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Nonlinear Rank Correlations

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Abstract: rank correlation is a fundamental tool to express dependence in case in which the data are arranged in order. There are, by contrast, circumstances where the ordinal association is of a nonlinear type. In this paper we have investigated the effectiveness of thirty measures of rank correlation for assessing agreement between two evaluators in the presence of non costant scale of terms. These measures have been divided into three classes: unweighted rank correlations, weighted rank correlations, correlations of scores. Our findings suggest that none is systematically better than the other in all circumstances. However, a simply weighted version of the Kendall's τ provides plausible answers to many special situations where intercategory distances are not the same.

Keywords: ordinal data, nonlinear association.

1 Introduction

Throughout this paper we will examine situations of the following type. Consider a fixed set of n distinct items ordered according to the different degree in which they possess two common attributes represented by X and Y. Let us suppose that each attribute consist of a host of intangibles that can be ranked but not measured and that the evaluations are expressed in terms of an ordinal scale of n ranks: $\mathbf{q} = q_1, q_2, \cdots, q_n$ for X and $\mathbf{s} = s_1, s_2, \cdots, s_n$

for Y. In this paper we consider only a complete linear ordering case so that s_i and q_i take on value in the set of integers $\{1, 2, \cdots, n\}$; moreover, evaluators are asked to decide on a definite rank order for each attribute so that no two objects are given the same rank, even if they seem equally acceptable. In practice, the vectors $\mathbf{s}$ and $\mathbf{q}$ are elements of ${}_nP_n$, the set of all $n!$ permutations. With no essential loss of generality we may assume that s_i is the rank of y_i after $\mathbf{q}$ has been arranged in its natural order (sorted permutation) with the corresponding ranks s_i aligned beneath them.

$$\begin{array}{ccccccc} 1 & 2 & \cdots & i & \cdots & n-1 & n \\ s_1 & s_2 & \cdots & s_i & \cdots & s_{n-1} & s_n \end{array} \tag{1}$$

A rank correlation $r(\mathbf{q}, \mathbf{s})$ is a statistic summarizing the degree of association between two rankings $\mathbf{q}$ and $\mathbf{s}$ where $\mathbf{q}$ acts as a reference to the other. For comparability, the coefficients are usually constructed to vary between -1 and 1. Their magnitude increases as the association increases with a $+1$ (-1) value when there is perfect positive (negative) association from concordance (discordance) of all pairs. The value of zero is indicative of no association.

The inverse $\mathbf{s}'$ of $\mathbf{s}$ is the permutation obtained by interchanging the two rows and then sorting the column into increasing order of the new top row. The permutation inverse transforms an index vector into a rank vector and *vice versa*, that is $s'_{s_i} = i, i = 1, \cdots, n$. A plausible rank correlation statistic should be restricted to be symmetric under inversion

$$r(\mathbf{q}, \mathbf{s}) = r(\mathbf{q}, \mathbf{s}') \tag{2}$$

otherwise it would be possible to change the value of $r(\mathbf{q}, \mathbf{s})$ simply by exchanging the reference ordering.

Rankings in [1] are referred to a classification of n items with 1 assigned to the most preferred item, 2 to the next-to-most preferred and so forth. If an opposite orientation of the arrangement is applied, then a rank correlation statistic that change its sign, but not its absolute value is said to be antisymmetric under reversal

$$r(\mathbf{q}, \mathbf{s}) = -r(\mathbf{q}, \mathbf{s}^*) \tag{3}$$

where s_i^* is the antithetic ranking of s_i that is $s_i^* = n - s_i + 1, i = 1, \cdots, n$. The usefulness of this principle is that a classification of n items can be organized according to the types of problems that occur, enabling a user to better focus on and control problems and thereby providing more meaningful measurement. Condition [3] can easily be obtained by averaging a statistic computed on the ranks with the same statistic computed on the antithetic ranks (Salama and Quade [24], Genest and Plante [9], Cifarelli *et al.* [4]).

The main obiective of this paper is to examine a selection of rank correlations and identify limitations and merits of each relatively to several situations of nonlinear type. The contents of the various sections are as follows. Section 2 presents several cases of nonlinear association between rankings. In Section 3 we will concentrate mainly on analyzing the fundamental factors which affect the behavior of some unweighted rank correlation statistics under a nonlinear interaction. In particular, we will show the inadequacy of unweighted rank correlation coefficients to deal with such situations. Section 4 reviews the general formulation of weighted rank correlations in which the incorporation of a weight fucntion allows more flexibility in the tests. The function is to be chosen so as to weigh the comparisons according to the importance attached to various subsets of ranks. In this sense, Section 4 highlights the more salient features and the performance of several choices of the weight function. Section 5 reports on a class of correlation statistics obtained by computing the Pearson's product-moment correlation coefficient on suitably chosen scores which replace ordinary ranks. In Section 6 we obtain the critical values of the most promising coefficients to enable such statistics to be applied to real data.

2 Nonlinear association

Situations in which a coefficient of agreement/disagreement should take into account the contextual factors that affect judgement are common in real world. In this section we describe a number of tight but nonlinear relationships between two rankings.

Ceiling or floor effects. These represent cases of limited resource allocation because ascribing higher importance to one item reduces the importance of another. For example, it is more satisfactory the placing of the winner in a race in the first position than the placing of the worst contestant last. In other cases differences in low ranks would seem more critical. For example, when an admission office expunges the less qualified candidates.

Bipolarity conditions. The top-down and the bottom-up process may simultaneously affect the same attribute giving rise to a bi-directional effect. Let us consider, for example, the comparison of the final league tables with expert forecasts made before the start of the season. In league football, both the teams placed near the top (which gain promotion) and those placed near the bottom (which risk relegation) are relevant to evaluate the accuracy of the prediction. The teams placed in the middle part of the rankings have negligible influence.

Critical region phenomena. A relationship exists within the central part of the scale of measurement, but at the extremes no relationship is observed, either by virtue of insensitivity of the measures, or through some more intrinsic characteristic of the causal relationship, or because the errors of observations are greatest at the extremes. Only the central rankings are believed to have any practical importance. For instance, the unhealthy levels of body mass index are at both ends of the scale and the middle is relatively safe. This implies that both extremes of the ranking in one variable may be placed together at one extreme of the other variable.

Quadratic trends. One of the most common nonlinear pattern is a U-shaped or inverted U-shaped relationship in which the values in the ranking show an increase followed by a decrease or vice versa. An example of the former is the environmental Kuznets curve predicting that the environmental quality appears to deteriorate with countries' economic growth at low levels of income, and then to improve with economic growth at higher levels of income. An example of an inverted U-shaped pattern is the Yerkes-Dodson law relating the level of arousal and the expected quality of performance.

Bilinear association. Increasing degree of attribute Y are combined with increasing degree of attribute X, but in a bilinear ascending (descending) pattern the mean of the ranks to the left of the central rank of Y is significantly higher (lower) than the mean of the ranks on opposite side. These situations may occur, for instance, when the evaluators tend to separate the items into two distinct groups, but all the items in a group are considered superior, in some sense, to all the items in the other group.

In order to get a feeling as to the nature of nonlinear association, the relationships discussed above are illustrated in Table 1 with $n = 15$ fictitious rankings. The abbreviation LH (HL) indicates that the lowest (highest) points of the scale come first. The suffix A and D stand for ascending and descending respectively.

Table 1: examples of nonlinear rankings

A	Natural ordering	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
B	Inverse ordering	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1
C	Floor effect	1	2	3	4	15	14	13	12	11	10	9	8	7	6	5
D	Ceiling effect	11	10	9	8	7	6	5	4	3	2	1	12	13	14	15
E	Bipolarity/A	1	2	3	4	11	10	9	8	7	6	5	12	13	14	15
F	Bipolarity/D	15	14	13	12	11	6	7	8	9	10	5	4	3	2	1
I	U-shaped/LH	8	7	6	5	4	3	2	1	9	10	11	12	13	14	15
J	U-Shaped/HL	15	14	13	12	11	10	9	1	2	3	4	5	6	7	8
K	Inverted U/LH	1	2	3	4	5	6	7	8	15	14	13	12	11	10	9
L	Inverted U/HL	9	10	11	12	13	14	15	8	7	6	5	4	3	2	1
M	Bilinear/A	9	10	11	12	13	14	15	1	2	3	4	5	6	7	8
N	Bilinear/D	8	7	6	5	4	3	2	1	15	14	13	12	11	10	9

Naturally, such nonlinearities are not exhaustive, but we believe that they cover some of the most interesting cases.

3 Unweighted rank correlations

Moran [19] suggested a simple technique for devising a nonlinear rank correlation. First we define a ranking $\mathbf{q}$ which shows such behavior perfectly. Given some other ranking $\mathbf{s}$ we measure its departure from such an ideal behavior by quantifying the nonlinear deficit by an appropriate distance $\delta(.)$ which does not take explicitly into account a weighting scheme for ranking comparisons.

$$r(\mathbf{q}, \mathbf{s}) = 1 - 2\frac{\delta(\mathbf{q}, \mathbf{s})}{\max\limits_{\mathbf{s} \in {}_nP_n} \delta(\mathbf{q}, \mathbf{s})} \tag{4}$$

where ${}_nP_n$ is the set of all $n!$ permutations. Both Kendall's τ ([15]) and Spearman's ρ ([28]) can be expressed using [4].

Another example (see Cifarelli *et al.*[4]) is the ratio between distance of $\mathbf{s}$ from a reference permutation $\mathbf{q}$ and distance of $\mathbf{s}$ from the antithetic reference permutation $\mathbf{q}^*$

$$r(\mathbf{q}, \mathbf{s}) = \frac{\delta(\mathbf{q}^*, \mathbf{s}) - \delta(\mathbf{q}, \mathbf{s})}{\delta(\mathbf{q}^*, \mathbf{q})} \tag{5}$$

The Gini cogradation coefficient e the Gideon-Hollister maximal deviation belong to this class.Table 2 reports several examples of rank correlation coefficients based either on [4] or on [5].

The indices r_1 and r_4 are well-known. The cogradation coefficient r_2, proposed by Gini [11] as an improvement over the Spearman's footrule r_{14} ([29]), has been recently rediscovered by Salama and Quade [25] (see also Nelsen and Ubeda-Flores [20]). Formula [5] with the Hamming distance yields r_3 where $h(x)$ denotes the indicator function which equals 1 if x is true and 0 otherwise. The statistic r_5, given by Gideon and Hollister [10], derives from the principle of greatest deviation. The coefficient r_6 is based on the squared index of a permutation discussed by Knuth [16, p 16]. Salvemini [26] described the Fechner's coefficient r_7 and introduced the rank correlation r_8. The Fechner index r_7 can also be determined by the number of "runs up" in permutation $\mathbf{s}$; in fact, r_7 coincides with the test of randomness devised by Moore and Wallis [18] and with the rank correlation statistics based on rises discussed by Salama and Quade [24].

Table 2: unweighted rank correlations

Name	Formula
Spearman	$r_1 = 1 - \frac{6\sum_{i=1}^{n}(i-s_i)^2}{n^3-n}$
Gini	$r_2 = 2\frac{\sum_{i=1}^{n}\lvert i-s_i^*\rvert - \sum_{i=1}^{n}\lvert i-s_i\rvert}{n^2-k_n};\ \ k_n = n \mod 2$
Hamming distance	$r_3 = \frac{\sum_{i=1}^{n} h(s_i^*=i) - \sum_{i=1}^{n} h(s_i=i)}{n-k_n}$
Kendall	$r_4 = \frac{2\sum_{i<j} sgn(s_j-s_i)}{n(n-1)}$
Gideon-Hollister	$r_5 = 2\frac{\max\limits_{1\le i\le n}\sum\limits_{j=1}^{i} h(s_j^*>i) - \max\limits_{1\le i\le n}\sum\limits_{j=1}^{i} h(s_j>i)}{n-k_n}$
MacMahon	$r_6 = 1 - \frac{12\sum_{i=1}^{n-1} i^2 h(s_i>s_{i+1})}{2(n-1)^3\,3(n-1)^2+n-1}$
Fechner	$r_7 = \frac{\sum_{i=2}^{n} sgn(s_i-s_{i-1})}{n-1}$
Salvemini	$r_8 = \frac{s_n-s_1}{\sum_{i=2}^{n}\lvert s_i-s_{i-1}\rvert}$
Quadrant association	$r_9 = \frac{n_1-n_2}{n_1+n_2}$
Dallal-Hartigan	$r_{10} = \frac{\lambda_n-\gamma_n}{n-1}$
Average slope	$r_{11} = 2\frac{\sum_{i<j}\left(\frac{s_j-s_i}{j-i}\right)}{n(n-1)}$
Median slope	$r_{12} = median\left\{b_{ij}\lvert b_{ij} = \frac{s_j-s_i}{j-i}, 1\le i<j\le n\right\}$
Inversion table	$r_{13} = 1-2\sqrt{\frac{6\sum_{i=1}^{n} b_i^2}{2(n-1)^3+3(n-1)^2+(n-1)}}$
Spearman's footrule	$r_{14} = 1 - \frac{4\sum_{i=1}^{n}\lvert i-s_i\rvert}{(n^2-k_n)}$
Gordon	$r_{15} = 2\left(\frac{\lambda_n-1}{n-1}\right)-1$
Bhat-Nayar	$r_{16} = 1 - \frac{2\max_{1\le i\le n}\sum_{j=1}^{i} h(s_j'>i)}{\lfloor\frac{n}{2}\rfloor}$

Let the (q, s) plane be divided into four regions by the lines $q = (n+1)/2$ and $s = (n+1)/2$. The statistics r_9, developed by Blomqvist [3], is calculated on the basis of concordance/discordance in the number of pairs n_1 belonging to the first and third quadrants compared with the number n_2

belonging to the second and fourth quadrants. The coefficient r_{10} has been suggested by Dallal and Hartigan [7] as a measure of monotone association for a bivariate sample. The symbols λ_n and γ_n indicate the maximum length of a subsequence $\left(q_{i_j}, s_{i_j}\right), j = 1, \cdots, \lambda_n$ such that both q_{i_j} and s_{i_j} are increasing or decreasing respectively. The slopes statistic r_{11} is the average pairwise slope between observation (i, s_i) and (j, s_j). An analogous concept to r_{11} is the median r_{12} of the slopes between all combinations of two points in the data. Coefficient r_{13} depends on the Euclidean distance between the inversion table of the current ranking and the inversion table of the sorted permutation $s_i = i$ for $i = 1, 2, \cdots, n$. An inversion table uniquely determines the corresponding permutation (Knuth, [16, p. 12]). A similar coefficient based on the city-block distance gives the same values as the median slope statistic. The coefficient r_{15} is a linear transformation of the Gower's measure [13] of similarity for variables measured on an ordinal scale. Coefficient r_{16}, given by Bhat and Nayar [1], is based on the distance between the identity permutation $\mathbf{q}$ and the inverse permutation $\mathbf{s}'$ of $\mathbf{s}$.

The coefficients in Table 2 have been computed for the rankings in Table 1 and the results are reported in Table 3.

The findings reveal that unweighted rank correlation coefficients are not well suited to measure the association in nonlinear cases. The Spearman's r_1 and Kendall's r_4 obtain a high value for the bipolarities E, F and for the inverted U relationship, but the other nonlinearities turn out not to have a large impact on r_1 and r_4. The values of the Gini's r_2 are very similar to those produced by r_1. The Hamming distance r_3, the Gideon-Hollister coefficient r_5, the Salvemini's index r_8 and the index based on the inversion table r_{13} have low values for nearly all rankings. Coefficient r_6 describes properly the bilinear relationships M and N and the floor effect C, but gives low resolution over the set of all permutations.

The Fechner index r_7 focuses its attention on the bilinear configurations M and N. The quadrant association r_9 illuminates quadratic and bilinear relationships but the other patterns went undetected; in fact, r_9 has a large negative value for too many patterns which can be misleading. Coeffcient r_{10} indicates a fairly high degree of similarity between the sorted permutation

Table 3: values of unweighted correlation coefficients

	C	D	E	F	G	H	I	J	K	L	M	N
r_1	0.21	0.21	0.80	−0.93	0.01	−0.19	0.70	−0.70	0.80	−0.80	−0.50	0.50
r_2	0.25	0.25	0.57	−0.79	0.18	−0.25	0.71	−0.71	0.79	−0.79	−0.50	0.50
r_3	0.36	0.36	0.14	−0.43	0.43	−0.43	0.50	−0.50	0.57	−0.57	−0.07	0.07
r_4	−0.05	−0.05	0.60	−0.81	0.14	−0.26	0.47	−0.47	0.60	−0.60	−0.07	0.07
r_5	−0.14	−0.14	0.14	−0.43	0.43	−0.29	0.43	−0.43	0.57	−0.57	−0.43	0.43
r_6	−0.94	0.24	0.30	−0.55	0.41	−0.29	0.72	0.72	−0.60	−0.82	0.90	−0.87
r_7	−0.43	−0.43	0.14	−0.43	0.43	−0.43	0.00	0.00	0.14	−0.14	0.86	−0.86
r_8	0.17	0.17	0.54	−0.64	−0.04	−0.04	0.33	−0.33	0.40	−0.40	−0.04	0.04
r_9	0.00	0.00	0.14	−0.43	0.43	−0.43	0.86	−0.86	1.00	−1.00	−0.86	0.86
r_{10}	−0.43	−0.43	0.14	−0.43	0.36	−0.36	0.00	0.00	0.14	−0.14	0.43	−0.43
r_{11}	0.24	0.24	0.85	−0.95	−0.02	−0.17	0.66	−0.66	0.75	−0.75	−0.41	0.41
r_{12}	−1.00	−1.00	1.00	−1.00	0.67	−0.71	0.88	−0.88	1.00	−1.00	−0.25	0.25
r_{13}	−0.23	−0.23	0.40	−0.87	−0.04	−0.33	0.26	−0.38	0.40	−0.91	−0.24	0.05
r_{14}	−0.07	−0.07	0.57	−0.79	0.07	−0.36	0.43	−1.00	0.57	−1.00	−1.00	0.00
r_{15}	−0.43	−0.43	0.14	−0.43	0.14	−0.71	0.00	0.00	0.14	−0.14	0.00	−0.86
r_{16}	−0.43	−0.43	0.14	−0.43	0.43	−0.43	−0.14	−1.00	0.14	−1.00	−1.00	−0.14

A and the descending bilinear pattern N, but not for the bilinear ascending pattern M. The average slope r_{11} draws attention to the dual character in E and F and to the quadratic relationships I, J, K, L.

Satisfactory results have been obtained by the median slope r_{12} which allows a correct evaluation of most of the effects (it fails, however, to characterize the bilinear condition in M and N); more importantly, r_{12} has the same magnitude, but opposite sign for (I, J) and (K, L). The Spearman's footrule r_{14} understates ceiling, floor, and bilinear descending effects. Moreover, it is not very sensitive to change in ranks since it assigns the minimum value -1 non only to the association of A with the inverse of the natural order B, but also to other very different arrangements: J, L, M. The Bhat-Nayar coefficient r_{16} has a similar behavior. The Gordon [12] index r_{15} detects the sign and the magnitude of the bilinear descending pattern N and the descending critical region phenomenon H but it also indicates a false absence of association in I, J, M. In addition, r_{15} does not distinguish between ceeling and floor effects since $r_{15}(A, C) = r_{15}(A, D)$ making very difficult to offer an explanation.

The main drawbak of the rank correlations included in Table 2 is that they implicitly assume that the level of any one of the items is of equal importance with the level of any other item. In fact, the Kendall's τ corresponds to assign the same weight regardless of how nearer separated are the items. Hence we are crediting rankings with possessing more information than is intended. This perhaps explains their general failure to assess concordance in ranking affected by a critical region phenomenon.

4 Weighted rank correlation

The decision to weight or not to weight rank comparisons is a controversial issue. Those in favour of using "neutral" methods prefer not to weight comparisons; other recognize that giving more weight to agreement on certain comparisons and less weight to others increases flexibility. The use of the weights, as a matter of fact, avoids a direct assumption that there is a linear relation between two rankings and thus uncovers a potential nonlinear association, should it exist. By contrast, some measure of rank correlations has an implicit weighting scheme. For instance, the Salvemini's r_8 attributes zero weight to intermediate ranks. Also, the Spearman's r_1 gives greater weight to differences between items separated by more members of the ranking.

At least part of the problem is how to decide on a plausible set of weights. Quade and Salama [22] showed that the numerous statistical methods for measuring association when the magnitude of intercategory distances cannot be ignored, group naturally in two classes: weighted rank correlation and correlation of scores. This section is devoted to the first type, whereas the second one will be treated in the next section.

The following formula is a weighted version of Spearman's ρ that includes several special cases.

$$\rho_w = 1 - \frac{2\sum_{i=1}^{n} w_i \left(i - s_i\right)^2}{\max_{{}_nP_n} \left\{ \sum_{i=1}^{n} w_i \left(i - s_i\right)^2 \right\}} \tag{6}$$

An alternative generalization can be stated in the following terms

$$\rho'_w = \frac{\sum_{i=1}^{n} (w_i - \bar{w})\, s_i}{\sum_{i=1}^{n} (w_i - \bar{w})\, i} \qquad \text{with} \quad \bar{w} = n^{-1} \sum_{i=1}^{n} w_i \tag{7}$$

The Spearman's ρ is obtained for $w_i = i$.

Quade and Salama [22] discussed two weighted versions of Kendall's τ

$$\text{additive } \tau_{w,a} = \frac{\sum_{i<j}^{n} (w_i + w_j)\, h\,(s_i < s_j)}{\sum_{i=1}^{n} (n - i)\, w_i} - 1 \tag{8}$$

$$\text{multiplicative } \tau_{w,m} = \frac{2\sum_{i<j}^{n} w_i w_j h\,(s_i < s_j)}{\sum_{i<j}^{n} w_i w_j} - 1 \tag{9}$$

The usual Kendall's τ is obtained from [9] for $w_i = 0.5, i = 1, 2, \cdots, n$.

Table 4 shows some special cases of [6]-[9] that have already been considered in literature.

Coefficient r_{17} is based on the mean rate of change between the identity permutation $\mathbf{q}$ and the actual permutation $\mathbf{s}$. Index r_{18}, suggested by Salama and Quade [23], gives special attention to high-ranked items $(1, 2, \cdots,)$. r_{19} and r_{20} were proposed by Quade and Salama [22] as variants of the standard Spearman's ρ. Costa and Soares [6] recommend r_{21} which is based on the weights $w_i = [2\,(n+1) - (i + s_i)]$; such a scheme represents not only the importance of the sorted values but also the importance of the current ranking. The values of $r_{17} \cdots r_{21}$, however, are not antisymmetric under reversal.

The Mango index r_{22} is a special case of [7] with weights $w_i = i^2$ and thus places emphasis on the relative importance of high ranks $(, \cdots, n-2, n-1, n)$. From another point of view, r_{22} can be interpreted in terms of the sum of the ${}_nC_2$ second order minors extracted from the $(nx2)$ matrix having $\mathbf{s}$ as first column and the natural ordering $\mathbf{q}$ as second column. The index r_{23} proposed by Blest [2] derives from [7] with weights $w_i = (n + 1 - i)^2$ and it can be interpreted as the differences between the accumulated ranks of the two orderings $\mathbf{q}$ and $\mathbf{s}$. The weighting scheme of r_{23} favors low-ranked items $(1, 2, \cdots,)$. Furthermore, $(r_{22} + r_{23}) = 2r_1$.

In their important survey, Quade and Salama [22] formulated a new version r_{24} of the Kendall's τ which involves the harmonic weights [8]. Shieh

Table 4: weighted rank correlations

Name	Formula
Weighted Spearman	
Mean rate	$r_{17} = \dfrac{2\sum_{i=1}^{n}\dfrac{(i-s_i)}{s_i}}{(n+1)L_1 - 2n} - 1\ ; \quad L_1 = \sum_{i=1}^{n} i^{-1}$
Salama-Quade 82a	$r_{18} = 1 - \dfrac{2\sum_{i=1}^{n}(i-s_i)^2\left(i^{-1}+s_i^{-1}\right)}{(n+1)\sum_{i=1}^{n}\dfrac{[2i-(n+1)]^2}{is_i}}$
Salama-Quade 82b	$r_{19} = 1 - \dfrac{\sum_{i=1}^{n}\dfrac{(i-s_i)^2}{is_i}}{(n+1)L_1 - 2n}$
Salama-Quade 92	$r_{20} = 1 - \dfrac{6}{n(n-1)}\sum_{i=1}^{n}\dfrac{(i-s_i)^2}{i+s_i}$
Costa-Soares	$r_{21} = 1 - \dfrac{6\sum_{i=1}^{n}(i-s_i)^2\left[2(n+1)-(i+s_i)\right]}{n^4+n^3-n^2-n}$
Mango	$r_{22} = 1 - \dfrac{3\left[n^2(n+1)^2 - 4\sum_{i=1}^{n} i^2 s_i\right]}{n(n-1)(n+1)^2}$
Blest	$r_{23} = 1 - \dfrac{\left[12\sum_{i=1}^{n}(n+1-i)^2 s_i - n(n+2)(n+1)^2\right]}{n(n-1)(n+1)^2}$
Weighted Kendall	
Quade-Salama	$r_{24} = \dfrac{2\sum_{i<j}^{n}\left(i^{-1}+s_j^{-1}\right)h(s_i<s_j)}{\sum_{i<j}\left(i^{-1}+s_j^{-1}\right)} - 1$
Shieh/a	$r_{25} = \dfrac{2\sum_{i<j}^{n}(i*j)^2\, sgn(s_j-s_i)}{n\left(n^5/9+2n^4/15-5n^3/36-n^2/6+n/36+1/30\right)}$
Shieh/b	$r_{26} = \dfrac{2\sum_{i<j}^{n}\left[(n+1-i)(n+1-j)\right]^2 sgn(s_j-s_i)}{n\left(n^5/9+2n^4/15-5n^3/36-n^2/6+n/36+1/30\right)}$

[27] analyzed [9] with weights $w_i = h\left(i \leq \lfloor (n+1)p \rfloor\right)$ where $p = m/n$ and $m = \lfloor (n+1)p \rfloor$. The value of m must be determined on a case-by-case basis. For this reason, it appears to be unsuitable for a general use and we preferred using the weighting schemes applied to the Blest and Mango indices. The weighted rank correlation coefficients included in Table 4 have been computed for the rankings of Table 1.

Even in this case, the results leave something to be desired. The mean rate r_{17} depicts well the bipolarity conditions, the U-shaped/HL and the inverted U/LH pattern, but it fails to identify all the other structures. The

Table 5: values of weighted rank correlations

	C	D	E	F	G	H	I	J	K	L	M	N
r_{16}	−0.77	0.23	−0.91	0.97	0.37	0.46	−0.27	0.94	−0.96	0.42	0.36	−0.23
r_{17}	0.75	0.20	0.93	−0.71	−0.00	−0.08	0.64	−0.27	0.95	−0.35	0.12	0.60
r_{18}	0.77	−0.23	0.91	−0.97	−0.24	−0.32	0.27	−0.60	0.96	−0.69	−0.29	0.23
r_{19}	0.37	−0.05	0.80	−0.93	−0.01	−0.21	0.47	−0.77	0.87	−0.86	−0.62	0.33
r_{20}	0.41	0.02	0.80	−0.93	0.01	−0.19	0.57	−0.70	0.90	−0.80	−0.50	0.47
r_{21}	0.02	0.41	0.80	−0.93	0.05	−0.25	0.83	−0.83	0.70	−0.70	−0.53	0.53
r_{22}	0.41	0.02	0.80	−0.93	−0.08	−0.12	0.57	−0.57	0.90	−0.90	−0.47	0.47
r_{23}	0.47	−0.30	0.76	−0.89	−0.13	−0.37	0.18	−0.65	0.84	−0.45	−0.10	0.03
r_{24}	−0.89	0.68	0.72	−0.88	0.27	−0.01	0.95	0.52	−0.33	−0.98	0.55	−0.38
r_{25}	0.68	−0.89	0.72	−0.88	−0.12	−0.40	−0.52	−0.95	0.98	0.33	0.38	−0.55

Spearman formula with harmonic weights r_{18} is relatively large for C, E, and K where the high-ranked items are in the first positions. A moderate degree of anticorrelation is attributed to F but for the other permutations there is no tendency for the ranks to run with or against each other. Coefficients r_{19} and r_{20} are in line with r_{18} but show a wider variety of values than that of r_{18}. In addition, they assign a large negative value to J and L characterized by high-ranked items in the last positions. From another standpoint, r_{18}, r_{19}, r_{20} do not stress the inherent concordance for several nonlinearities or have the wrong sign or are confused (*e.g.* for the ceiling effect). The coefficient of Costa and Soares r_{21} yields values of the same type as the Blest index r_{23} but provides a better description of the U-shaped/HL permutation.

Since $r_{22}(\mathbf{q},\mathbf{s}) = -r_{23}(\mathbf{q},\mathbf{s}^*)$, the index of Mango and the index of Blest act as complementary statistics in cases of limited resource allocation because ascribing higher importance to one item reduces the importance of another. In facts, the quadratic and the bilinear patterns are consistently reflected by the two coefficients. The signs of r_{22} and r_{23} are concordant with the sole exception of G where both are near to zero. Nevertheless, the Mango and Blest coefficients misstate the actual amount of agreement due to a ceiling or to a floor effect. On the other hand, a high value in both indices constitutes a clear symptom that the permutation is ruled by antagonistic forces.

The weighted Kendall index r_{24} takes into account the bipolarities E, F and, at least partially, the quadratic interactions J and K, but other patterns are almost completely ignored (*e.g.* H and I). The coefficient r_{25} omits the strength of the linkage in G, K and N, but all the other values are well above 0.5. The coefficient r_{26} captures the floor effect in C, the ceiling effect in D, the duality effect in E and F, and the quadratic relationship in J and K. In addition, r_{25} and r_{26} give the same sign (but a different magnitude) to the pairs (I, J) and (K, L). Note that none among the indices has been able to give an accurate image of the relationship between the central levels of the sorted permutation A and the corresponding ranks in G or H. Moreover, $r_{17}, \cdots, r_{26}$ do not take into account the fact that the highest five ranks and the lowest five ranks are put together in the first ten positions.

5 Correlation of scores

Scoring methods have been developed specifically for the analysis of ordered categorical data. A common procedure to measure agreement between two observers consists of first assigning arbitrary equal-interval scores to the ordinal levels (unless the particular case requires otherwise) and then applying classical statistical methods based on these scores. See, among the others, Nikitin and Stepanova [21]).

For fixed n, consider the set of sample pairs $\{(x_i, y_i), i = 1, 2, \cdots, n\}$ from an absolutely continuous bivariate distribution function $H(X, Y)$ having marginal distribution functions $F(X)$ and $G(Y)$. Assume further that F and G have a finite mean and do not contain unknown parameters. The hypothesys to be tested is $H_0 : H(X, Y) = F(X) F(Y)$. The rank correlation statistics which suggest themselves to test H_0 are of the form

$$T_n = \sum_{i=1}^{n} a_{q_i:n} b_{s_i:n} \tag{10}$$

where q_i (s_i) is the i-th order stastistic for a sample of size n drawn from F (G). Constants $\{a_{i:n}\}$, $\{b_{i:n}\}$, $i = 1, 2, \cdots, n$ are two sets of real numbers depending on the ranks and satisfying the constraints

$$\sum_{i=1}^{n} a_{q_i:n} = \sum_{i=1}^{n} b_{s_i:n} \tag{11}$$

$$a_i \leq a_{i+1}, \quad b_i \leq b_{i+1} \quad i = 1, 2, \cdots, n \tag{12}$$

In this section we are interested in exploring [10] when $\{a_{i:n}\}$ and $\{b_{i:n}\}$ are the expected value of order statistics $E(x_{i:n}) =_F m_{i:n}$ and $E(y_{i:n}) =_G m_{i:n}$ respectively. In particular, we consider constants in the form

$$a_{q_i:n} = \frac{{}_F m_{i:n} - \mu_F}{\sqrt{\sum_{i=1}^{n} \left({}_F m_{i:n} - \mu_F\right)^2}}; \qquad b_{s_i:n} = \frac{{}_G m_{i:n} - \mu_G}{\sqrt{\sum_{i=1}^{n} \left({}_G m_{i:n} - \mu_G\right)^2}} \tag{13}$$

with $\mu_F = E(X), \mu_G = E(Y)$. The degree of concordance/discordance between two rankings is determined by calculating Pearson's product moment coefficient of correlation with $\{{}_F m_{i:n}\}$ and the $\{{}_G m_{i:n}\}$ in place of the ranks

$$r_n(F, G) = \frac{\sum_{i=1}^{n} \left({}_F m_{i:n} - \mu_F\right)\left({}_G m_{i:n} - \mu_G\right)}{\sqrt{\sum_{i=1}^{n} \left({}_F m_{i:n} - \mu_F\right)^2 \sum_{i=1}^{n} \left({}_G m_{i:n} - \mu_G\right)^2}} \tag{14}$$

The statistic $r_n(F, G)$ has a maximum value of 1 achieved when the model F and G are a linear transform of each other. The minimum possible value is attained if the rankings are exactly invertered, but it is not necessarily -1 because it depends on F and G. For intermediate values, $r_n(F, G)$ provides a measures of the dependence between the two rankings. In general, in carrying out the test, we reject the hypothesis of independence if the absolute value of [14] is too large.

The models F and G generate the scores and may be chosen to conform to one's judgement about the general charateristics of the measurement. Iman and Conover [**?**] proposed a rank correlation coefficient which emphasizes the concordance for the top-ranked items $(1, 2, 3, \cdots)$

$${}_F m_{i:n} =_G m_{i:n} = -\sum_{j=i}^{n} j^{-1} = M_i \rightarrow r_{27} = \frac{\sum_{i=1}^{n} M_i M_{s_i} - n}{n + M_1} \tag{15}$$

which is generated by reflected exponential distributions $F(x) = G(x) = e^x, x < 0$. Conversely, by using a positive exponential distribution $F(X) =$

$G(x) = 1 - e^{-x}, x > 0$ we obtain

$$_F m_{i:n} =_G m_{i:n} = -\sum_{j=n+1-i}^{n} j^{-1} = L_i \rightarrow r_{28} = \frac{\sum_{i=1}^{n} L_i L_{s_i} - n}{n + L_n} \tag{16}$$

which can be interpreted as a bottom-up rank correlation because it is especially sensitive to the concordance for low-ranked items $(\cdots, n-2, n-1, n)$. The coefficients $r_{27} and r_{28}$ are not antisymmetric under reversal and have a mean value different from zero.

Crathorne [5], Fieller *et al.* [8], and many other used the expected values of the standard normal order statistics (or approximations of them) to define a measure of rank correlation corresponding to the Pearson's correlation coefficient, that is, the Fisher-Yates coefficient.

$$_F m_{i:n} =_G m_{i:n} = L_i \rightarrow r_{29} = \frac{\sum_{i=1}^{n} N_i N_{s_i}}{\sum_{i=1}^{n} N_i^2} \tag{17}$$

Where N_i are the expected value of the i-th standard normal order statistic. Since F and G are symmetric about $x = 0$ and $y = 0$, respectively, then $_F m_{i:n} +_F m_{n-i+1:n} = 0$ and $_G m_{i:n} +_G m_{n-i+1:n} = 0$ implying that a similar score is attached to ordered position at equal depths from the extremes for each distribution. Furthermore, the absolute value of the scores increases as we go from the mediocre item to extreme items so that [17] is equally sensitive to agreement in both extremes but not in the center.

Table 6 reports the value of r_{27} and r_{28}. The scores [15] and [16] have been rescaled so that the extreme values of [14] will lie in the interval from -1 to 1. In addition, we have computed r_{29} for three different approximations to the normal scores.

$$\begin{cases} \text{Van der Waerden} & _F m_{i:n} =_G m_{i:n} = \Phi^{-1}\left(\dfrac{s_i}{n+1}\right) \\ \text{Blom} & _F m_{i:n} =_G m_{i:n} = \Phi^{-1}\left(\dfrac{s_i - 0.375}{n+0.25}\right) \\ \text{Tukey} & _F m_{i:n} =_G m_{i:n} = \Phi^{-1}\left(\dfrac{s_i - 1/3}{n+1/3}\right) \end{cases} \tag{18}$$

Where $\Phi^{-1}(.)$ is the inverse cumulative normal distribution.

The values of r_{27} convey the information that there is agreement for configurations dominated by the concordance between low-ranked items (*e.g.*

Table 6: values of rank correlation of scores

	C	D	E	F	G	H	I	J	K	L	M	N
r_{27}	0.72	−0.39	0.89	−0.96	−0.24	−0.35	0.07	−0.74	0.96	−0.82	−0.56	0.02
r_{28}	−0.39	0.72	0.89	−0.96	−0.20	−0.31	0.93	−0.74	0.22	−0.82	−0.56	0.15
$r_{29,a}$	0.25	0.25	0.86	−0.95	−0.08	−0.22	0.65	−0.65	0.74	−0.74	−0.39	0.39
$r_{29,b}$	0.26	0.26	0.87	−0.96	−0.10	−0.23	0.64	−0.64	0.73	−0.73	−0.37	0.37
$r_{29,c}$	0.26	0.26	0.87	−0.96	−0.10	−0.23	0.64	−0.64	0.73	−0.73	−0.37	0.37

K). Coefficient r_{28} emphasizes the dependence in configuration characterized by the concordance between top-ranked items (*e.g.* I). A high positive value for both r_{27} and r_{28} is a signal of a bipolarity ascending pattern whereas a large negative value for both r_{27} and r_{28} may indicate either a bipolarity descending pattern or a quadratic link. Rank correlations based on the normal distribution r_{29} depict sufficiently well bipolarity conditions and quadratic relationships but perform ineffectively for the other comparisons. The last three columns of Table 6 show that do not exist meaningful differences between the three alternatives in [18].

6 Choice of a rank correlation

The setting up of a rank correlation statistic for all the alternative simultanously is, perhaps, an impossible task since the regions of rejection tend to overlap. Although the notion of a specific measure of dependence for each reference permutation has some merits, the comparability of different scenarios would be enhanced by a rather general coefficient. The easiest method to overcome this dilemma is to limit ourserlves arbitrarily to a definite class of statistics which has desirable properties over a broad range of hypotheses of association and then to choose an optimum statistic from this class.

When the value of any typing method for $r(\mathbf{q}, \mathbf{s})$ is assessed, the two main characteristics that need to be considered are the robustness and the sensitivity. The former determines the degree of rank order inconsistency that can be withstood by the method before mismatches begin to occur. The sensitivity of $r(\mathbf{q}, \mathbf{s})$ is an estimate of its ability to differentiate between rankings. Robustness and sensitivity are antithetical requirements where more robust indices give greater stability against random change of the ranks whereas more sensitive coefficients offer a richer source of information on association patterns. In order, therefore, to choose a "good" index, some balancing of conflicting objectives will be required. A reasonable solution can be obtained by considering that ranking is an intrinsically robust process; thus, the choice of a rank correlation should privilege its discriminatory power.

It is plain that a given value of a rank correlation coefficient does not in general define a unique permutation to be associated with the natural order permutation, except perhaps the maximum value of the coefficient. Nevertheless, the values of the unweighted rank correlations included in Table 2 have a "resistance-to-change" property which appears to be of little value for the purposes of rank comparisons. Moreover, only $r_1, \cdots, r_4, r_{11}$, r_{12} are antisymmetric under reversal. The weighted coefficients seem more flexible and can discriminate more easily between permutations than the unweighted rank correlations. It remains unclear, however, how to effectively choose among the various indices of the second type. As a preliminary matter, we note that the sensitivity possessed by $r_{17}, \cdots, r_{21}$, also in consideration that they fail to verify condition [3], seems inadequate to evaluate the majority of the situations described in Table 1. Finally, the weighted Kendall coefficient r_{24} has a negative bias that precludes its usage and application. As a consequence, the indices $r_1, \cdots, r_{21}$ and r_{24} are not considered suitable statistics to use when the capacity of an index to respond to changes in a permutation is of concern.

Correlations of scores have received attention in a wide range of research disciplines because their definition gives us the freedom to choose a suitable rank scores. We studied strengh and weakness of some correlations of scores defined as product-moment correlation between the expected value of the order statistics from two identical distribution. Our analysis would suggest that this approach is less satisfactory than weighed rank correlation in reflecting certain patterns of agreement/disagreement between rankings. It must be noted, however, that the two approaches: rank correlation and correlation of scores are not necessarily different. The Spearman's ρ, in fact, can be obtained from [14] by using the order statistics from the uniform ditribution. The Blest index can be well approximated (Genest and Plante [9]) by a reflected power-function distribution $F(x) = 1 - \sqrt{-x}$ for $-1 < x < 0$, and $G(x) = x$ for $0 < x < 1$. On the other hand, the Mango index can be approximated by a power-function distribution $F(x) = \sqrt{x}$ and $G(x) = x$ for $0 < x < 1$. In this sense, the results achieved with correlations based on scores do not appear an effective improvement over weighted rank correlations. Moreover, a specification of realiable models is required and any such choice implies a further variant of the index which may be discouraging for a non-expert user. Consequently, even the indices $r_{26}, \cdots, r_{28}$, are not considered further here. In summary, we have restricted our attention to $r_{22}, r_{23}, r_{25}, r_{26}$ which are the most promising indices discussed in the previous sections.

Let us suppose that the values of $r(\mathbf{q}, \mathbf{s})$ are rounded after the m-th decimal place

$$\frac{\lfloor r(\mathbf{q}, \mathbf{s}) \, 10^m + 0.5 \rfloor}{10^m} \tag{19}$$

The discriminatory power of $r(\mathbf{q}, \mathbf{s})$ can be quantified by the fraction of values assumed by [19] in relation to the maximum potential number of values.

$$\psi = \frac{m}{\min\left\{{}_nP_n, 2(10^m) + 1\right\}} \tag{20}$$

where m is the number of distinct values that [19] takes on over ${}_nP_n$. Thus $\psi = 1$ would indicate that $r(\mathbf{q}, \mathbf{s})$ has the minimum number of repeated values at the given level of approximation. Conversely, $\psi \approxeq 0$ would indicate that virtually all members of ${}_nP_n$ are considered of an identical type from the point of view of $r(\mathbf{q}, \mathbf{s})$. A value of ψ around 0.50 would mean that if one ranking is chosen at random then there would be a 50% probability that the next ranking chosen at random would be indistinguishable from the first.

A summary of [20] for $r_{22}, r_{23}, r_{25}, r_{26}$ is given in Table 7 for $n = 8, \cdots, 11$. Included in the table, for comparative purposes, are the results of r_1, r_4, r_{29a}. In particular, column 3-6 show the mean, the standard deviation, the standardized third moment γ_1 and the standardized coefficient of kurtosys γ_2.

The last column reports the ratio [20] where the values have been rounded after the 4th decimal place to keep computations at a feasible level.

Table 7: summary statistics

n	Coefficient	μ	σ	γ_1	γ_2	ψ
8	r_1	−0.0416	0.344	0.02	2.80	0.42
	r_4	−0.0312	0.260	−0.05	3.13	0.14
	r_{22}	−0.0509	0.345	0.01	2.81	3.66
	r_{23}	−0.0324	0.359	0.03	2.72	3.70
	r_{25}	−0.0682	0.340	−0.12	2.78	56.82
	r_{26}	−0.0015	0.377	−0.00	2.69	61.03
	r_{29a}	−0.0467	0.339	0.01	2.74	27.44
9	r_1	−0.0333	0.326	0.02	2.82	0.60
	r_4	−0.0247	0.244	−0.04	3.11	0.18
	r_{22}	−0.0411	0.329	0.01	2.82	5.87
	r_{23}	−0.0256	0.359	0.02	2.72	5.92
	r_{25}	−0.0557	0.320	−0.10	2.80	87.42
	r_{26}	−0.0010	0.351	−0.00	2.73	92.86
	r_{29a}	−0.0382	0.322	0.01	2.76	69.65
10	r_1	−0.0273	0.310	0.02	2.84	0.83
	r_4	−0.0200	0.230	−0.03	3.10	0.23
	r_{21}	−0.0339	0.314	0.01	2.78	8.94
	r_{22}	−0.0207	0.323	0.02	2.78	9.00
	r_{24}	−0.0464	0.304	−0.09	2.82	95.37
	r_{25}	−0.0006	0.329	−0.00	2.77	98.33
	r_{29a}	−0.0319	0.307	0.01	2.78	88.55
11	r_1	−0.0227	0.297	0.01	2.85	1.10
	r_4	−0.0165	0.218	−0.02	3.09	0.28
	r_{22}	−0.0284	0.301	0.01	2.85	13.05
	r_{23}	−0.0170	0.308	0.01	2.80	13.12
	r_{25}	−0.0392	0.289	−0.08	2.83	97.93
	r_{26}	−0.0004	0.310	−0.00	2.79	99.60
	r_{29a}	−0.0271	0.294	0.01	2.80	93.71

The results suggest that the rank correlations are negatively biased although the bias diminishes as the number of ranks increases. For large n, the distributions is nearly normal. The best rate of convergence is achieved by r_{26}. It may be also observed that the sensitivity of all the indices increases as n increases, but r_{26} can discriminate most easily between individual permutations.

7 Conclusion

From the discussion and evaluation presented in the previous sections it is clear that important factors such as the context in which we do association analysis, the properties of the items to be ranked, the purpose of the study, may influence the choice of a particolar weighting scheme for a rank correlation coefficient. There are many methods of rank correlation, from simple ones such as the Blomqvist's coefficient to relatively complicated ones invoking one or two system of weights and/or special rank transformations. Any of these methods describes a different aspect of ordinal association. The flexibility of the formula and the high resolution over the set of all rankings are primary factors for a general coefficient of rank correlation.

We analyzed several measures which emphasize or de-emphasize certain part of the scale by considering special characteristics of the ranks or by attaching to each comparison a weight that reflects the judgement of the evaluator about how much a rank matters.

In response to the special needs arising from the peculiar situations discussed in our paper, a reasonably general answer could be given by a modified weighted Kendall's τ. However, the question of how to weight and integrate impacts on the different ranks is not trivial. After all, if one needs to know the proper set of weights before one can choose the proper measure of rank correlation, the strategy of avoiding bias seems circular.

References

[1] Bhat D.N., Nayar S.K. (1997). Ordinal measures for visual correspondence. CUCS-009-96

[2] Blest D.C. (2000). Rank correlation - An Alternative measure. *Australian and New Zealand Journal of Statistics*, 42, 101-111.

[3] Blomqvist N. (1950). On a measure of dependence between two tandom variables. *The Annals of Mathematical Statistics*, 21, 593-600.

[4] Cifarelli D.M., Conti P.L., Regazzini E. (1996). On the asymptotic distribution of a general measure of monotone dependence. *The Annals of Statistics*, 25, 1386-1399.

[5] Crathorne A.R. (1925). A weighted rank correlation problem. *Metron*, 12, 47-52.

[6] Costa J., Soares C. (2005): A weighted rank measure of correlation. *Australian and New Zealand Journal of Statistics*, 47, 515-529.

[7] Dallal G.E., Hartigan J.A. (1980): Note on a test of monotone association insensitive to outliers. *Journal of the American Statistical Association*, 75, 722-725.

[8] Fieller E.C., Hartley H.O., Pearson E.S. (1957): Tests for rank correlation coefficients. I. *Biometrika*, 44, 470-481.

[9] Genest C., Plante J. (2003). On Blest's measure of rank correlation. The Canadian Journal of Statistics, 31, 1-18.

[10] Gideon R.A. Hollister A. (1987). A rank correlation coefficient resistant to outliers. *Journal of the American Statistical Association*, 82, 656-666.

[11] Gini C. (1914). *Di una misura delle relazioni tra le graduatorie di due caratteri*. Tipografia Cecchini, Roma.

[12] Gordon A.D. (1979). A measure of the agreement beween rankings. *Biometrika*, 66, 7-15.

[13] Gower J.C. (1971): A general coefficient of similarity and some of its properties. *Biometrics*, 27, 857-874.

[14] Iman R.L., Conover W.J. (1987). A measure of top-down correlation. *Technometrics*, 29, 351-357.

[15] Kendall M.G. (1938). A new measure of rank correlations. *Biometrika*, 30,91-93,

[16] Knuth D. E. (1973). *The art of computer programming, 2nd edition. Vol 1: Fundamel algorithms*. Addison-Wesley Publishing Company, Reading Mas.

[17] Mango A. (1997): Rank correlation coefficients: A new approach. *Computing Science and Statistics. Computational Statistics and Data Analysis on the Eve of the 21st Century. Proceedings of the Second World Congress of the IASC*, 29, 471-476.

[18] Moore G.H., Wallis A. W. (1943) Time series significance test based on sign differences. *Journal of the American Statistical Association*, 38, 153-165.

[19] Moran P.A.P. (1950). A curvilinear ranking test. *Journal of the Royal Statististical Society B*, 12, 292-295.

[20] Nelsen R.B. Ubeda-Flores M. (2004). The symmetric footrule is Gini's rank association coefficient. *Communications in Statistics, Part A-Theory and Methods*, 33, 195-196.

[21] Nikitin Y.Yu., Stepanova N.A. (2003). Pitman efficiency of independence tests based on weighted rank statistics. *Journal of Mathematical Science*, 118, 5596-5606.

[22] Quade D., Salama I.A. (1992). A survey of weighted rank correlation. P.K. Sen and I.A. Salama (Eds.). *Order statistics and nonparametrics: theory and applications.* Elsevier Science Publishers B.V., Amsterdam, 213-225.

[23] Salama I.A., Quade D. (1982). A nonparametric comparison of two multiple regressions by means of a weighted measure of correlation. *Communications in Statistics, Part A-Theory and Methods*, 11,1185-1195.

[24] Salama I.A., Quade D. (1997). The asymptotic normality of a rank corrrelation statistic based on rises. *Statistics and Probability Letters*, 32, 201-205.

[25] Salama I.A, Quade D.(2001). The symmetric footrule.*Communications in Statistics, Part A-Theory and Methods*, 30,1099-1109.

[26] Salvemini T. (1951). Sui vari indici di concentrazione, *Statistica*, 2, 133-154.

[27] Shieh G.S. (1998). A weighted Kendall's tau statistic. *Statistics and Probability Letters*, 39, 17-24.

[28] Spearman C. (1904). The proof and measurement of association between two things. *American Journal of Psichology*, 15, 72-101.

[29] Spearman C. (1906). The proof and measurement of association between two things. *British Journal of Psichology*, 2, 89-108.