Pub. Mat. UAB
N° 25, Juny 1981

ON THE NUMBER OF PERIODIC ORBITS OF CONTINUOUS MAPPINGS OF THE INTERVAL

Jaume Llibre and Agustí Reventós

Secció de Matemàtiques, Facultat de Ciències, Universitat Autònoma de Barcelona, Bellaterra, Barcelona, Spain.

Rebut l'1 de Juny del 1981

Abstract. Let f be a continuous map of a closed interval into itself, and let P(f) denote the set of positive integers k such that f has a periodic point of period k. Consider the following ordering of positive integers: 3,5,7,...,2.3,2.5,2.7,...,4.3,4.5,4.7,...,8,4,2,1. Sarkovskii's theorem states that if $n \in P(f)$ and m is to the right of n in the above ordering then $m \in P(f)$. We may ask the following question: if $n \in P(f)$ and m is to the right of n in the above ordering what can be said about the number of periodic orbits of f of period m ?. We give the answer to this question if n is either odd or a power of 2.

1. Introduction

This paper is concerned with the periodic orbits of continuous mappings of the interval into itself. Let I denote a closed interval on the real line and let $C^0(I,I)$ denote the space of continuous maps of I into itself. For $f \in C^0(I,I)$, let $P(f)$ denote the set of positive integers k such that f has (at least) a periodic point of period k (see section 2 for definition). One may ask the following question: If $k \in P(f)$, what other integers must be elements of $P(f)$?.

This question is answered by a theorem of Sarkovskii. Consider the following ordering of the set of positive integers N:

$3,5,7,\ldots,2.3,2.5,2.7,\ldots,4.3,4.5,4.7,\ldots,8,4,2,1.$

Thus, in this ordering the smallest element of N is 3 and the greatest is 1. Sarkovskii's theorem states that if $n \in P(f)$ and m is to the right of n in the above ordering (Sarkovskii ordering) then there is at least one periodic orbit of period m (see [2] or [3]). Furthermore, if m is to the left of n in the Sarkovskii ordering, then there is a map $f \in C^0(I,I)$ with $n \in P(f)$ and $m \notin P(f)$.

For $f \in C^0(I,I)$, let $N(f,m)$ denote the number of periodic orbits of f of period m. In this paper, we ask the following question: If $n \in P(f)$ and m is to the right of n in the Sarkovskii ordering, what can be said about $N(f,m)$?. Our main result is the following.

Theorem A. *Let $f \in C^0(I,I)$ and let n denote the minimum of $P(f)$ in the Sarkovskii ordering. Suppose n is odd, $n > 1$ and m is to the right of n in the Sarkovskii ordering. Then the following hold.*

(i) *There is an integer $N_n(m)$ (easily computable, see section 3) such that $N(f,m) \geq N_n(m)$.*

(ii) *There is a map $g \in C^0(I,I)$ such that $P(g) = P(f)$ and $N(g,m) = N_n(m)$.*

Note, for example, that if $f \in C^0(I,I)$ and $3 \in P(f)$, then f has at least $N_3(m)$ periodic orbits of period m. We have compute $N_3(m)$ and $N_5(m)$ for $m = 1,2,\dots,50$ in Tables I and II, respectively (for details see section 3). We remark that Sarkovskii's theorem only says $N_n(m) \geq 1$.

Proposition B. <u>Let</u> $f \in C^0(I,I)$ <u>and let</u> n <u>denote the minimum of</u> $P(f)$ <u>in the Sarkovskii ordering. Suppose</u> n <u>is a power of</u> 2 <u>and</u> m <u>is to the right of</u> n <u>in the Sarkovskii ordering. Then the integer</u> $N_n(m)$ <u>which satisfies conditions</u> (i) <u>and</u> (ii) <u>of</u> Theorem A <u>is the unity</u>.

Proposition B follows immediately from the fact that for each power of 2, let 2^r, there is a map $f \in C^0(I,I)$ such that $P(f) = \{1,2,4,\dots,2^r\}$ and $N(f,2^k) = 1$ for $k = 0,1,\dots,r$ (see Lemma 16 of [1]).

In proving Theorem A, we use a result of Stefan (see section 2). This result describes how a mapping $f \in C^0(I,I)$ must act on a periodic orbit $\{p_1,\dots,p_n\}$ of odd period $n > 1$, where n is the minimum of $P(f)$ in the Sarkovskii ordering.

We note the algorithm described in order to compute the integer $N_n(m)$ defined in Theorem A (see section 3) can be used for all $n \in P(f)$ not necessarily odd. But we need to know how f must act on a periodic orbit of f of period n. That is, if $\{p_1,\dots,p_n\}$ is a periodic orbit of f of period n, who is $f(p_i)$ for each $i = 1,\dots,n$?.

We are grateful to Ramon Reventós who have helped us in the preparation of this note.

2. Preliminary definitions and results

Let $f \in C^0(I,I)$. For any positive integer n, we define f^n inductively by $f^1 = f$ and $f^n = f \circ f^{n-1}$. We let f^0 denote the identity map of I.

Let $p \in I$. We say p is a fixed point of f if $f(p) = p$. If p is a fixed point of f^n, for some $n \in \mathbb{N}$, we say p is a periodic point of f. In this case the smallest element of $\{n \in N : f^n(p) = p\}$ is called the period of p.

We define the orbit of p to be $\{f^n(p) : n = 0,1,2,\dots\}$. If p is a periodic point of f of period n, we say the orbit of p is a periodic orbit of period n. In this case the orbit of p contains exactly n points each of which is a periodic point of period n.

We will use the following theorem (see Theorem 2 of Stefan [3]).

Theorem 1. Let $f \in C^0(I,I)$ and let n denote the minimum of $P(f)$ in the Sarkovskii ordering. Suppose n is odd and $n > 1$. Let $\{p_1,\dots,p_n\}$ be a periodic orbit of period n with $p_1 < p_2 < \dots < p_n$. Let $t = (n+1)/2$. Then either (a) or (b) holds (see (a) in fig.1 for n = 3,5,7,9):

(a) $f(p_{t-k}) = p_{t+k+1}$ for $k = 0,\dots,t-2$,

$f(p_{t+k}) = p_{t-k}$ for $k = 1,\dots,t-1$, and

$f(p_1) = p_t$.

(b) $f(p_{t-k}) = p_{t+k}$ for $k = 1,\dots,t-1$,

$f(p_{t+k}) = p_{t-k-1}$ for $k = 0,\dots,t-2$, and

$f(p_n) = p_t$.

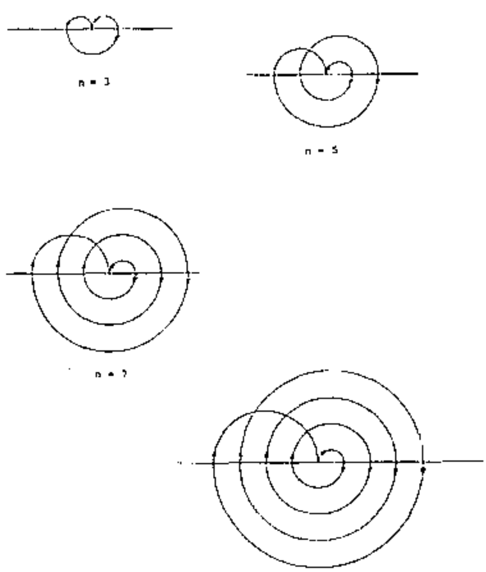


Fig. 1

3. Proof of Theorem A

Let $f \in C^0(I,I)$ and let n denote the minimum of $P(f)$ in the Sarkovskii ordering. Suppose n is odd and $n > 1$. Let $\{p_1,\dots,p_n\}$ be a periodic orbit of f of period n. We can assume that we are in the case (a) of Theorem 1 (the case (b) is similar).

Now, we study the map $g: [p_1,p_n] \longrightarrow [p_1,p_n]$ defined by

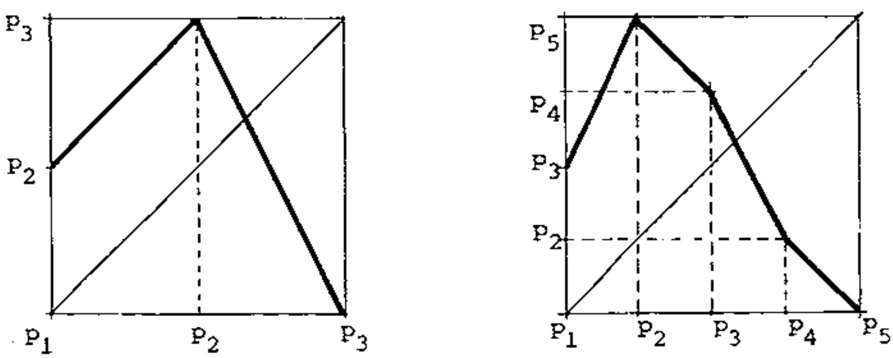
$$g(p_{t-k}) = p_{t+k+1} \quad \text{for } k = 0,\dots,t-2,$$

$$g(p_{t+k}) = p_{t-k} \quad \text{for } k = 1,\dots,t-1, \text{ and}$$

$$g(p_1) = p_t$$

where $t = (n+1)/2$, and on each interval $[p_i,p_{i+1}]$, $i = 1,\dots,n-1$, assume g is linear (see fig.2 for $n = 3$ and $n = 5$).

Suppose m is to the right of n in the Sarkovskii ordering. By continuity, $N(f,m) \geq N(g,m)$. Let $N_n(m) = N(g,m)$. Now, we shall give an algorithm to compute $N_n(m)$ and Theorem A will follow. We only describe the algorithm to compute $N_n(m)$ for $n = 3$ and $n = 5$, since for the other values of n (odd), it is similar.



$n = 3$ $\qquad\qquad$ $n = 5$

Fig. 2

Suppose $n = 3$. Let $\{q_1, q_2, \ldots, q_{k(m)}\}$ denote the set of points of $[p_1, p_3]$ where g^m has a maximum or a minimum. It is easy to see that $q_1 = p_1, q_{k(m)} = p_3$, $p_2 \in \{q_1, \ldots, q_{k(m)}\}$, $g^m(\{q_1, \ldots, q_{k(m)}\}) = \{p_1, p_2, p_3\}$ and $g^m([q_i, q_{i+1}])$ is either $[p_2, p_3]$ or $[p_1, p_3]$ for each $i = 1, \ldots, k(m)-1$.

Let $a_{23}(m)$ (respectively $b_{23}(m)$) be the number of intervals $[q_i, q_{i+1}] \subset [p_1, p_2]$ (respectively $[p_2, p_3]$) such that $g^m([q_i, q_{i+1}]) = [p_2, p_3]$. Let $a_{13}(m)$ (respectively $b_{13}(m)$) be the number of intervals $[q_i, q_{i+1}] \subset [p_1, p_2]$ (respectively $[p_2, p_3]$) such that $g^m([q_i, q_{i+1}]) = [p_1, p_3]$.

From the definition of g it is clear that

$$a_{23}(1) = 1, \qquad a_{13}(1) = 0,$$

$$b_{23}(1) = 0, \qquad b_{13}(1) = 1,$$

and

$$a_{23}(m+1) = a_{13}(m), \qquad a_{13}(m+1) = a_{23}(m) + a_{13}(m),$$

$$b_{23}(m+1) = b_{13}(m), \qquad b_{13}(m+1) = b_{23}(m) + b_{13}(m),$$

for $m = 1, 2, \ldots$

Since the fixed points of g^m are the points of the graphic of g^m which are on the diagonal of the square $[p_1, p_3] \times [p_1, p_3]$, we obtain that g^m has

$$a_{13}(m) + b_{23}(m) + b_{13}(m) = \left(\frac{1+\sqrt{5}}{2}\right)^m + \left(\frac{1-\sqrt{5}}{2}\right)^m,$$

fixed points. Then it is easy to compute $N_n(m)$ for $n = 3$ (see Table I). Note the number of fixed points of g^m is a Fibonacci number.

Now, suppose $n = 5$. Let $\{q_1, \ldots, q_{k(m)}\}$ denote the set of points of $[p_1, p_5]$ where g^m has a maximum or a minimum. It is easy to see that $q_1 = p_1$, $q_{k(m)} = p_5$, $\{p_2, p_3, p_4\} \subset \{q_1, \ldots, q_{k(m)}\}$, $g^m(\{q_1, \ldots, q_{k(m)}\}) = \{p_1, p_2, p_3, p_4, p_5\}$ and $g^m([q_i, q_{i+1}])$ is one of the following intervals: $[p_3, p_5]$, $[p_2, p_5]$, $[p_1, p_4]$, $[p_1, p_5]$, for each $i = 1, \ldots, k(m)-1$.

Let $a_{rs}(m)$ (respectively $b_{rs}(m)$, $c_{rs}(m)$, $d_{rs}(m)$) be the number of intervals $[q_i, q_{i+1}] \subset [p_1, p_2]$ (respectively $[p_2, p_3]$, $[p_3, p_4]$, $[p_4, p_5]$) such that $g^m([q_i, q_{i+1}]) = [p_r, p_s]$. From the definition of g we have that

$$a_{35}(3) = 1, \quad a_{25}(3) = 1, \quad a_{14}(3) = 0, \quad a_{15}(3) = 0,$$

$$b_{35}(3) = 1, \quad b_{25}(3) = 0, \quad b_{14}(3) = 0, \quad b_{15}(3) = 0,$$

$$c_{35}(3) = 0, \quad c_{25}(3) = 0, \quad c_{14}(3) = 0, \quad c_{15}(3) = 1,$$

$$d_{35}(3) = 0, \quad d_{25}(3) = 0, \quad d_{14}(3) = 1, \quad d_{15}(3) = 0,$$

and

$$x_{35}(m+1) = x_{14}(m) + x_{15}(m),$$

$$x_{25}(m+1) = x_{14}(m),$$

$$x_{14}(m+1) = x_{35}(m)$$

$$x_{15}(m+1) = x_{25}(m) + x_{15}(m),$$

for $m = 3, 4, \ldots$ and $x \in \{a, b, c, d\}$.

Because the fixed points of g^m are the points of the graphic of g^m which are on the diagonal of the square $[p_1, p_5] \times [p_1, p_5]$, we obtain that g^m has

$$a_{14}(m) + a_{15}(m) + b_{25}(m) + b_{14}(m) + b_{15}(m) + c_{35}(m) + c_{25}(m) + c_{14}(m) + c_{15}(m) + d_{35}(m) + d_{25}(m) + d_{15}(m)$$

fixed points. Hence it is easy to compute $N_n(m)$ for $n = 5$ (see Table II).

Table I

m	$N_3(m)$	m	$N_3(m)$
1	1	26	10420
2	1	27	16264
3	1	28	25350
4	1	29	39650
5	2	30	61967
6	2	31	97108
7	4	32	152145
8	5	33	238818
9	8	34	374955
10	11	35	589520
11	18	36	927200
12	25	37	1459960
13	40	38	2299854
14	58	39	3626200
15	90	40	5720274
16	135	41	9030450
17	210	42	14263078
18	316	43	22542396
19	492	44	35644500
20	750	45	56393760
21	1164	46	89262047
22	1791	47	141358274
23	2786	48	223955235
24	4305	49	354975428
25	6710	50	562871705

Table II

m	$N_5(m)$	m	$N_5(m)$
1	1	26	1814
2	1	27	2646
3	0	28	3858
4	1	29	5644
5	1	30	8246
6	2	31	12088
7	2	32	17706
8	3	33	25992
9	4	34	38155
10	6	35	56102
11	8	36	82490
12	11	37	121474
13	16	38	178902
14	23	39	263776
15	32	40	389033
16	46	41	574304
17	66	42	848069
18	94	43	1253344
19	136	44	1852926
20	195	45	2741164
21	282	46	4056706
22	408	47	6007042
23	592	48	8898261
24	856	49	13187750
25	1248	50	19551952

REFERENCES

1. Block, L. : The periodic points of Morse–Smale endomorphisms of the circle, Trans.Amer.Math.Soc. 266, 77–88 (1977).

2. Sarkovskii, A.N. : Coexistence of cycles of a continuous map of a line into itself, Ukr.Mat.Z 16, 61–71 (1964).

3. Stefan, P. : A theorem of Sarkovskii on the existence of periodic orbits of continuous endomorphisms of the real line, Comm.Math.Phys. 54, 237–248 (1977).