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H-SPACES OF SELF-EQUIVALENCES OF FIBRATIONS

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Let G(p) be the space of all equivariant automorphisms of a principal G-bundle $p : E \to B$, topologized as a subspace of $M(E,E)$, the space of maps from E to itself. Composition of automorphisms gives G(p) a group structure and indeed, G(p) is a topological group. The topological group G(p) has been used quite frequently in connection with certain problems of Theoretical Physics; for example, it appears in the Feynman approach to Quantum Mechanics as the group of all gauge transformations of a smooth principal G-bundle p, with G a Lie group. In these problems, it is necessary on several occasions to know more about the space G(p) or about certain of its homotopy groups (see [8]). Clearly, if p is a trivial G-bundle over a space B, then G(p) is homeomorphic to the space $M(B,G)$. In general, if $f \in$ G(p) and $x \in E$, because G acts effectively and transitively on fibres there is a unique $g \in G$ such that $f(x) = gx$. This gives rise to a homeomorphism θ from G(p) to the space $M_C(E,G)$ of all maps φ from E to G, such that $\varphi(gx) = g\varphi(x)g^{-1}$ for all $g \in G$ and all $x \in E$, in practice, a difficult space to deal with. Note that if G is abelian, θ : G(p) $\cong M(B,G)$ [6]. This is a better result but, of course, it is too limited. A more general result was obtained by D.H.Gottlieb in 1972 [5]: if BG is the classifying space for G, $k : B \to BG$ is the classifying map for the principal G-bundle $p : E \to B$ and $M(B,BG;k)$ is the path-component of $M(B,BG)$ containing k, then

Proposition 1–G(p) $\simeq_w$ $\Omega M(B,BG;k)$ *($\simeq_w$ = weak homotopy equivalence).*

As for other types of fibrations, probably the first result along the lines of Proposition I was also obtained by Gottlieb [4]. To describe it, we must recall the following classification theorem due to A.Dold:"let $E^F(B)$ be the set of all fibre-homotopy equivalence classes of Hurewicz fibrations over a path-connected CW-complex B and with fibres of the homotopy type of a fixed space F; then, there is a CW-complex B_∞ such that the functors E^F and $[\ ,B_\infty]$ of CW into Set are naturally equivalent" (here $[X,Y]$ represents the set of all homotopy classes of maps from X into Y ; see [3], Corollary 16.9).

Proposition 2–*If* $p : E \to B$ *is a Hurewicz fibration with fibre* F, B *is a path-connected CW-complex and* $k : B \to B_\infty$ *is the classifying map, the space* G(p) *of all self-fibre homotopy equivalences of* p *is such that*

$$\pi_0(\mathrm{G}(p)) \cong \pi_0(\Omega M(B,B_\infty;k)).$$

The purpose of this note is to report results of a joint work with P.Booth, P.Heath and C.Morgan, concerning the study – in a unified fashion – of the homotopy type and certain homotopy groups of the space G(p), where p is an object of an arbitrary *category of fibrations* over CW-complexes. Proofs will be given elsewhere.

The main examples of categories of fibrations we have in mind are the following (note that all fibrations considered have a path-connected CW-complex as a base space).

(i) Dold fibrations with fibres of the homotopy type of a fixed space F (we define a *Dold fibration* as a fibration satisfying the *Weak Covering Homotopy Property* [2]);

(ii) Hurewicz fibrations with fibres of the homotopy type of a fixed space F;

(III) principal G-bundles, G a topological group;

(IV) smooth principal G-bundles, G a Lie group;

(V) vector bundles with fibres isomorphic to a fixed vector space V;

(VI) fibre bundles with fibre F, corresponding to a given effective action of a compact topological group G;

(VII) principal H-fibrations with fibres of the homotopy type of a strictly associative H-space with strict identity (see [1], Ex.3).

All these categories have in common the fact that each has a Universal Object $(E_\infty, p_\infty, B_\infty)$ from which one deduces a Classification Theorem of Dold's type; furthermore, in each one of these examples, $(E_\infty, p_\infty, B_\infty)$ also satisfies another type of universality which we shall describe later on and which plays a crucial role in our considerations.

In order to unify these ideas we begin by taking a category **F** with a distinguished object F and a faithful underlying space functor **F** → **K**, where **K** is the *convenient category of k-spaces*, that is to say, **K** is the image of **Top** under the functor k : **Top** → **Top** – called the *k-ification functor* – obtained as a left Kan-extension of the imbedding **C** → **Top** over itself, where **C** is the category of all compact Hausdorff spaces.It is also assumed that for any two objects $X, Y \in$ **F**, **F**(X,Y) is non empty . We then define an *F-space* as a triple (E,p,B) such that B is a CW-complex, $E \in$ **K** , $p: E \to B$ is a map in **K** and finally, for every $b \in B$, $E_b = p^{-1}(b) \in$ **F**. An **F**-*map* $(f_1, f_0): (E,p,B) \to (E',p',B')$ is given by two maps $f_1: E \to E'$, $f_0: B \to B'$ such that $p' f_1 = f_0 p$ and the restriction of f_1 *to any fibre* E_b is a morphism of **F**. If $B = B'$ and $f_0 = 1_B$, an **F**-map $(f_1, 1_B)$ is said to be an **F**-*map over* B. An **F**-*homotopy* is an **F**-map (H,h) such that $pH = h\, q \times 1$. If $A = B$ and h is the projection map, we have the notion of **F**-*homotopy over* B. An **F**-map $g: X \to E$ over B is an **F** -*homotopy equivalence* if there exists an **F**-*map* $g': E \to X$ *over* B such that gg' and $g'g$ are **F**-homotopic over B to the respective identity maps. We now once more restrict the category **F** by requiring that every morphism of **F** is an **F**-homotopy equivalence over a point.

We are now prepared to define formally what we intend for a *category of fibrations* relatively to a category **F**.

Definition – A *category of fibrations* is a non-empty, full subcategory **A** of the category of **F**-spaces and **F**-maps such that:

[1] $(F, c, *) \in \mathbf{A}$, where $*$ is a singleton space and c is the constant map;

[2] if $(E, p, B) \in \mathbf{A}, A \in \mathbf{CW}$ and $f: A \to B$ is a map, the pullback $(f^{*}(E), p_f, A) \in \mathbf{A}$;

[3] **A** is closed under **F**-isomorphisms over a fixed base space;

[4] if $(E, p, B) \in \mathbf{A}$, there is a numerable open covering (U) of B such that, for every $U \in (U)$ $p: p^{-1}(U) \to U$ is **F**-homotopy equivalent to $pr: U \times F \to U$.

As examples of categories of fibrations we quote the categories [I] to [VII] described earlier; we limit ourselves to define the category **F** in each case. For the examples numbered [I] and [II], **F** is the category of all spaces of the same homotopy type as F and all homotopy equivalences between these spaces. For [III] and [IV], **F** is the category whose objects are right G-spaces Y such that, for all $y \in Y$, the function $\tilde{y}: G \to Y$ defined by $\tilde{y}(g) = yg$ is a homeomorphism; its morphisms are G-maps. For [V], **F** consists of all vector spaces isomorphic to V and all isomorphisms between such vector spaces. For [VI], we first assume that G acts effectively on the left of F; then, we define **F** by taking for its objects all pairs (X, ψ) such that X is a left G-space and $\psi: F \to X$ is a homeomorphism of left G-spaces; the set of morphisms from (X, ψ) to (X', ψ') is given by

$$\mathbf{F}((X, \psi), (X', \psi')) = \{\psi' g \psi^{-1} \mid g \in G\}$$

with the obvious operation of composition. Finally, for [VII], **F** is similar to that of [III] ([1], Example 3).

We continue as in [1] by defining, for two arbitrary **F**-spaces (X, q, A) and

(Y,r,B) , the *functional space*

$$X \circledast Y = \bigcup_{\substack{a\in A \\ b\in B}} F(X_a, Y_b)$$

and the function

$$q \circledast r : X \circledast Y \to A \times B \, , q \circledast r(f : X_a \to Y_b) = (a,b).$$

The topology of $X \circledast Y$ is given as follows.Let $Y^+ = Y \cup \{\infty\}$ be the k-ification of the topology defined by requiring that C is closed in Y^+ if either $C = Y^+$ or if C is closed in Y. Now define the function $j : X \circledast Y \to M(X, Y^+)$ by $j(f)(x) = f(x)$ if $x \in X_a$, $f : X_a \to Y_b$ and $j(f)(x) = \infty$,otherwise. ($M(X,Y^+)$ is endowed with the compact open topology).Then we give $X \circledast Y$ the k-ification of the initial topology with respect to j and $q \circledast r$. In general, $(X \circledast Y, q \circledast r, A \times B)$ is not an F-space; however, the following holds.

Theorem 1-*If* $(X,q,A),(Y,r,B) \in$ **A** *then,* $q \circledast r : X \circledast Y \to A \times B$ *is a Dold fibration.*

As we have mentioned before, each one of the categories described in the examples [i] to [vii] has a (free)*universal* object $(E_\infty, p_\infty, B_\infty)$.It also happens that in these examples, the Dold fibration $(F \circledast E_\infty, c \circledast p_\infty, * \times B_\infty)$ has a weakly contractible total space (i.e., for every non-negative integer n, $\pi_n(F \circledast E_\infty)=0$); in this case, we say that $(E_\infty, p_\infty, B_\infty)$ is ***Weakly Contractible Universal.*** We wish to observe, at this point, that if a category of fibrations has a weakly contractible universal object, then such object is also free universal; however the converse is not necessarily true ([1], Theorem 3.2 and Example 4).

For a given object (E,p,B) of the category of fibrations **A** let $G(p)$ be the space of all F-homotopy equivalences of p into itself over B, topologized as a subspace of $M(E,E)$; notice that the composition of F-maps of p into p over B gives to $G(p)$ a continuous product under which $G(p)$ becomes an associative H-space with a strict two-sided unit defined by the identity morphism of p into itself over B.

Theorem 2–*Let* $\mathbf{A}$ *be a category of fibrations with a weakly contractible object* $(E_\infty, p_\infty, E_\infty)$ *and let* (E,p,B) *be an arbitrary element of* $\mathbf{A}$; *suppose that* $k: B \to B_\infty$ *is a classifying map for* (E,p,B). *Then, there exists an H–map*

$$\delta : \Omega M(B, B_\infty; k) \to G(p)$$

which is a weak homotopy equivalence.

Observe that the Dold fibration $F \boxtimes E_\infty \to B_\infty$ has fibre $F \boxtimes F$ and so, if $F \boxtimes F$ has the homotopy type of a CW–complex, $F \boxtimes E_\infty$ is contractible; this, in turn, will imply that the H–map δ of Theorem 2 is a homotopy equivalence. This is precisely the situation of Example [IV], since $G \boxtimes G$ is homeomorphic to G.

From now on, we shall assume for technical reasons that (E,p,B) is an object of the category of fibrations $\mathbf{A}$ which satisfies a strenghtened version of axiom [4] in the definition of a category of fibrations, implying that if (X,q,A) and (Y,r,B) are objects of $\mathbf{A}$ then, $(X \boxtimes Y, q \boxtimes r, A \times B)$ is a Hurewicz fibration; furthermore, we suppose that $\mathbf{A}$ has a weakly contractible universal object $(E_\infty, p_\infty, B_\infty)$. This is the case of examples [II] and [VI]. Let F be the fibre of (E,p,B) over a point $\boxtimes \in B$ and define $G^1(p)$ to be the subspace of $G(p)$ of all F–homotopy equivalences of p over itself over B which extend the identity map $1_F: F \to F$. The space $G^1(p)$ has proved itself very useful in certain problems of Mathematical Physics, where (E,p,B) is an object of the category [IV] (see [8]). We wish to observe that the relation between $G^1(p)$ and $G(p)$ is deeper than just the relation "subspace–space"; in fact,

Theorem 3–*There is a Hurewicz fibration* $G(p) \to F \boxtimes F$ *with fibre* $G^1(p)$ *over* 1_F

A result similar to Theorem 2 holds for $G^1(p)$: in what follows $M_{\boxtimes}(B, B_\infty; k)$ denotes the space of all based maps from B to B_∞.

Theorem 4–*There is an H-map*

$$\delta : \Omega M_*(B, B_\infty ; k) \to \mathbf{G}^1(p)$$

which is a weak homotopy equivalence (or a strong homotopy equivalence if $F*E_\infty$ is contractible).

Next, consider the Hurewicz fibration $F*E_\infty \to B_\infty$ (with fibre $F*F$ over $b = k(*) \in B_\infty$) and its long homotopy sequence

$$\cdots \to \Omega(F*E_\infty) \to \Omega B_\infty \to F*F \to F*E_\infty \to B_\infty ;$$

because $F*E_\infty$ is weakly contractible,

$$\delta : \Omega B_\infty \to F*F$$

is a weak homotopy equivalence (strong homotopy equivalence if $F*E_\infty$ is contractible). This fact is used to prove the following.

Theorem 5– *Suppose that all path-components of $M(B, B_\infty)$ (resp. $M_*(B, B_\infty)$) have the same homotopy type. Then*

$$\mathbf{G}(p) \simeq_w M(B, F*F) \quad (resp.\ \mathbf{G}^1(p) \simeq_w M_*(B, F*F))$$

(strong homotopy equivalence if $F*E_\infty$ is contractible); furthermore, these weak (strong) equivalences preserve the H-space structures.

In connection to the previous theorem the reader should recall that if B is an H-cogroup (e.g., B is a suspension space) then the hypothesis of Theorem 5 hold for $M_*(B, B_\infty)$ and if B_∞ is an H-group (e.g., $B_\infty = BU, BO, BSp$) then these hypothesis hold for both $M(B, B_\infty)$ and $M_*(B, B_\infty)$.

Theorem 6–*If* $F*F$ is $(n-1)$-connected (n positive) and $\dim B = m < 2n$, *then* for $0 \leqslant j < 2n - m$

$$\pi_j(\mathbf{G}(p)) \cong \pi_j(M(B, F*F ; c))$$

$$\pi_j(\mathbf{G}^1(p)) \cong \pi_j(M_*(B, F*F ; c))$$

where $c : B \to F*F$ *is the constant map to* 1_F

We complete these notes with a few computations. If (E,p,B) is a smooth principal $Sp(1)$-bundle and B is a manifold of dimension $m \leqslant 5$, since $Sp(1) \cong S^3$ and $Sp(1)*Sp(1) \cong Sp(1)$, theorem 6 shows that if $0 \leqslant j < 6-m$,

$$\pi_j(\mathrm{G}(p)) \cong \pi_j(M(B,Sp(1))$$

and

$$\pi_j(\mathrm{G}^1(p)) \cong \pi_j(M_*(B,Sp(1)).$$

If p is a smooth principal G-bundle over a sphere S^n, $n>0$, then

$$\mathrm{G}^1(p) \cong M_*(S^n,G)$$

and thus the homotopy groups of $\mathrm{G}^1(p)$ are totally determined by the homotopy groups of G, since, for every $j>0$, $\pi_j(\mathrm{G}^1(p)) \cong \pi_{j+n}(G)$. If p is a smooth principal U-bundle over a manifold B, then $\mathrm{G}(p) \simeq M(B,U)$ and $\mathrm{G}^1(p) \simeq M_*(B,U)$; if, in particular, $B=S^n$ and $n>0$, then

$$\pi_j(\mathrm{G}^1(p)) \cong \pi_{j+n}(U) \cong \begin{cases} 0, \text{if } j=\text{even}, n=\text{even} \quad \text{or} \quad Z, \text{if } j=\text{even}, n=\text{odd} \\ Z, \text{if } j=\text{odd}, n=\text{even} \quad \text{or} \quad 0, \text{if } j=\text{odd}, n=\text{odd} \end{cases}$$

On the other hand, Theorem 2.2 of [7] shows that $M(S^n,U) \cong U \times M_*(S^n,U)$ and so,

$$\pi_j(\mathrm{G}(p)) \cong \begin{cases} 0, \text{if } j=\text{even}, n=\text{even} \quad \text{or} \quad Z, \text{if } j=\text{even}, n=\text{odd} \\ Z \oplus Z, \text{if } j=\text{odd}, n=\text{even} \quad \text{or} \quad Z, \text{if } j=\text{odd}, n=\text{odd} \end{cases}$$

Finally, we recall from the Milnor construction of universal bundles that each countable, connected CW-complex X can be viewed as the base space of a universal G-bundle (G is constructed from X); let us take X to be S^4 and let $k : S^4 \to S^4$ be a degree k function and let (E_k, p_k, S^4) be the corresponding principal G-bundle. Then,

$$\pi_2(G(p_k)) \cong \pi_2(\Omega M(S^4, S^4; k)) \cong \pi_3(M(S^4, S^4; k)) \cong Z_{24|k|} \oplus Z_{12},$$

according to ([7], Lemma 3.10). Since $G^1(p) \simeq M_*(S^4, G)$ it follows that

$$\pi_2(G^1(p)) \cong \pi_6(G) \cong \pi_7(S^4) \cong Z \oplus Z_{12},$$

independently of k.

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