Pub. Mat. UAB
Nº 16, Novembre 1979

THE NORMAL AND MACKEY TOPOLOGIES ON CO-ECHELON SPACES

by

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Abstract. A necessary and sufficient condition for the coincidence of the normal and Mackey topologies on a co-echelon space of order one is studied.

Introduction. Let $a_n^{(k)}$, $n=0,1,\dots$, $k=1,2,\dots$ be such that the following conditions are satisfied.

a) $a_n^{(k)} > 0$, for each k and n.

b) $a_n^{(1)} \leq a_n^{(2)} \leq a_n^{(3)} \dots$ $n=0,1,\dots$

Let E and E^x be the echelon and co-echelon spaces respectively, corresponding to the steps $a_n^{(k)}$ [1, p.419].

In this paper, the following theorem is proved.

Theorem. *The normal and Mackey topologies in* E^x *coincide if and only if E with the normal topology is nuclear.*

It is known that the normal and Mackey topologies in E^x coincide if and only if for every sequence (x_n) satisfying $\lim_{n\to\infty} x_n \cdot a_n^{(k)} = 0$ for $k=1,2,\dots$, it follows that $\sum_{n=0}^{\infty} |x_n| \cdot a_n^{(x)} < \infty$, for every $k=1,2,\dots$ [3].

On the other hand, the Grothendieck-Pietsch criterium establishes that E with the normal topology is nuclear if and only if for every k, there exists an N(k) such that $\sum_{n=0}^{\infty} a_n^{(k)}/a_n^{(N(k))} < \infty$ [2, p.98].

The previous theorem is, then, an immediate consequence of the following proposition.

Proposition. *Let k_o be such that for each $j=1,2\ldots$, we have $\sum_{n=0}^{\infty} a_n^{(k_o)}/a_n^{(j)} = \infty$. There exists a sequence (a_n), $a_n > 0$, $n=1,2\ldots$, such that $\lim_{n\to\infty} a_n . a_n^{(k)} = 0$, $k=1,2,\ldots$ while $\sum_{n=0}^{\infty} a_n . a_n^{(k_o)} = \infty$.*

Proof. $\sum_{n=0}^{\infty} a_n^{(k_o)}/a_n^{(j)} = \infty$, $j=1,2\ldots$ implies that there exists $0 < n_1 < n_2 < \ldots$ such that

$$\sum_{n=0}^{n_1} a_n^{(k_o)}/a_n^{(1)} \geq 2$$

$$\sum_{n_i + 1}^{n_i + 1} a_n^{(k_o)}/a_n^{(i+1)} \geq 2^{i+1}, \qquad i=1,2\ldots$$

Consider, now, the sequence

$$a_n = \begin{cases} \frac{1}{2}\,(a_n^{(1)})^{-1} & , \quad 0 \leq n \leq n_1 \\ \frac{1}{2^{i+1}} \cdot a_n^{(i+1)\ -1} & , \quad n_i + 1 \leq n \leq n_{i+1}, \quad i=1,2\ldots \end{cases}$$

It is obvious that $\lim_{n \to \infty} a_n . a_n^{(k)} = 0$, $k = 1,2,\ldots$ because given k, we have

$$a_n^{(j)^{-1}} . a_n^{(k)} \leq 1, \quad j \geq k, \quad n = 1,2,\ldots .$$

Howewer $\sum_{n=0}^{\infty} a_n . a_n^{(k_0)} = \infty$.

REFERENCES

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2 A. Pietsch, Nuclear locally convex spaces, Springer-Verlag. Berlin. Heidelberg. New York, 1972.

3 M. Tort Pinilla, Consideraciones sobre la topología normal en los espacios de Köthe, Actas de las primeras jornadas matemáticas luso-españolas, 1973.

Appendix. The referee has kindly pointed out to us that it is not necessary to take $a_n^{(k)} > 0$ for each k and n. Supposing $a_n^{(k)} \geq 0$ for each k and n, and that for each $k \in N$ there exists an $n \in N$ such that $a_n^{(k)} \neq 0$, then we may proceed in an analogous way finding the same results.