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BLOWING UP FIXED POINTS

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This note shows how to use the technique of blowing up submanifolds to give easier proofs of some theorems concerning fixed point sets of toral actions on smooth manifolds.

None of the results given here is new. The main theorem (theorem 3) as well as the applications (8), (9) and (10) are well known. Nevertheless I believe proofs presented in this note are not the usual ones.

(1) Let us begin by describing the *equivariant blowing up of invariant submanifolds.*

Let G be a compact Lie group acting smoothly on a smooth manifold M and let B be a G-stable closed smooth submanifold of M.

Consider the induced action of G on the tangent bundle τ_M of M. It is given by

$$a.v = (dT_a)_x(v) \quad x \in M,\ a \in G,\ v \in T_x(M)$$

where $T_x(M)$ denotes the tangent space of M at x and T_a is the diffeomorphism of M defined by $T_a(x) = a.x$.

The above action of G on τ_M restricts to actions of G on $\tau_{M|B}$ and τ_B, since B is G-invariant. Thus we have an induced action of G on the normal bundle, $\nu: E \xrightarrow{\pi} B$, of B in M.

Let $P(\nu):P(E) \to B$ be the projective bundle associated to ν(its fibre over $x \in B$ is the projective space associated to the quotient $T_x(M)/T_x(B)$). The bundle $P(\nu)$ inherits an obvious action from the action of G on ν.

We also consider the canonical line bundle, $\hat{E} \to P(E)$, on P(E) (its fibre over $z \in P(E)$ consists of all vectors of z).

The above actions of G on P(E) and E induce an action of G on E such that the map $\alpha:\hat{E} \to E$, given by $\alpha(z,v)=v$, is G-equivariant.

Observe that α is surjective and restricts to a diffeomorphism $\hat{E}-P(E) \xrightarrow{\alpha} E-B$ (we identify the base B to its image under the zero cross-section).

It is not difficult to construct a G-invariant open neighbourhood U of B in M together with a G-equivariant diffeomorphism $E \xrightarrow{\varphi} U$ such that φ restricts to the identity on B. Set $\sigma = \varphi \circ \alpha: \hat{E} \to U \subset M$. Thus $\sigma: \hat{E}-P(E) \to$ $\to U-B$ is a G-equivariant diffeomorphism.

Let $\hat{M}$ be the space obtained by attaching $\hat{E}$ to M-B via the map $\hat{E}-P(E) \xrightarrow{\sigma} M-B$ (i.e. $\hat{M}$ is obtained from the disjoint union of $\hat{E}$ and M-B by identifying the points of $\hat{E}-P(E)$ to their images on M-B under σ). Endow $\hat{M}$ with the obvious smooth structure for which the inclusions $\hat{E} \xrightarrow{i} \hat{M}$, $M-B \xrightarrow{j} \hat{M}$ are diffeomorphisms onto their images. It is clear that the actions of G on $\hat{E}$ and M-B induce an action of G on $\hat{M}$.

Define $\hat{\sigma}:\hat{M} \to M$ by $\hat{\sigma}(i(x)) = \sigma(x)$ if $x \in \hat{E}$ and $\hat{\sigma}(j(x)) = x$ if $x \in M-B$.

The map $\hat{\sigma}$ is G-equivariant and surjective. Furthermore $\hat{\sigma}:\hat{M}-P(E) \to$ $\to M-B$ is a G-equivariant diffeomorphism.

Observe that $\hat{M}$ is obtained from M by bowing up B onto a G-invariant hypersurface P(E). In particular a point of B has been blown up on to a real projective space.

In case B is a single point, $\tilde{M}$ is simply the connected sum of M and a real projective space.

Blowing up the fixed point set of a group action.

It is easily seen that the connected components of the fixed point set, $F_G(M)$, of the action of G on M are closed smooth submanifolds of M. Blowing up in turn each of them we obtain a smooth manifold $\hat{M}$, acted on by G, together with a G-equivarinat surjective smooth map $\hat{\sigma}:\hat{M} \to M$.

(2) Lemma.

Let G° be the connected component of the unit in G and suppose that G/G° has an odd number of elements. Then the action of G on $\hat{M}$ has no fixed point.

Proof:

Clearly $\hat{M} - \hat{\sigma}^{-1}(F_G(M))$ does not have fixed points since it is G-equivariantly diffeomorphic to $M-F_G(M)$. Therefore it is enough to show that $\hat{\sigma}^{-1}(F_G(M))$ does not have fixed points.

Let B be one of the connected components of $F_G(M)$ and let $\nu:E \xrightarrow{\pi} B$ denote its normal bundle. Endow ν with a G-invariant Riemannian metric and assume that an element $z \in P(E)$ exits such that $a.z = z$ for all $a \in G$. Fix a vector v of norm 1 belonging to z. We must have $a.v = \varepsilon(a).v$ where $\varepsilon:G \to \{1,-1\}$ is a group homomorphism, constant on each connected components of G.

Define $\bar{\varepsilon}:G/G^{\circ} \to \{1,-1\}$ by $\bar{\varepsilon}(\bar{a}) = \varepsilon(a)$ ($\bar{a}$ denoting the class of a in G/G°). Set $\bar{\varepsilon}^{-1}(1) = \{\alpha_1,\dots,\alpha_r\}$ and $\bar{\varepsilon}^{-1}(-1) = \{\beta_1,\dots,\beta_s\}$. We know that $\bar{\varepsilon}^{-1}(-1) \neq \emptyset$ because if not one had $a.v = v$ for all $a \in C$ and, if we identify E to a G-invariant tubular neighbourhood of B in M, this would yield a contradiction, since B is a connected component of $F_G(M)$.

Therefore $\bar{\varepsilon}^{-1}(-1) = \{\alpha_1\beta_1,\dots,\alpha_r\beta_1\} = \{\beta_1,\dots,\beta_s\}$. Hence $r = s$ and G/G° has an even number of elements contradicting the hypothesis of the lemma.

(3) Theorem.

Let G be a torus acting smoothly on a compact smooth manifold M. Then the Euler-Poincaré characteristic, $\chi(F_G(M))$, of the fixed point set of the action of G on M coincides with the Euler Poincaré characteristic, $\chi(M)$, of M.

Proof:

Let $B_1,\ldots,B_r$ be the connected components of $F_G(M)$ with normal bundles $\nu_i:E_i \xrightarrow{\pi_i} B_i$ $(i=1,\ldots,r)$ and let $\varphi_i:E_i \to U_i$ $(i=1,\ldots,r)$ be G-equivariant diffeomorphisms where U_i is a G-invariant open neighbourhood of B_i in M. Set $U = \bigcup_{i=1}^{r} U_i$.

The corresponding Mayer-Vietoris sequences of the open sets $\{M-F_G(M), U\}$ of M and the open sets $\{\hat{M}-\hat{\sigma}^{-1}(F_G(M)) = M-F_G(M), \hat{\sigma}^{-1}(U)\}$ of $\hat{M}$ yield

(4) $\chi(M) + \chi(U-F_G(M)) = \chi(U) + \chi(M-F_G(M))$

(5) $\chi(\hat{M}) + \chi(U-F_G(M)) = \chi(\hat{\sigma}^{-1}(U)) + \chi(M-F_G(M))$

where χ denotes the Euler-Poincaré characteristic and $\hat{\sigma}:\hat{M} \to M$ is obtained, as explained before, by bowing up in turn each of the B_i. Hence $\hat{\sigma}$ is G-equivariant and the action of G on $\hat{M}$ does not have fixed points.

Choose next an element h of the Lie algebra of G such that expth is dense in G and let Z_h be its corresponding fundamental vector field in M. Explicitly Z_h is given by $Z_h(x) = (dA_x)_e(h)$ where $A_x:G \to \hat{M}$ is given by $A_x(a) = a.x$ and e is the unit element of G.

The vector field Z_h has no zeros since the action of G on $\hat{M}$ has no fixed point. Therefore $\chi(\hat{M}) = 0$ because of Hopf theorem (see corollary 3, page 399 of [1]).

On the other hand $\chi(\hat{\sigma}^{-1}(U)) = \sum_{i=1}^{r} \chi(P(E_i))$ and since the action of G on $P(E_i)$ has no fixed point the same argument as before yield $\chi(P(E_i)) = 0$ $(i=1,\ldots,r)$.

Therefore (5) can be written as follows

(6) $\chi(U-F_G(M)) = \chi(M-F_G(M))$.

Finally we deduce from (4) and (6), using also the obvious fact that $\chi(U) = \chi(F_G(M))$,

(7) $\chi(M) = \chi(F_G(M))$.

We show now some applications of theorem (3).

(8) Proposition.

Let T be a maximal torus of a compact connected Lie group G. Then $\chi(G/T)$ is the number of elements of N(T)/T where N(T) is the normalizer of T in G.

Proof:

The fixed point set of the obvious left action of T in G/T, is given by $F_T(G/T) = \{xT \mid x \in N(T)\}$. Therefore $F_T(G/T)$ has the same number of elements as N(T)/T. The proof in now finished by applying (3).

(9) Corollary.

Any two maximal tori of a compact connected Lie group are conjugate.

Proof:

Let T,T' be maximal tori of a compact connected Lie group G and consider the left action of T' on G/T given by $t'.xT = (t'.x).T (t' \in T', x \in G)$.

We know from proposition (8) and theorem (3) that $\chi(F_{T'}(G/T))$ = number of elements of N(T)/T. In particular $F_{T'}(G/T) \neq \emptyset$. Therefore there exists $x \in G$ such that $t'xT = xT$ for all $t' \in T'$. Thus $x^{-1}T'x = T$.

(10) Theorem.

Let K be a closed connected subgroup of a compact connected Lie group G. Then $\chi(G/K) = 0$ if rank K < rank G and $\chi(G/K) = n_G/n_K$ if rank K = rank G (n_G is the number of elements of N(T)/T for T being a maximal torus of G and n_K is the corresponding number for K).

Proof:

a) Suppose first that rank $K < \text{rank } G$ and let T be a maximal torus of G. The left action of T on G/K has no fixed points because if $txK = xK$ for all $t \in T$ then $x^{-1}Tx \subset K$ which is imposible since rank $K <$ rank G. We use then theorem (3) to conclude that $\chi(G/K) = 0$.

b) Assume now that rank K = rank G and let T be a maximal torus of K and hence of G. We have $F_T(G/K) = \{xK \mid x^{-1}Tx \subset K\}$. But $x^{-1}Tx$ is a maximal torus in K, if $x^{-1}Tx \subset K$. Thus by corollary (9) there exists $k \in K$ such that $x^{-1}Tx = k^{-1}Tk$. Therefore $xk^{-1} \in N_G(T)$ (normalizer of T in G). Hence $xK = xk^{-1}.kK = xk^{-1}K$ with $xk^{-1} \in N_G(T)$. Therefore

$$F_T(G/K) = \{yK \mid y \in N_G(T)\} = N_G(T)/N_G(T) \cap K) = N_G(T)/N_K(T) = \frac{N_G(T)/T}{N_K(T)/T}.$$

Therefore $\chi(F_T(G/K)) = n_G/n_K$ and we finish now the proof by using (3).

Bibliography.

1.- W. Greub, S. Halperin, R. Vanstone; Connections, Curvature and Cohomology. Vol.I. Academic Press.