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A NOTE ON THE ITERATION OF EXPONENTIALS

Carles Simó
Secció de Matemàtiques
Universitat Autònoma de Barcelona

We consider the sequence $y_{i+1} = z^{y_i}$, $i \geq 0$, $z \in R_+$, with $y_0 = z$. In [1], the question of the behavior of such sequence is posed. Subsequently, many references to solutions are given (see [2]), for instance [3]. In this paper we obtain a full description of these iterates as functions of the parameter z, for every value of y . Our technique just uses the discrete dynamical system in R_+ defined by $f_z(x) = z^x$. The properties of the curves of fixed points and of two-periodic points are also given.

§ 1.- Fixed points.

a) If $z > 1$, $f_z(x)$ is concave; $z^x = x$ has solution iff $z \leq b$, where b must satisfy $x = b^x$, $1 = b^x \ln b \Longrightarrow b = e^{1/e}$. If $z \in (1,b)$ there are two fixed points $x_1(z) < x_2(z)$. They coincide for $z = b$.

For $x_1(z)$ we have $0 < f_z'(x_1(z)) < 1$. Then it is stable. Instability occurs for $x_2(z)$.

b) If $z < 1$, $f_z(x)$ is monotonically decreasing. Then $x = z^x$ has only one solution $x_1(z)$.

Stability: $f_z'(x) < 0$ implies $x_1(z)$ stable if $f_z'(x_1) > -1$. The limit of stability is found at $x \ln z = -1 \Longrightarrow x = 1/e$ and $z_{lim} = a = e^{-e}$.

For $z=1-\varepsilon$, ε small enough: $z^{x_1} > 1-\varepsilon \Longrightarrow f_z'(x_1) = O(\varepsilon)$. Then the fixed points is stable for $z \in [a,1)$. The negative character of f' implies that the iterates alternate around the fixed point.

c) Curve of fixed points: Consider the curve $x = x(z)$, $z \in (0,b]$ given by $x = z^x$ (two branches if $z>1$); $x' = \frac{dx}{dz} = \frac{x}{z(1-\ln x)} = \frac{\ln x}{z \ln z(1-\ln x)}$. One has $x' = \infty$ at $z=b$. The upper branch has $x_2' < 0$, $z \in (1,b)$, and lower one gives $x_1'(z) > 0$ in $(0,b)$. We get as limiting values: $\lim_{z\to 0^+} x_1' = -\lim_{z\to 0^+} \frac{1}{z \ln z} = \infty$;

$\lim_{z\to 1^+} x_2' = -\infty$; $\lim_{z\to 1} x_1' = 1$. We obtain for the second derivative

$x'' = \frac{x \ln x + x/(1-\ln x)}{z^2(1-\ln x)^2}$ zero values iff $\ln x = (1\pm\sqrt{5})/2$. Then

there are only two turning points: one, x_2^i, in $x_2(z)$ and the other, x_1^i, in $x_1(z)$ for some $z<1$. With this information we can plot $x(z)$. This is done in fig.1.

§ 2.- Periodic points.

a) Being $f_z(x)$ increasing if $z>1$, there are no periodic points. For $z<1$, $f_z^2(x)$ is also increasing. Then there are only fixed points under f_z (studied in §1)) or 2-periodic points $x_3(z)$, $x_4(z)$.

b) We consider the function $g_z(x) = z^x - \log_z x$ for $z<a$. We have $g_z(x_1) = 0$, $g_z'(x_1) < 0$ and $g_z(1) > 0$. Then there are points $y \in (x_1, 1)$ fixed under g_z. Let us now show their uniqueness.

It is enough to proof that there is a unique point x such that $g_z'(x) = 0$. Then $x z^x \ln^2 z = 1$. We define $\Psi(x) = xz^x$. As $\Psi'(x) = (1+\ln x)\, z^x$, we have for $x > x_1$:

$|x \ln z| > |x_1 \ln z| = |\ln x_1| > 1 \Rightarrow \Psi'(x) < 0$ for $x \in (x_1, 1)$. But $f_z'(x_1) = z^{x_1} \ln z$, $(f_z'(x_1))^2 = x_1 z^{x_1} \ln^2 z \Rightarrow \Psi'(x_1) =$ $= (f_z'(x_1))^2/\ln^2 z$. Then g_z' has a zero in $(x_1, 1)$ iff x_1 is unstable for f_z, i.e., iff $z \in (0,a)$. So there is only one 2-periodic point in $(x_1, 1)$ which is $x_4(z)$. The image under f_z, $x_3(z)$, is also 2-periodic and belongs to $(0, x_1)$. The stability of 2-periodic points is guaranteed because $g_z'(x_i) > 0$, $i = 3,4$. Furthermore, if $h_z(x) = z^{z^x}$ we have $h_z'(x_i) > 0$, $i = 3,4$.

c) Curve of two-periodic points: There are two branches for $z \in (0,a]$ which coincide if $z = a$. From $z^x = \frac{\ln x}{\ln z}$ we derive $z \ln z (x \ln x \ln z - 1) x' = -x \ln x (1 + x \ln z)$. For $x \ln x \ln z = 1$ we get $x' = \infty$. This happens if $x = e^{-1}$, $z = a$. The signs of the factors allow us to state that $x_3' > 0$, $x_4' < 0$. Indeed, we begin by proving that $1 + x \ln z$ has only one zero: $z^x = e^{-1} = \ln x / \ln z$ and $x \ln z = -1$ imply $x \ln x = -e^{-1}$, i.e., $x = e^{-1}$.

The same happens for $x \ln x \ln z - 1$, but the proof is more tedious: $z^x = e^{1/\ln x} = \ln x/\ln z$ and $x \ln x \ln z = 1$ give us $x \ln^2 x = e^{1/\ln x}$. The change $t = 1/\ln x$ transforms the above given condition to $\xi(t) = \xi(t^{-1})$, where $\xi(t) = t\, e^t$, $t < 0$. We must verify that $t = -1$ is the unique solution. This is equivalent to find the positive solutions of $\varphi(t) = t$, where $\varphi(t) = \exp(\frac{1}{2}(t - 1/t))$. Obviously 0, 1 are solutions. But $\varphi'(t) = (t^2 + 1)\varphi(t)/(2t^2)$; $\varphi''(t) =$ $= (t^4 + 2t^2 - 4t + 1)\varphi(t)/(4t^4)$; $\varphi'''(t) = (t^6 + 3t^4 - 12t^3 + 27t^2 - 12t + 1)\varphi(t)/(8t^6)$. Then, $\varphi'(0) < 1$, $\varphi'(1) = 1$, $\varphi''(1) = 0$, $\varphi'''(1) > 0$ implies that the number of zeros of $\varphi(t) = t$ in $(0,1)$ counted with their multiplicities is even. If that number is positive, $\varphi''(t)$ must have at least two zeros in $(0,1)$, but such zeros satisfy $t^4 + 2t + 1 = 4t$. Since $(t^2+1)^2$ is concave, there are exactly 2 solutions and one of them is 1. Then there are no solutions of $\varphi(t) = t$ in $(0,1)$. On the other side $\varphi''' > 0$ if $t > 1$, implies $\varphi(t) > t$, $\forall t > 1$. This ends the proof.

The behavior of the two branches near $z = 0$ is found by asymptotic expansions: Let $x_3 = z(1 + \alpha(z))$, $\alpha(z) = o(1)$. We try to satisfy $z^x = \ln x / \ln z$. Then $\alpha(z) = z \ln^2 z + O(z^2 \ln^2 z)$. The image under f_z gives $x_4 = 1 + z \ln z + O(z^2 \ln^2 z)$. This allows us to plot $x_i(z)$, $i = 3,4$. See fig.1.

§ 3.- Behavior of the iterates.

Let be $y_0, y_1, y_2, y_3, \ldots$ the successive iterates.

a) If $z > b$ one has $y_n \uparrow \infty$.

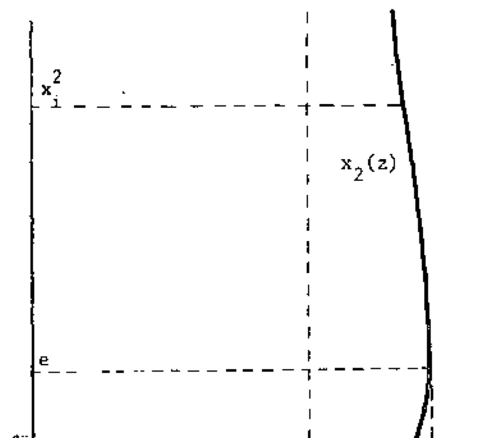

Fig.1

b) For $z = b$ and $y_0 \leq e$, we have $y_n \uparrow e$. For $y_0 > e \Rightarrow y_n \uparrow \infty$.

c) If $z \in (1,b)$ and $y_0 \leq x_1(z)$ we get $y_n \uparrow x_1(z)$; $y_0 \in (x_1, x_2) \Rightarrow y_n \downarrow x_1$; $y_0 > x_2 \Longrightarrow y_n \downarrow \infty$.

d) When $z \in [a,1)$, for every initial value y_0 we have $y_n \to x_1(z)$, but the iterates alternate in $(0, x_1)$, (x_1, ∞). So, $y_0 \in (0, x_1) \Longrightarrow y_{2k} \uparrow x_1$, $y_{2k+1} \downarrow x_1$. For the critical value $z = a$ we have a bifurcation: $x_1(z)$ losses the stability and a two-point stable cycle appears.

e) If $z \in (0,a)$ we have also the fixed point $x_1(z)$, but any $y_0 \neq x_1(z)$ gives iterates converging to the cycle $x_{3,4}(z)$ (and them they do not properly converge).

$y_0 \in (x_3, x_1) \Longrightarrow y_{2k+1} \uparrow x_4,\ y_{2k} \downarrow x_3;\ y_0 \in (0, x_3) \Longrightarrow y_{2k+1} \downarrow x_4,\ y_{2k} \uparrow x_3.$

Similar results are obtained for $y_0 \in (x_1, x_4)$ or $y_0 \in (x_4, \infty)$.

In particular, if $y_0 = z$ the iterates converge to x_1 iff $z \in [a,b]$ and to the cycle $\{x_3 x_4\}$ iff $z < a$.

References

[1] Notices Amer. Math. Soc. 25(1978), 197.

[2] Notices Amer. Math. Soc. 25(1978), 253, 335.

[3] Bromwich, T.J.I'A.: "An Introduction to the Theory of Infinite Series", MacMillan, 1965, p. 23.