STABILITY OF PARABOLIC POINTS OF AREA PRESERVING ANALYTIC DIF-
FEOMORPHISMS

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Abstract. Theorems characterizing stable parabolic points are proved. Essen-
tially, stability is equivalent to the fact that the generating function of
the differomorphism, taking out the part which generates the identity, has a
strict extremum at the fixed point. With these results, the study of the sta
bility of fixed points of analytic area preserving mappings (APM) is ended.
Some examples are included, specially the case of elliptic points whose ei--
genvalues are cubic or fourth roots of unity.

§1. Introduction and results. Let T an analytic APM. The (Lyapunov) stabili-
ty of fixed points of T is a method usually employed for the study of the qua
litative properties of periodic orbits (P.O.) in hamiltonian systems with two
degrees of freedom, via the Poincaré mapping with respect to a surface trans
versal to the P.O. in the energy level $H = h$. If the fixed point, that we take
as the origin, is hyperbolic, the inestability of the linear part remains
when nonlinear terms are taken into account. If the fixed point is elliptic
the stability of the linear part is preserved provided that the eigenvalues
$\lambda, \bar{\lambda}$ are not third or fourth roots of unity and that suitable coefficients of
the Birkhoff Normal Form (B.N.F.) are not zero. When the fixed point is dege
nerated or parabolic, i.e., Spec $DT(0) \subset \{\pm 1\}$, the stability is a more subtle
question. It is not always enough to consider only the lower degree nonli- -
near terms to decide about stability. Besides the cases $\lambda^3 = 1$ and $\lambda^4 = 1$, di--
fficulties can appear for every λ, k-th root of unity if all the determined
coefficients (the first $\left[\frac{k-2}{2}\right]$ ones) in the B.N.F. are zero. Examples of ines
tability exist for all k [9]. The case of λ being a k-root of unity is redu-

ced to the parabolic one taking T' instead of T. Without loss of generality we can suppose that in the parabolic case the eigenvalues are equal to one. (Take T^2 if necessary. This accounts also for T orientation reversing).

In [11] the following results are proven for the parabolic case when DT(0) can not be reduced to diagonal form, i.e., $DT(0) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ in a suitable basis:

1.1. Lemma. Let $T(x,y) = (x + f(x,y), x+y+g(x,y))$ *be an analytic APM with* f,g *beginning with terms of degree at least two. Then there exists a near the identity polynomial change of variables* C *such that the transformed mapping* $T^* = C^{-1}TC$ *is given by* $T^*(x,y) = (x+F_n(x,y)+0_{n+1}, x+y+0_{n+1})$ *where* F_n *is a degree n polynomial without linear terms and* 0_s *stands for a series with terms of lower degree at least s.*

1.2. Theorem. In the hypothesis of 1.1 let $F_n(z) = a_m z^m + 0_{m+1}$, $a_m \neq 0$. *Then the origin is stable under* T^* *(and therefore under* T*) iff m is odd and* $a_m < 0$.

The object of the communication is to give a theorem characterizing the stable parabolic points for the remaining case, i.e., when DT(0) can be put in diagonal form. (Then T is near the identity in a neighbournhood U of the fixed point).

Let $(x',y') = T(x,y)$ a canonical mapping. If $D_y y'$ is regular (as happens in our case) we can define an analytic generating function (see [1]) $\hat{G}(x,y')$ such that $\hat{G}(x,y') = xy' + G(x,y')$ and $x' = D_{y'}\hat{G}$, $y = D_x\hat{G}$. For the nondiagonal case of 1.2. we get $G(x,y') = -x^2/2 + \int_0^{y'} F_n(u)\, du + 0_{n+2}(x,y')$. Theorem 1.2 can be reformulated as: stability is equivalent to $G(x,y')$ having a strict extremum at the origin. That this characterization is applicable to the diagonal case is stated in the main result:

1.3. Theorem. Let P *be a parabolic fixed point of an analytic APM,* T, *and* $\hat{G}(x,y') = xy' + G(x,y')$ *a generating function for* T. *Then* P *is Lyapunov stable iff* G *has a strict extremum at* P.

As far as instability is concerned the results obtained here extend the ones of McGehee [7] but only for the conservative case.

§2. Sketch of the proof. Only the case $DT(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ of 1.3. remains to be proved. Instead of using the fact that G has a strict extremum at the origin we can equivalently consider that, near the origin, the sets G=g with $|g|$ small and suitable sign, are closed curves around the origin. The algorithm to decide whether or not G has a strict extremum at the origin using the Newton polygon is defered to the next section.

Let Φ_1 be the time unit flow associated to the hamiltonian system with hamiltonian G: $\Phi_1(x,y) = (\bar{x},\bar{y})$. We intend to use Φ_1 as an approximation of T in U. By the way, if G_i is the partial derivative of G w.r.t. the i-th argument, G_{ij}, G_{ijk}, ... the second, third, ... partial derivatives, better approximations to T can be obtained with modified hamiltonians:

$$H = G - \frac{1}{2} G_1 G_2 + \frac{1}{12}(G_{11}G_2^2 + 4G_{12}G_1 G_2 + G_{22}G_1^2) - \frac{1}{6} (G_{112}G_1 G_2^2 + G_{122}G_2 G_1^2 + G_{11}G_{12}G_2^2 +$$
$$+ G_{22}G_{12}G_1^2 + G_{11}G_{22}G_1 G_2 + 3G_{12}^2 G_1 G_2) + \cdots .$$

We get increasing approximation taking terms of increasing order. However H=G is enough for the proof.

In $U - \{0\}$ we define $r=G(x,y)$, $\alpha=2\pi t/\tau(r)$ where $\tau(r)$ is the period of the flow of hamiltonian G along the closed curve $\gamma=\{G(x,y)=r\}$. Here t stands for the time interval in going from $(x_o,0)$ to (x,y) along γ, with $x_o>0$ (one shows that γ is star-shaped w.r.t. the origin if $|r|$ is small enough). In the (r,α) variables one has $\Phi_1(r,\alpha) = (r,\alpha+2\pi/\tau(r))$. A computation gives $d\,\tau(r)/dr = 0(r^\beta)$, $\beta<0$. Therefore Φ_1 is a twist. The initial map can be expressed in the (r,α) variables as $T(r,\alpha) = (r+\Delta r, \alpha +2\pi/\tau(r)+\Delta\alpha)$, where $\Delta r,\Delta\alpha$ begin with terms of relative high order. Hence T can be seen as a perturbed twist [8] and this guarantees the existence of invariant curves from where the stability follows.

If $G=g$, $|g|$ small, does not define closed curves in U but G=0 has seve--ral branches through the origin then we get instability under Φ_1 [6] and, therefore, under T. Complete proofs appear in [12].

§3. An algorithm to decide about stability. First we plot the Newton poly--gon associated to G. A necessary condition for stability is that all the vertices have even coordinates. Let $m+k\alpha$, $n-k\beta$, $\alpha,\beta\in Z_+$, g.c.d. $(\alpha,\beta)=1$, $k=0 \div r$ be points in one side of the polygon. There we get $G=\ldots+\sum_o^r a_k x^{m+k\alpha}$. $y^{n-k\beta} + \ldots$. Let $\varphi = \sum_o^r a_k z^k$. An additional necessary condition is that all the real zeros of φ be of even multiplicity. If the multiplicity is zero this is enough for stability. If it is not zero, three cases are posible, associated to each of such zeros: $y = 0(x)$; $x = 0(y^\alpha)$, $\alpha >1$; $y = 0(x^\alpha)$, $\alpha >1$. The first and second cases can be reduced to the third one through a rotation or a relabelling of the axes, respectively. Therefore, we can suppose $y = mx^{p/q} + \ldots$, $p/q >1$. Introducing $x = u^q$, $y = mu^p + z$, we get a new Newton polygon and we proceed to the analysis of terms of the form $z = 0(u^\beta)$, $\beta>q$.

§4. Some examples.

a) We consider the case λ a fourth root of unity. A simple map is $T(x,y) =$
$= (-y, x+f(y))$ (a de Jonquière map, normal form if T is a Cremona map of pri_
me degree [3]). Taking T^4 we can apply 1.3 and §3. If f begins with terms
of degree k and k is odd, the origin is stable. If k is even and f has only
one term (y^k after scaling) the origin is stable. This is the case for the
classical Hénon map [5,10] with rotation angle $\alpha = \pi/2$ (k=2). The invariant
curves are latin cross shaped. However, higher order terms can produce ins-
tability. For instance, $T(x,y) = (-y, x+y^2+\alpha y^3)$ is unstable for $\alpha \in [-1,0)$. See
[12].

b) If λ is a cubic root of unity and we restrict ourselves to $T(x,y) =$
$= R_{2\pi/3} \circ (x, y - x^k)$, where R_δ is a rotation of angle δ around the origin, we
get stability (unstability) if k is odd (even).

c) Concerning the restricted three-body problem, the stability of $\mathcal{L}_4$ for ma_
sses μ equal to the critical values of Routh (see [2]) only the values μ_1,
μ_2, μ_3 remain to settle the question. With the help of some lengthly computa_
tions the results will appear elsewhere [13]. Other applications to the sta_
bility of bifurcation orbits can be found in [4].

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