N° 22 Nov. 1980
Actes VII JMHL

ON THE DOMAIN OF ATTRACTION OF STABLE LAWS

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Abstract:

Let f be a function defined on R, and assume we may consider in R a finite number s+1 of intervals such that f is monotone in each of them. The minimum value r that s may assume is the variation index of f. Let f be a non-negative, integrable, unimodal function possessing k-th order derivative. If the variation order of $f^{(i)}$ is i+1, $0 \leq i \leq k$, we shall say that f is unimodal of order k. If f is unimodal of order k for all $k \in N$, we shall say that f is totally unimodal.
We shall prove that any stable distribution possesses a totally unimodal distribution in its domain of attraction.

Let f(.) be a function defined on the real line , and assume that we may consider in $\mathbb{R}$ a finite number s+1 of intervals such that in each of them f(x) is monotone. The minimum value r of the possible values s may assume is called the variation index of f.

Let f(.) be a non-negative, integrable, unimodal function possessing k-th order derivative. If the variation indices of $f', f'', \ldots, f^{(k)}$ are respectively 2,3,...,k+1, we shall say that f is unimodal of order k. If f is unimodal of order k for k=1,2,..., we shall say that f is totally unimodal.

Consider the function $f_p(x;\alpha;c_1,c_2;a_1,a_2)$, where $c_1,c_2 \geq 0$ and $c_1+c_2,\alpha,a_1,a_2>0$, defined as follows:

Research partially supported by Calouste Gulbenkian Foundation.

i) For $x<-a_1$, $f_p(x;\alpha;c_1,c_2;a_1,a_2)=c_1|x|^{-(\alpha+1)}$

ii) For $x>a_2$, $f_p(x;\alpha;c_1,c_2;a_1,a_2)=c_2|x|^{-(\alpha+1)}$

iii) For $x\in[a_1,a_2]$, $f_p(x;\alpha;c_1,c_2;a_1,a_2)$ is identical to the polynomial (of degree 2p if $a_1=a_2$ e $c_1=c_2$, and otherwise of degree 2p+1) defined by the condition that $f_p(x), f_p'(x), f_p''(x), \dots, f_p^{(p)}(x)$ are continuous at $-a_1$ and at a_2.

It is immediate that $f_p(x;\alpha;c_1,c_2;a_1,a_2)$ is unimodal of order p. On the other hand it is easy to check that

$$f_p((x;\alpha;1,1;\sqrt{p},\sqrt{p})= p^{-(\alpha+1)/p}\ f_p(x/\sqrt{p};\alpha;1,1;1,1)$$

and hence, when p is large,

$$p^{(\alpha+1)/2}\ f_p(x;\ \alpha;\ 1,1;\sqrt{p},\sqrt{p}) \simeq \sum_{k=0}^{p} \gamma_k \exp\left(-kx^2/p\right)$$

where γ_k are the coefficients of the developement in Taylor's series of $(1-x)^{-(\alpha+1)/2}$ and hence

$$\gamma_k = \frac{\Gamma\left(\frac{\alpha+2k+1}{2}\right)}{k!\,\Gamma\left(\frac{\alpha+1}{2}\right)} \xrightarrow[k\to\infty]{} \frac{k^{(\alpha-1)/2}}{\Gamma\left(\frac{\alpha+1}{2}\right)}$$

From this, letting $p\to\infty$, $f_p(x;\ \alpha;\ 1,1;\sqrt{p},\sqrt{p})$ converges to the integral

$$f(x) = \frac{1}{\Gamma\left(\frac{\alpha+1}{2}\right)} \int_0^1 y^{(\alpha-1)/2} \exp(-yx^2)\,dy =$$

$$= \frac{1}{\Gamma\left(\frac{\alpha+1}{2}\right)\ |x|^{\alpha+1}} \int_0^{x^2} y^{(\alpha-1)/2} \exp(-y)\,dy$$

Since $f_p(x;\ \alpha;\ 1,1;\sqrt{p},\sqrt{p})$ is unimodal of order k for any $k<p$, $f(x)$ is totally unimodal.

On the other hand

$$\lim_{x \to \pm\infty} |x|^{\alpha+1} f(x) = \frac{1}{\Gamma(\frac{\alpha+1}{2})} \int_0^\infty y^{(\alpha-1)/2} \exp(-y)\, dy = 1$$

and hence f(x) is in the domain of attraction of the symmetric stable law with index parameter $a^* = \min(\alpha,2)$.

Along similar lines, it is possible to prove that the function

$$g(x)=|x|^{-(\alpha+1)} \int_0^{|x|} (1/\Gamma((\alpha+1)/2)+ \frac{x}{|x|} \frac{y}{\beta\Gamma((\alpha+1)/2)})\, y^{\alpha} \exp(-y^2)dy,$$

which belongs to the domain of attraction of the stable law with parameters parameters a^*, $\beta(0<a^*=\min(2,\alpha), |\beta|\leq 1)$, is totally unimodal.

References

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