists a $G \overset{\rho}{\hookrightarrow} GL(n; \mathbb{Z}/p)$ of order $d_1 \ldots d_n$ generated by pseudo reflections such that $H^* \simeq P(V)^G$ where $V := \oplus_n \mathbb{Z}/p$ is the representation space of ρ.
(Thus H^* is in the Clark-Ewing list)

<u>Cory 11:</u>

(a) ANASC that a polynomial algebra over $\mathbb{Z}/p$

$$H^* \simeq P[x_1,\ldots,x_n]; \deg x_i = 2d_i; i = 1,\ldots,n$$
$$p \nmid d_1 \ldots d_n$$

be the $\mathbb{Z}/p$ cohomology ring of a space is that H* admit an unstable action of the mod p Steenrod algebra.

(b) If $H^* \cong P[x_1,\ldots,x_n] \in \text{UnAl}/\mathcal{P}^*$

$$\deg x_i \not\equiv 0 \ (2p) \quad i = 1,\ldots,n$$

occurs as a cohomology ring then it occurs as the cohomology of a Clark-Ewing space.

Recently Bob Switzer and I have found that by rearranging the ideas of Adams and Wilkerson one can achieve a much shorter proof of their main theorem than that given in their Annals paper [2]. Let me try to sketch this new arrangement. As in [2] we will base our proof on the theory of algebraic extensions-Galois theory in the category

$$\text{UnId}/\mathcal{P}^* = \{I^* \in \text{UnAl}/\mathcal{P}^* \mid I^* \text{ is a graded integral domain}\}$$

Here then is a breif sketch of some of the main ideas.

Definition: An injective morphism $\varphi : A^* \hookrightarrow B^*$ in $\text{UnId}/\mathcal{P}^*$ is called an algebraic extension if every element of B* is a root of a polynomial equation with coefficients in A*.

N.B. A bit of care must be taken with the grading here.

If $b \in B^{2d}$ is a root of a polynomial $p(X) \in A^*[X]$, where X has degree 2d, then we call b:

separable $\Leftrightarrow$ p(X) can be chosen separable

integral $\Leftrightarrow$ p(X) can be chosen with lead coefficient 1

etc. $\Leftrightarrow$ ....

The following result shows that UnId/$\mathcal{P}^*$ contains algebraically closed objects [22]

Algebraic Closure Theorem (Serre-Wilkerson): If $P^* \simeq P[t_1,\ldots,t_n]$ is an unstable polynomial algebra over $\mathcal{P}^*$ on 2 dimensional generators than P* is algebraically closed in UnId/$\mathcal{P}^*$.

Wilkerson [21] also showed how to extend the action of $\mathcal{P}^*$ from an $A^* \in$ UnId/$\mathcal{P}^*$ to its graded field of fractions F(A*), viz, if

$$P_\xi : A^* \to A^*[[\xi]] \,\Big|\, P_\xi(a) = \sum_{k=0}^{\infty} P^k(a)\xi^k$$

is the giant reduced power and $^a/_b \in F(A^*)$, then

$$P_\xi(\tfrac{a}{b}) = \frac{P_\xi(a)}{P_\xi(b)} \in F(A^*)[[\xi]] .$$

This makes sense as a power series in ξ because $P_\xi(b) = 1 +$ higher terms.

N.B.: The graded field of fractions is graded over $\mathbb{Z}$, i.e. contains elements of negative degrees. This we agree to allow, but it means we *loose* the unstability con-

condition, which because of

$$x = P^{o}x = 0 \; : \; \deg x < 0$$

implies unstable $\mathcal{P}^*$ algebras are non-negatively graded.

Wilkerson further proves:

Separable Extension Lemma (Wilkerson): Let K^* be a graded field over $\mathcal{P}^*$ and let $L^* > K^*$ be a separable field extension. Then there exists a unique extension to L^* of the $\mathcal{P}^*$ algebra structure on K^*.

The entire theory devolves in a crucial manner around derivations. In particular the primitive elements

$$P^{\Delta_i} \in \mathcal{P}^{2p^i-2} = \begin{cases} P^{\Delta_1} & := P^1 \\ P^{\Delta_{i+1}} & := [P^{p^i}, P^{\Delta_i}] \quad i > 1, \end{cases}$$

which act as derivations, together with the special derivation P^{Δ_o} defined by

$$P^{\Delta_o}(a) = \begin{cases} (\deg a) \cdot a & p \neq 2 \\ (\frac{1}{2} \deg a) & p = 2 \end{cases}$$

play an important role. Several Lemmas of Adams and Wilkerson concerning these derivations can be summarized as follows [2; § 5]:

Δ-Theorem (Adams-Wilkerson): Suppose $H^* \in UnId/\mathcal{P}^*$ has finite transcendence degree over $\mathbb{Z}/p$. Then there exists an integer $n \geq 0$ with the following property:

(Δ) any n-distinct derivations $P^{\Delta_{i_1}},\ldots,P^{\Delta_{i_n}}$ are linearly independent over H^* and any n+1 derivations are linearly dependent.

Thus there exists elements $h_o,\ldots,h_n \in H^*$, all non-zero, such that

$$h_o P^{\Delta_o} + \ldots + h_n P^{\Delta_n} \equiv 0$$

vanishes identically on H^* .

Finally the following classical result is usefull in establishing the algebraic independence of elements.

∂-Lemma: Suppose $A^* \hookrightarrow B^*$ is an inclusion of graded algebras over $\mathbb{Z}/p$ and $\partial_1,\ldots,\partial_n : A^* \to B^*$ are derivations. If $a_1,\ldots,a_n \in A^*$ satisfy

$$\det(\partial_i a_j) \neq 0$$

then $a_1,\ldots,a_n$ are algebraically independent.

Proof of the Main Theorem (joint with R.M. Switzer): Let

$$\Delta(X) := h_o X + h_1 X^p + \ldots + h_n X^{p^n} \in H^*[X]$$

be the magic polynomial, i.e.

$$\delta := h_o P^{\Delta_o} + \ldots + h_n P^{\Delta_n}$$

is the derivation of the Δ-Theorem (magic differential). Regard $\Delta(X)$ as a polynomial over $F(H^*)$:= field of fractions of H^* and form $E^* > F(H^*)$ the splitting field of $\Delta(X)$ over $F(H^*)$. Since

$$\frac{d}{dX}\Big|_{X=o} \Delta(X) = h_o \neq 0,$$

$\Delta(X)$ is separable and thus $E^* > F(H^*)$ is a separable extension. By the separable extension lemma there is a unique $\mathcal{P}^*$ algebra structure on E^* extending the given structure on $F(H^*)$. Let

$$V := \{v \in E^2 | \Delta(v) = 0\}.$$

Because $\Delta(X)$ is additive, V is a vector space. In fact V is n-dimensional, consisting of precisely the p^n roots of $\Delta(X)$. Thus

$$\Delta(X) = h_n \prod_{v \in V} (X-v) .$$

We collect some facts about V [2; § 5].

(1) If $t_1,\dots,t_n$ is a $\mathbb{Z}/p$ basis for V then $t_1,\dots,t_n$ are algebraically independent.

(2) The elements of V are unstable, so

$$P^* := P[t_1,\dots,t_n] \in \mathrm{UnId}/\mathcal{P}^*$$

(3) the action of $\mathcal{P}^*$ on P^* commutes with the action of $GL(n; \mathbb{F}_p)$, so

$$D^*(n) := P^{*\,GL(n;\,\mathbb{F}_p)} < P^* < E^*$$

are inclusions in $\mathrm{UnId}/\mathcal{P}^*$

(4) every $x \in P^*$ is integral over H^*.

Let A^* be the algebra obtained from H^* by adjoining $t_1,\dots,t_n$. Then $A^* \in \mathrm{UnId}/\mathcal{P}^*$ and we have the inclusions

$$H^* < A^* > P^* > D^*(n).$$

Suppose that we knew that $A^* > D^*(n)$ were an algebraic extension. Then $A^* > P^*$ is algebraic. But by the algebraic closure theorem P^* is algebraically closed and hence $A^* = P^*$ whence $H^* < A^* = P^*$ is a separable algebraic extension. Consider the Galois group $G := \mathrm{Gal}(E^* > F(H^*))$.

Clearly $G < GL(n; \mathbb{F}_p)$ because the elements of G define linear transformations of V and an automorphism of E* fixing F(H*) is uniquely determined by its action on V. Now we claim

$$\circledast \qquad H^* = P^{*G} = \mathbb{F}_p[t_1, \dots, t_n]^G .$$

To see this note the inclusion $H^* < P^{*G}$ is clear. On the other hand because $E^* > F(H^*)$ is a Galois extension every $x \in P^{*G}$ lies in F(H*). Furthermore x is integral over H* by (4) above. H* is however a polynomial algebra, hence integrally closed, and thus $x \in H^*$, i.e. $P^{*G} < H^*$ so $\circledast$ follows.

Hence our problem reduces to showing

$$A^* > D^*(n)$$

is an algebraic extension. In fact we show that it is an integral extension by an argument lifted from Adams and Wilkerson [2]. First of all recall $D^*(n) \simeq \mathbb{F}_p[y_1, \dots, y_n]$. Let $(y_1, \dots, y_n)$ denote the ideal of A* generated by $y_1, \dots, y_n$. If we can show that $A^*/(y_1, \dots, y_n)$ is finite dimensional as a vector space over $\mathbb{F}_p$, then in fact an easy argument via induction over the grading, shows that every element of A* is integral over D*(n). To see that

$A^*/(y_1,\dots,y_n)$ is finite dimensional over $\mathbb{F}_p$ we proceed as follows. Let $z \in A^{2d}$ with $d \not\equiv 0 \bmod p$. Then [2 ; 2.3] there is an element $b \in \mathcal{O}^*$ such that

$$P^{\Delta_n}(bz) = z^{p^n} .$$

Now the derivation

$$y_n P^{\Delta_o} + y_{n-1} P^{\Delta_1} + \dots + y_1 P^{\Delta_{n-1}} + P^{\Delta_n}$$

is

$$\frac{1}{h_n} \cdot \partial = \frac{1}{h_1} (h_o P^{\Delta_o} + \dots + h_n P^{\Delta_n})$$

and hence vanishes on E* and therefore certainly on $A^* < E^*$. Thus we have

$$z^{p^n} = P^{\Delta_n}(bz) = -y_n P^{\Delta_n}(bz) - \dots - y_1 P^{\Delta_{n-1}}(bz) \in (y_1,\dots,y_n)$$

By construction A^* is a finitely generated $\mathbb{F}_p$ algebra, with generators $a_1,\dots,a_m$ whose degrees are relatively prime to p. By the preceeding calculation the natural map

$$\frac{\mathbb{F}_p[a_1,\dots,a_m]}{(a^p\ ,\dots,a_m^p)} \to A^*/(y_1,\dots,y_n)$$

is surjective. But

$$\mathbb{F}_p[a_1,\dots,a_m]/(a_1^p\ ,\dots,\ a_m^p)$$

is visably finite dimensional. □

REFERENCES

1. J.F. Adams, On the non-existence of elements of Hopf Invariant One, Ann. of Math. 72(1960), 20-104

2. J.F. Adams and C. Wilkerson, Finite H-Spaces and Algebras over the Steenrod Algebra, Ann. of Math. 78(1980), 95-143

3. J. Aguadé, Cohomology Algebras with Two Generators, Math. Z. 177(1981), 289-296

4. N. Bourbaki, Groupes et Algebrès de Lie Ch V Hermann Paris 1968

5. C. Chevalley, Invariants of Finite Groups generated by Reflections, Am. J. of Math. 77(1955) 778-782

6. A. Clark and J. Ewing, The Realization of Polynomial Algebras as Cohomology Rings, Pac. J. of Math. 50 (1974), 425-434

7. G.E. Cooke, Constructing Spaces with Interesting Cohomology via ϵ-Actions on Loop Spaces Am. J. of Math. 101(1979),515-542

8. G.E. Cooke and L. Smith, Mod p Decompositions of co-H Spaces and Applications, Math. Z. 157(1977), 155-177

9. R. Holzsager, H Spaces of Category ≤ 2, Topology 9 (1970), 211-216

10. H. Nakajima, Modular Representations of p-Groups with Regular Rings of Invariants, Proc. Jap. Acad. 56 (1980), 469-473

11. G.C. Shephard and J.A. Todd, Finite Unitary Reflection Groups, Canadian J. of Math. 6(1954) 274-304

12. L. Smith, A Note on the Realization of Graded Complete Intersection Algebras by the Cohomology of a Space, Quart. J. of Math. (to appear)

13. L. Smith, On the Realization and Classification of Symmetric Algebras as Cohomology Rings, IHES Preprint 1981

14. L. Smith, On the Homotopy Classification of Maps between certain spaces with Polynomial Cohomology, Göttingen Preprint 1981

15. L. Smith, On the Non Realization of Modular Rings of Invariants as Cohomology Algebras PAMS (to appear)

16. L. Smith and R.M. Switzer, Polynomial Algebras over the Steenrod Algebra, Variations on a Theme of Adams and Wilkerson, Göttingen Preprint 1981

17. L. Smith and R.M. Switzer, On the Realization and Non Realization of Dickson Algebras as Cohomology Rings, IHES Preprint 1981

18. N.E. Steenrod, Polynomial Algebras over the algebra of Cohomology Operations, H-Spaces (Neuchatel) Springer LNM 196(1971), 85-99

19. D. Sullivan, Genetics of Homotopy Theory and the Adams Conjectures, Ann. of Math. 100(1974), 1-78

20. C. Wilkerson, Some Polynomial Algebras over the Steenrod Algebra $\mathcal{A}_p$, BAMS 79(1973), 1274-1276

21. C. Wilkerson, Classifying Spaces, Steenrod Operations and Algebraic Closure, Topology 16(1977), 227-237

22. C. Wilkerson, Integral Closure of Unstable Steenrod Operations, J. Pure and Applied Algebra 13(1978), 49-55

23. A. Zabrodsky, On the Realization of Invariant Subgroups of $\pi_*(X)$, Jerusalem Preprint

I want to thank Bob Switzer for his continuous support, encouragement, and collaboration, without which this, and many other projects, would have been impossable.

Mathematisches Institut der Universität
Bunsenstraße 3/5
D-3400 Göttingen
Bundesrepublik Deutschland