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Weak solutions and blow-up for wave equations of *p*-Laplacian type with supercritical sources

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This paper investigates a quasilinear wave equation with Kelvin-Voigt damping, $u_{tt} - \Delta_p u - \Delta u_t = f(u)$, in a bounded domain $\Omega \subset \mathbb{R}^3$ and subject to Dirichlét boundary conditions. The operator Δ_p, $2 < p < 3$, denotes the classical p-Laplacian. The nonlinear term $f(u)$ is a source feedback that is allowed to have a *supercritical* exponent, in the sense that the associated Nemytskii operator is not locally Lipschitz from $W_0^{1,p}(\Omega)$ into $L^2(\Omega)$. Under suitable assumptions on the parameters, we prove existence of local weak solutions, which can be extended globally provided the damping term dominates the source in an appropriate sense. Moreover, a blow-up result is proved for solutions with negative initial total energy. [http://dx.doi.org/10.1063/1.4927688]

I. INTRODUCTION

A. The model

The study of quasilinear wave equations with the p-Laplacian operator originates from the nonlinear Voigt model for the longitudinal vibrations of a rod made from a viscoelastic material. In particular, it can be shown that the so-called Ludwick materials obey such equations under the effect of an external forcing (e.g., see Refs. 37 and 42). This paper addresses the existence of local and global solutions to the following quasilinear problem:

$$\begin{cases} u_{tt} - \Delta_p u - \Delta u_t = f(u) \quad \text{in} \quad \Omega \times (0,T), \\ (u(0),\, u_t(0)) = (u_0,\, u_1) \in W_0^{1,p}(\Omega) \times L^2(\Omega), \qquad 2 < p < 3, \\ u = 0 \quad \text{on} \quad \Gamma \times (0,\, T), \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded open domain with a C^2-boundary Γ. The p-Laplacian for $p \geq 2$ is defined by

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)\,.$$

This operator can be extended to a monotone operator between the space $W_0^{1,p}(\Omega)$ and its dual as follows:

$$\begin{cases} -\Delta_p : W_0^{1,p}(\Omega) \longrightarrow W^{-1,p'}(\Omega), \\ \left\langle -\Delta_p u, \phi \right\rangle_p = \displaystyle\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \phi dx, \;\; 2 \leq p < \infty, \end{cases} \tag{1.2}$$

where $\langle \cdot, \cdot \rangle_p$ denotes the duality pairing between $\mathrm{W}^{-1,p'}(\Omega)$ and $W_0^{1,p}(\Omega)$, $\frac{1}{p} + \frac{1}{p'} = 1$.

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B. Literature overview and new contributions

The semilinear case with the classical Laplace operator (when $p = 2$) was studied by Webb,[41] but only under the influence of a conservative ("restoring") source $f(u)$ that is globally Lipschitz continuous from $H_0^1(\Omega)$ into $L^2(\Omega)$. In addition, the case $p = 2$ with a *supercritical* (as defined below) source has been recently treated in Ref. 35, and later in Ref. 33 where the authors provide a complete analysis of that problem including local and global well-posedness, uniqueness, continuous dependence on the initial data, and decay of energy.

Other related works include

- Biazutti[10] investigated global existence and asymptotic behavior of weak solutions to the abstract Cauchy problem (with a solution-independent forcing),
$$u_{tt} - \Delta_p u - \Delta u_t + |u_t|^\rho \operatorname{sgn}(u_t) = f(t,x). \tag{1.3}$$
- In Ref. 16, Gao and Ma studied the global existence and asymptotic behavior of solutions to such equations under fractional-Laplacian Kelvin-Voigt damping,
$$u_{tt} - \Delta_p u + (-\Delta)^\alpha u_t + g(u) = f(t,x), \tag{1.4}$$
where $\alpha \in (0,1]$. The source-feedback map g is subject to the growth condition
$$|g(x,u)| \le a|u|^{\sigma-1} + b,$$
for $1 < \sigma < \frac{pn}{n-p}$. The first existence theorem in Ref. 16 deals with the case $\sigma < p$ which is *subcritical* in our formulation below. And the supercritical scenario was only addressed under a smallness assumption on the initial data Ref. 16, (2.6) and (2.7) p. 3.
- Ma and Soriano in Ref. 28 considered the equation
$$u_{tt} - \Delta_n u - \Delta u_t + g(u) = f(t,x),$$
where $g(u)$ is a dissipative source term, $g(s)s \ge 0$ with the growth bound of the form $e^{|s|^{n/(n-1)}}$ with $\Omega \subset \mathbb{R}^n$, $n \ge 2$.

Remark 1.1. The authors of Refs. 10 and 16 cited the above considered wave equations with the pseudo-Laplacian operator $\Delta_p u = \sum_j \frac{\partial}{\partial x_j}(|\frac{\partial u}{\partial x_j}|^{p-2}\frac{\partial u}{\partial x_j})$ which is slightly different from (1.2).

In this paper, we prove well-posedness results for a more general range of nonlinear sources. We focus on the more interesting situation of dimension $n = 3$ and exponent $2 < p < 3$. For $p = 2$, the principal part becomes linear, whereas for $p \ge 3$, the topology of $W_0^{1,p}(\Omega)$ permits consideration of source terms with arbitrary large exponent, in accordance with the classical Sobolev embedding results in 3D. The source $f(u)$ considered in (1.1) need not be dissipative and is allowed to be of a *supercritical* order. The "criticality" classification is chosen with respect to the associated Sobolev embeddings. In particular, we assume that $f \in C^1(\mathbb{R})$ with the following polynomial growth rate at infinity:
$$|f(s)| \le c|s|^r \quad \text{for all} \quad |s| \ge 1.$$
We follow the terminology introduced in Refs. 12 and 13:

- In view of the 3D embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{\frac{3p}{3-p}}(\Omega)$, with $2 \le p < 3$, the source, if regarded as a Nemytski operator $f : W_0^{1,p}(\Omega) \to L^2(\Omega)$, is locally Lipschitz for the values $1 \le r \le \frac{3p}{2(3-p)}$. When the exponent r satisfies $1 \le r < \frac{3p}{2(3-p)}$, we call the source **subcritical**, and **critical** if $r = \frac{3p}{2(3-p)}$.
- If the exponent r satisfies
$$\frac{3p}{2(3-p)} < r \le \frac{4p-3}{3-p},$$
the source will be called **supercritical**, and in this case f is no longer locally Lipschitz continuous from $W_0^{1,p}(\Omega)$ into $L^2(\Omega)$. However, for this range of exponents, the associated potential energy functional

$$\frac{1}{p}\|\nabla u\|_p^p - \int_\Omega F(u(t))dx$$

is well-defined on the finite energy space $(u,u_t) \in W_0^{1,p}(\Omega) \times L^2(\Omega)$; here, F denotes a primitive of f.

- When $\frac{4p-3}{3-p} < r < \frac{3p}{3-p}$, the source is referred to by a somewhat lengthy, but now frequently used term, **super-supercritical**. In this scenario, the potential energy may not be defined in the finite energy space and the problem itself is no longer well-posed within the framework of potential well theory (see Refs. 4, 26, 30, 39, and 40).

Remark 1.2. It is worth noting here that when the damping term $-\Delta u_t$ is absent, a source term of the form $|u|^{r-1}u$, with $r > r_p$ for some $r_p > 1$ dependent on p, should drive the solution of (1.1) to a blow-up in finite time. In such a scenario, by appealing to a variety of methods (going back to the works of Glassey,[18] Levine,[25] and others), one can show that a large class of solutions to the problem develop a singularity with respect to the natural energy topology in finite time. On the other hand, if the source $f(u)$ is removed from the equation, then it should possess global solutions (cf. Refs. 2, 5, 7, and 22). However, when both damping and source are present, the analysis of their interaction and their influence on the global behavior of solutions becomes more difficult and has served as a motivation for this study (see also other related results[8,17,26,31,34,35] and the references therein).

Results on existence of solutions to the above problem have a non-trivial history. For second-order problems, the quasi-nonlinearity of the p-Laplacian makes the system non-monotonic, which prevents a direct application of nonlinear semigroup techniques. So the process of Galerkin approximation will be invoked similarly to Refs. 10, 16, and 28. We adapt the strategy in Refs. 11, 14, 19–21, 28, 33, and 35 and show that (1.1) has local weak solutions even in the presence of supercritical sources. The details of the proof, however, are non-trivial when it comes to identifying the weak limits associated to the p-Laplacian nonlinearities.

> Despite this solution strategy being used previously (for more restrictive sources) in this context, we believe that this is the first manuscript supplying comprehensive details of such an existence argument. Our detailed approach also highlights the crucial difficulty that would arise if the Kelvin-Voigt damping was replaced with an m-Laplacian term $\Delta_m u_t$, $m > 2$. In that case, the simultaneous identification procedure for the two weak limits, one for the p-Laplacian of u and the other for the m-Laplacian of u_t (even if $m = p$) cannot be carried out by the same methods. It had been assumed in some previous works, e.g., Ref. 9 (which attempts to rely on Refs. 29 and 10 that deal with a *single* p-Laplace operator in the equation), that the Galerkin approach might trivially extend to the m-p model. That is not the case and rigorous analysis of well-posedness for p-Laplacian/m-Laplacian (with $m,\ p > 2$) second-order equation is presently missing from the literature, remaining a challenging open problem.

In addition to the existence theorem with supercritical sources, the concluding result of this article also verifies that solutions blow up in finite time provided the initial total energy is negative and the source "dominates" the damping term in an appropriate sense.

C. Strategies and technical difficulties

Here, we outline the paper highlighting some of the technical steps in the arguments.

Section I D summarizes the standing assumptions on the nonlinear source term f and presents the main results of the paper. In addition, the auxiliary result of Lemma 1.1 is nontrivial and plays an important role in the proof of the energy inequality (see Section II E) and in the verification of the existence result for more general sources (see Sections IV).

Sections II–IV focus on proving the existence of the local solutions. The strategy used to verify the solvability of (1.1) can be summarized as follows:

Step 1: For a source that corresponds to a globally Lipschitz Nemytski operator from $W_0^{1,p}(\Omega)$ into $L^2(\Omega)$, we can construct a local solution by using Galerkin approximations. The particular Galerkin approximations used here are similar to those in Refs. 10, 16, and 28, but this manuscript supplies additional crucial details which cannot be omitted.

Step 2: Extend the existence result in Step 1 to confirm local existence of solutions in the case of sources that are *locally* Lipschitz from $W_0^{1,p}(\Omega)$ into $L^2(\Omega)$ by using a standard truncation argument (e.g., see Refs. 15 and 23). It is essential to demonstrate that the local existence time T does not depend on the (local) Lipschitz constant of f, regarded as a map from a bounded subset of $W_0^{1,p}(\Omega)$ into $L^2(\Omega)$, but rather depends on the corresponding constant of the source f as a mapping from a subset of $W_0^{1,p}(\Omega)$ into $L^{6/5}(\Omega)$ (again, the exponents are for 3D setting).

Step 3: Construct approximations of the original source that obey the requirements of Step 2 by using smooth cut-off functions introduced in Ref. 32. Finally, pass to the limit in the weak variational form for Galerkin approximations to conclude the existence of a local weak solution to the original problem. An important ingredient in this argument is the energy inequality

$$E(t) + \int_0^t \|\nabla u_t(s)\|_2^2 \, ds \le E(0),$$

where $E(t)$ denotes the *total energy* of the system

$$E(t) := \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{p}\|\nabla u(t)\|_p^p - \int_\Omega F(u(t))dx,$$

and $F(u) = \int_0^u f(s)ds$.

Section V proves global existence of solutions as claimed in Theorem 1.4 for the case when the exponent of the damping dominates that of the source. Here, the energy inequality also plays a crucial role as well as the fact that $\int_0^t \int_\Omega f(u(\tau))u_t(\tau)dxd\tau = \int_\Omega F(u(t))dx - \int_\Omega F(u(0))dx$, which is non-trivial due to the lack of regularity.

Finally, Theorem 1.6 presented in Section VI verifies that solutions blow up in finite time provided the initial total energy is negative and the exponent of the source dominates that of the damping term. Due to the lack of regularity, we have to separately verify the product rule for derivatives (Proposition A.1 in the Appendix) in general functional spaces which is one of the technical challenges addressed in Section VI.

D. Preliminaries and main results

We begin by introducing some basic notation that will be used in the subsequent discussion. Define the usual Lebesgue norms and the L^2-inner-product

$$\|u\|_r = \|u\|_{L^r(\Omega)} \quad \text{and} \quad (u,v)_\Omega = (u,v)_{L^2(\Omega)}.$$

The duality pairing between the space $W_0^{1,p}(\Omega)$ and its dual $\mathrm{W}^{-1,p'}(\Omega)$ will be denoted using the form $\langle\cdot,\cdot\rangle_p$. According to Poincaré's inequality, the standard norm $\|u\|_{W_0^{1,p}(\Omega)}$ is equivalent to the norm $\|\nabla u\|_p$ on $W_0^{1,p}(\Omega)$. Henceforth, we put

$$\|u\|_{W_0^{1,p}(\Omega)} = \|\nabla u\|_p.$$

The following Sobolev embedding theorem in 3D will be invoked frequently:

$$W_0^{1,p}(\Omega) \hookrightarrow L^{\frac{3p}{3-p}}(\Omega), \quad \text{for } 2 < p < 3. \tag{1.5}$$

Throughout the paper, we assume the validity of the following assumption.

Assumption 1.1. Assume that

- the exponent of the p-Laplacian belongs to the range $2 < p < 3$,
- the source feedback function $f \in C^1(\mathbb{R})$ satisfies for $|s| \ge 1$

$$|f(s)| \leq c_0|s|^r, \quad |f'(s)| \leq c_1|s|^{r-1}$$

for some constants c_0, $c_1 > 0$ where

$$1 \leq r < \frac{5p}{2(3-p)}, \tag{1.6}$$

- the initial data reside in the following function spaces: $u_0 \in W_0^{1,p}(\Omega)$ and $u_1 \in L^2(\Omega)$.

The assumption on source term is very general and f need not be locally Lipschitz continuous form $W_0^{1,p}(\Omega)$ into $L^2(\Omega)$. However, one can still take advantage of the following lemma:

Lemma 1.1. Under Assumption 1.1, then for all sufficiently small $\epsilon,\delta \in (0,1/2)$ *satisfying* (1.8) *and* (1.17) *below, respectively, we have*

- $f : W^{1-\epsilon,p}(\Omega) \to L^{\frac{6}{5}}(\Omega)$ *is locally Lipschitz continuous.*
- $f : W_0^{1,p}(\Omega) \to L^{\frac{6}{5}(1+\delta)}(\Omega)$ *is locally Lipschitz continuous.*

Proof. Let us identify $\epsilon > 0$ such that for a given $R > 0$ and all $u, v \in W^{1-\epsilon,p}(\Omega)$ with $\|u\|_{W^{1-\epsilon,p}(\Omega)}, \|v\|_{W^{1-\epsilon,p}(\Omega)} \leq R$, the following inequality holds:

$$\|f(u) - f(v)\|_{6/5} \leq C_R\|u - v\|_{W^{1-\epsilon,p}(\Omega)}, \tag{1.7}$$

where the constant $C_R > 0$ is independent of u and v.

First note that by the restriction on r in (1.6), we can choose $\epsilon \in (0, \frac{1}{2})$ such that

$$0 < \epsilon < \frac{\frac{5p}{2(3-p)} - r}{\frac{p}{3-p}r} = \frac{5}{2r} - \frac{3-p}{p}. \tag{1.8}$$

This inequality readily implies

$$1 \leq r < \frac{5p}{2(p\epsilon + 3 - p)}. \tag{1.9}$$

It follows from the mean value theorem and Assumption 1.1 that

$$\begin{aligned}\|f(u) - f(v)\|_{6/5}^{6/5} &= \int_\Omega |f(u) - f(v)|^{\frac{6}{5}}dx = \int_\Omega \left|f'(\xi_{u,v})(u - v)\right|^{\frac{6}{5}}dx \\ &\leq C\int_\Omega |u - v|^{\frac{6}{5}}\left(|u|^{\frac{6}{5}(r-1)} + |v|^{\frac{6}{5}(r-1)} + 1\right)dx.\end{aligned} \tag{1.10}$$

Having the embedding $W^{1-\epsilon,p}(\Omega) \hookrightarrow L^{\frac{3p}{p\epsilon+3-p}}(\Omega)$ in mind (for instance, see Ref. 1), we employ Hölder's inequality in (1.10) with the conjugate exponents: $\alpha = \frac{3p}{\frac{6}{5}(p\epsilon+3-p)}$ and $\alpha' = \frac{3p}{3p-\frac{6}{5}(p\epsilon+3-p)}$ to obtain

$$\|f(u) - f(v)\|_{6/5}^{6/5} \leq C\,\|u - v\|_{\frac{3p}{p\epsilon+3-p}}^{\frac{6}{5}}\left(\|u\|_{\frac{6}{5}(r-1)\alpha'}^{\frac{6}{5}(r-1)} + \|v\|_{\frac{6}{5}(r-1)\alpha'}^{\frac{6}{5}(r-1)} + C_1\right), \tag{1.11}$$

where $C_1 > 0$ depends on Ω. It is easy to check that $\frac{3p}{\frac{6}{5}(p\epsilon+3-p)} > 1$ for all $p \in [2,3)$ and all $\epsilon \in [0,1/2]$. According to inequality (1.9),

$$\begin{aligned}\frac{6}{5}(r-1)\alpha' &= \frac{3p \cdot \frac{6}{5}(r-1)}{3p - \frac{6}{5}(p\epsilon + 3 - p)} \\ &< \frac{6}{5}\left(\frac{5p}{2(p\epsilon + 3 - p)} - 1\right)\frac{3p}{3p - \frac{6}{5}(p\epsilon + 3 - p)} = \frac{3p}{p\epsilon + 3 - p}.\end{aligned} \tag{1.12}$$

Therefore, for any $R > 0$, and for all u, $v \in W^{1-\epsilon,p}(\Omega)$ with $\|u\|_{W^{1-\epsilon,p}(\Omega)}$, $\|v\|_{W^{1-\epsilon,p}(\Omega)} \leq R$, inequality (1.11) and the embedding $W^{1-\epsilon,p}(\Omega) \hookrightarrow L^{\frac{3p}{3-(1-\epsilon)p}}(\Omega)$ imply

$$\|f(u)-f(v)\|_{6/5}^{6/5} \le C\,\|u-v\|_{W^{1-\epsilon,p}(\Omega)}^{\frac{6}{5}}\left(\|u\|_{W^{1-\epsilon,p}(\Omega)}^{\frac{6}{5}(r-1)}+\|v\|_{W^{1-\epsilon,p}(\Omega)}^{\frac{6}{5}(r-1)}+C_1\right)$$
$$\le C(2R^{\frac{6}{5}(r-1)}+C_1)\,\|u-v\|_{W^{1-\epsilon,p}(\Omega)}^{\frac{6}{5}}, \tag{1.13}$$

which proves (1.7).

We now address the second bullet in the statement of the lemma. To this end, we will find $\delta > 0$ such that for any $R > 0$, and for all u, $v \in W_0^{1,p}(\Omega)$ with $\|u\|_{W_0^{1,p}(\Omega)}$, $\|v\|_{W_0^{1,p}(\Omega)} \le R$, the source f satisfies

$$\|f(u)-f(v)\|_{6(1+\delta)/5} \le C_R\|u-v\|_{W_0^{1,p}(\Omega)}, \tag{1.14}$$

where $C_R > 0$ is independent of u and v.

From the mean value theorem and Assumption 1.1, we conclude that

$$\|f(u)-f(v)\|_{6(1+\delta)/5}^{6(1+\delta)/5} = \int_\Omega |f(u)-f(v)|^{\frac{6}{5}(1+\delta)}dx \le \int_\Omega \left|f'(\xi_{u,v})(u-v)\right|^{\frac{6}{5}(1+\delta)}dx$$
$$\le C\int_\Omega |u-v|^{\frac{6}{5}(1+\delta)}\big(|u|^{\frac{6}{5}(1+\delta)(r-1)}+|v|^{\frac{6}{5}(1+\delta)(r-1)}+1\big)dx. \tag{1.15}$$

Invoke the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{\frac{3p}{3-p}}(\Omega)$ and Hölder's inequality with exponents $\alpha = \frac{3p}{\frac{6}{5}(1+\delta)(p\epsilon+3-p)}$ and $\alpha' = \frac{3p}{3p-\frac{6}{5}(1+\delta)(p\epsilon+3-p)}$ to derive

$$\|f(u)-f(v)\|_{6(1+\delta)/5}^{6(1+\delta)/5}$$
$$\le C\,\|u-v\|_{\frac{3p}{[p\epsilon+3-p]}}^{\frac{6}{5}(1+\delta)}\left(\|u\|_{\frac{6}{5}(1+\delta)(r-1)\alpha'}^{\frac{6}{5}(1+\delta)(r-1)}+\|v\|_{\frac{6}{5}(1+\delta)(r-1)\alpha'}^{\frac{6}{5}(1+\delta)(r-1)}+C_1\right), \tag{1.16}$$

where $C_1 > 0$ depends only on the domain Ω.

Inequality (1.9) (with $0 < \epsilon < 1/2$ being the same as in (1.8)) implies that there exists some $\delta \in (0,1/2)$ such that

$$0 < \frac{6}{5}(1+\delta)(r-1)\alpha' < \frac{3p}{3-p}\,. \tag{1.17}$$

Therefore, for any $R > 0$ and for all u, $v \in W_0^{1,p}(\Omega)$ with $\|u\|_{W_0^{1,p}(\Omega)}$, $\|v\|_{W_0^{1,p}(\Omega)} \le R$, the Sobolev embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{\frac{3p}{3-p}}(\Omega)$ along with (1.17) imply

$$\|f(u)-f(v)\|_{6(1+\delta)/5}^{6(1+\delta)/5} \le C\,\|u-v\|_{W_0^{1,p}(\Omega)}^{\frac{6}{5}(1+\delta)}\left(\|u\|_{W_0^{1,p}(\Omega)}^{\frac{6}{5}(1+\delta)(r-1)}+\|v\|_{W_0^{1,p}(\Omega)}^{\frac{6}{5}(1+\delta)(r-1)}+C_1\right)$$
$$\le C\big(2R^{\frac{6}{5}(1+\delta)(r-1)}+C_1\big)\,\|u-v\|_{W_0^{1,p}(\Omega)}^{\frac{6}{5}(1+\delta)}, \tag{1.18}$$

which completes the proof of Lemma 1.1. □

In order to state our main results, we begin with a precise definition of a weak solution to (1.1).

Definition 1.2. A function u is said to be a weak solution of (1.1) on $[0,T]$ if $u \in C_w([0,T], W_0^{1,p}(\Omega))$, $u_t \in C_w([0,T],L^2(\Omega)) \cap L^2(0,T,W_0^{1,2}(\Omega))$, $(u(0),u_t(0)) = (u_0,u_1) \in W_0^{1,p}(\Omega)\times L^2(\Omega)$, and u verifies the identity

$$(u'(t),\phi)_\Omega - (u'(0),\phi)_\Omega + \int_0^t\int_\Omega |\nabla u(t)|^{p-2}\nabla u(t)\cdot\nabla\phi\, dxd\tau$$
$$+\int_0^t\int_\Omega \nabla u_t(t)\cdot\nabla\phi\, dxd\tau = \int_0^t\int_\Omega f(u(t))\phi\, dxd\tau, \tag{1.19}$$

for all test functions $\phi \in W_0^{1,p}(\Omega)$, and for almost everywhere $t \in [0,T]$.

The first result establishes the existence of a weak solution of (1.1).

Theorem 1.3 (Local solutions). *Under the validity of Assumption 1.1, problem (1.1) has a local weak solution u defined on* $[0,T]$ *(in the sense of Definition 1.2) for some* $T > 0$ *which depends only on the norms* $\|u(0)\|_{W_0^{1,p}(\Omega)}$, $\|u_t(0)\|_2$ *and p. In addition, u satisfies the following energy inequality:*

$$E(t) + \int_0^t \|\nabla u_t(s)\|_2^2 \, ds \le E(0), \tag{1.20}$$

for all $t \in [0,T]$, *where* $E(t)$ *denotes the total energy of the system*

$$E(t) := \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{p}\|\nabla u(t)\|_p^p - \int_\Omega F(u(t))dx, \tag{1.21}$$

and $F(u) = \int_0^u f(s)ds$. *Moreover, for all* $t \in [0,T]$ *the following identity holds:*

$$\int_0^t \int_\Omega f(u(\tau))u_t(\tau)dxd\tau = \int_\Omega F(u(t))dx - \int_\Omega F(u(0))dx. \tag{1.22}$$

Remark 1.3. Note that no claims of uniqueness are made here.

The next theorem states that the weak solution described by Theorem 1.3 can be extended globally to the time interval $[0,\infty)$ provided the source exponent is at most $p/2$.

Theorem 1.4 (Global solutions). *In addition to Assumption 1.1, assume that* $r \le \frac{p}{2}$. *Then, the weak solution u furnished by Theorem 1.3 is a global solution and the existence time T can be taken arbitrarily large.*

In order to state our blow-up result, we need to impose additional assumptions on the source term.

Assumption 1.5. Assume that the source map is given by

$$f(s) = (r+1)|s|^{r-1}s, \quad \text{where } r \ge 1.$$

In particular, $f(s) = \frac{d}{ds}F(s)$, *where* $F(s) = |s|^{r+1}$, *and* $sf(s) = (r+1)F(s)$, *for all* $s \in \mathbb{R}$.

Theorem 1.6 (Blow up in finite time). *In addition to Assumptions 1.1 and 1.5, suppose that* $r > p-1$ *and the initial total energy is negative,* $E(0) < 0$. *Then, any weak solution u to (1.1) provided by Theorem 1.3 blows up in some finite time. More precisely,* $\limsup_{t\to T^-} \mathscr{E}(t) = \infty$ *for some* $T < \infty$, *where* $\mathscr{E}(t)$ *is the positive energy given by*

$$\mathscr{E}(t) = \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{p}\|\nabla u(t)\|_p^p.$$

II. LOCAL SOLUTIONS FOR GLOBALLY LIPSCHITZ SOURCES

The first step towards a proof Theorem 1.3 is Proposition 2.1 which deals with the case of globally Lipschitz sources.

Proposition 2.1. In addition to Assumption 1.1, assume that $f : W_0^{1,p}(\Omega) \to L^2(\Omega)$ *is globally Lipschitz with a Lipschitz constant* $L > 0$. *Then, system (1.1) has a local weak solution u defined on time-interval* $[0,T]$, *for some* $T > 0$ *which depends on the norms of the initial data* $\|u(0)\|_{W_0^{1,p}(\Omega)}$, $\|u_t(0)\|_2$. *In addition, u satisfies energy inequality (1.20) and identity (1.22).*

Remark 2.1. Due to lack of regularity, energy inequality (1.20) is not a trivial corollary of the existence result and will be proved in Section II E.

The proof of Proposition 2.1 will be carried out in five steps outlined in Subsections II A–II E.

A. Approximate solutions

Our strategy here is to use suitable Galerkin approximations. Consider $\mathscr{A} = -\Delta$ as an unbounded operator on $L^2(\Omega)$ with the domain $\mathscr{D}(\mathscr{A}) = W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$ (which relies on the fact that Ω is bounded of class C^2). It is well known that $\mathscr{A}$ is positive, self-adjoint, and $\mathscr{A}^{-1}$ is compact. Moreover, $\mathscr{A}$ has an infinite sequence of positive eigenvalues $(\lambda_j)_{j=1}^{\infty}$ and a corresponding sequence of eigenfunctions $(w_j)_{j=1}^{\infty}$ that forms an orthonormal basis for $L^2(\Omega)$. Namely, if $u \in L^2(\Omega)$, then $u = \sum_{j=1}^{\infty} u_j w_j$, with $u_j = \left(u, w_j\right)_\Omega$ and the convergence in the $L^2(\Omega)$-sense; the norm of u is given by $\|u\|_2^2 = \sum_{j=1}^{\infty} \left|u_j\right|^2$.

Thus, $\mathscr{A} u = \sum_{j=1}^{\infty} \lambda_j u_j w_j$ and its domain can be equivalently characterized as

$$D(\mathscr{A}) = \left\{ u = \sum_{j=1}^{\infty} u_j w_j \in L^2(\Omega), \quad \|\mathscr{A} u\|_2^2 = \sum_{j=1}^{\infty} \lambda_j^2 |u_j|^2 < \infty \right\}.$$

This domain is itself a Hilbert space with the inner product

$$(u, v)_{D(\mathscr{A})} = \sum_{j=1}^{\infty} \lambda_j^2 u_j v_j \text{ and } \|u\|_{D(\mathscr{A})} = \|\mathscr{A} u\|_2, \tag{2.1}$$

where $u = \sum_{j=1}^{\infty} u_j w_j$ and $v = \sum_{j=1}^{\infty} v_j w_j$. Furthermore, the sequence $(w_j)_{j=1}^{\infty}$ forms an orthogonal basis for $D(\mathscr{A})$. Hence, for initial data $(u_0, u_1) \in W_0^{1,p}(\Omega) \times L^2(\Omega)$, we can find sequences of scalars $\left(u_{N,j}^0; j = 1, 2, \ldots, N\right)_{N=1}^{\infty}$ and $(u_j^1)_{j=1}^{\infty}$ such that

$$\lim_{N\to\infty} \sum_{j=1}^{N} u_{N,j}^0 w_j = u_0, \quad \text{in} \quad W_0^{1,p}(\Omega), \tag{2.2}$$

$$\sum_{j=1}^{\infty} u_j^1 w_j = u_1, \quad \text{in} \quad L^2(\Omega). \tag{2.3}$$

Let V_N denote the linear span of $(w_1, \ldots, w_N)$, and $\mathscr{P}_N$ be the orthogonal projection from $L^2(\Omega)$ to V_N. Let $u_N(t) = \sum_{j=1}^{N} u_{N,j}(t) w_j$ be the approximate solution of (1.1) in V_N, i.e., u_N satisfies the following system of ordinary differential equations:

$$\begin{aligned} &\left(u_N''(t), w_j\right)_\Omega + \left(|\nabla u_N(t)|^{p-2} \nabla u_N(t), \nabla w_j\right)_\Omega \\ &\qquad + \left(\nabla u_N'(t), \nabla w_j\right)_\Omega = \left(\mathscr{P}_N(f(u_N(t))), w_j\right)_\Omega, \end{aligned} \tag{2.4}$$

$$u_{N,j}(0) = u_{N,j}^0, \quad u_{N,j}'(0) = u_j^1; \quad \text{for} \quad j = 1, \ldots, N. \tag{2.5}$$

It is clear that (2.4) and (2.5) are an initial value problem for a second order $2N \times 2N$ system of ordinary differential equations with continuous nonlinearities in the unknown functions $u_{N,j}$ and their time derivatives. Therefore, it follows from the Cauchy-Peano Theorem that for every $N \geq 1$, (2.4) and (2.5) have a solution $u_{N,j} \in C^2[0, T_N]$, $j = 1, \ldots N$, for some $T_N > 0$.

Remark 2.2. The projection $\mathscr{P}_N$ can be discarded in (2.4) since

$$\left(\mathscr{P}_N(f(u_N(t))), w_j\right)_\Omega = \left(f(u_N(t)), w_j\right)_\Omega, \; j = 1, \ldots, N.$$

However, we will keep $\mathscr{P}_N$ in (2.4) to handle a technical step in obtaining estimate (2.13) below.

B. *A priori* estimates

Next we show that the existence time T_N for system (2.4) can be replaced by some $T > 0$ independent of $N \in \mathbb{N}$.

Proposition 2.2. Assume $2 < p < 3$ and $f : W_0^{1,p}(\Omega) \to L^2(\Omega)$ is globally Lipschitz continuous with a Lipschitz constant $L > 0$. Then, there exists a constant $T > 0$, independent of $N \in \mathbb{N}$, such *that the sequence of approximate solutions $(u_N)_N$ to (2.4) and (2.5) satisfies the following:*

$$\begin{cases} (u_N)_N \text{ is a bounded sequence in } L^\infty(0,T;W_0^{1,p}(\Omega)),\\ (u_N')_N \text{ is a bounded sequence in } L^\infty\left(0,T;L^2(\Omega)\right),\\ (u_N')_N \text{ is a bounded sequence in } L^2(0,T,W_0^{1,2}(\Omega)),\\ (u_N'')_N \text{ is a bounded sequence in } L^2(0,T;W^{-1,p'}(\Omega)). \end{cases} \tag{2.6}$$

Proof. For any fixed $N \in \mathbb{N}$, multiply (2.4) by $u_{N,j}'(t)$ and sum over $j = 1,\ldots,N$ to obtain

$$\frac{1}{2}\frac{d}{dt}\left\|u_N'(t)\right\|_2^2 + \frac{1}{p}\frac{d}{dt}\left\|\nabla u_N(t)\right\|_p^p + \left\|\nabla u_N'(t)\right\|_2^2 = \int_\Omega f(u_N(t))u_N'(t)dx. \tag{2.7}$$

Replace t by τ in (2.7) and integrate over $\tau \in [0,t]$ to arrive at

$$\begin{aligned} &\frac{1}{2}\left\|u_N'(t)\right\|_2^2 + \frac{1}{p}\left\|\nabla u_N(t)\right\|_p^p + \int_0^t \left\|\nabla u_N'(\tau)\right\|_2^2 d\tau \\ &\qquad = \frac{1}{2}\left\|u_N'(0)\right\|_2^2 + \frac{1}{p}\left\|\nabla u_N(0)\right\|_p^p + \int_0^t \int_\Omega f(u_N(\tau))u_N'(\tau)dxd\tau \\ &\qquad \le C\left(\|u_0\|_{W_0^{1,p}(\Omega)}, \|u_1\|_2\right) + \int_0^t \int_\Omega f(u_N(\tau))u_N'(\tau)dxd\tau. \end{aligned} \tag{2.8}$$

By using Hölder's inequality and the fact that the map $f : W_0^{1,p}(\Omega) \to L^2(\Omega)$ is globally Lipschitz continuous, the last term on right-hand side of (2.8) can be estimated as follows:

$$\begin{aligned} \left|\int_\Omega f(u_N(\tau))u_N'(\tau)dx\right| &\le \int_\Omega \left(|f(u_N(\tau)) - f(0)| + |f(0)|\right)\left|u_N'\right| dx \\ &\le \left(\|f(u_N(\tau)) - f(0)\|_2 + \|f(0)\|_2\right)\left\|u_N'(\tau)\right\|_2 \\ &\le \left(L\|\nabla u_N(\tau)\|_p + c_1\right)\left\|u_N'(\tau)\right\|_2, \end{aligned} \tag{2.9}$$

where $c_1 = \|f(0)\|_2$. By Poincaré's and Young's inequalities, we have

$$\begin{aligned} &\left|\int_\Omega f(u_N(\tau))u_N'(\tau)dx\right| \le \\ &\quad \le L\left(\frac{1}{2\lambda}\|\nabla u_N(\tau)\|_p^2 + \frac{\lambda}{2}\left\|\nabla u_N'(\tau)\right\|_2^2\right) + \left(\frac{1}{2\lambda}c_1^2 + \frac{\lambda}{2}\left\|\nabla u_N'(\tau)\right\|_2^2\right) \\ &\quad \le C(L,\lambda,f(0))\left(\|\nabla u_N(\tau)\|_p^2 + 1\right) + \frac{\lambda}{2}(L+1)\left\|\nabla u_N'(\tau)\right\|_2^2, \end{aligned} \tag{2.10}$$

where (2.9) hold for all $\lambda > 0$. Let $\lambda = \frac{1}{L+1}$ and $y_N(t) = \frac{1}{2}\left\|u_N'(t)\right\|_2^2 + \frac{1}{p}\|\nabla u_N(t)\|_p^p$, then (2.8) and (2.10) imply that

$$y_N(t) + \frac{1}{2}\int_0^t \left\|\nabla u_N'(\tau)\right\|_2^2 d\tau \le C_0 + C\int_0^t (y_N(\tau))^{\frac{2}{p}} d\tau, \tag{2.11}$$

where $C_0 = C\big(\|u_0\|_{W_0^{1,p}(\Omega)}, \|u_1\|_2, L, f(0)\big) > 0$ and $C > 0$ is some constant dependent on p, L, and $f(0)$. In particular, y_N satisfies the inequality

$$y_N(t) \le C_0 + C\int_0^t (y_N(\tau))^{\frac{2}{p}} d\tau. \tag{2.12}$$

By a standard comparison theorem (see, for instance, Ref. 24), inequality (2.12) yields

$$y_N(t) \le z(t), \quad \textit{where} \quad z(t) = \left[C_0^{1-\frac{2}{p}} + C\left(1 - \frac{2}{p}\right)t\right]^{\frac{1}{1-\frac{2}{p}}}$$

is the solution of the Volterra integral equation

$$z(t) = C_0 + C\int_0^t (z(\tau))^{\frac{2}{p}} d\tau.$$

Let us note that since $2 < p < 3$, then $z(t)$ is defined for all t. Therefore, we can select $T > 0$ such that $y_N(t) \le z(t) \le C_T < \infty$ for all $t \in [0,T]$, where C_T is independent of N. Hence, for all $N \ge 1$, the bound $y_N(t) \le C_T$ holds on $[0,T]$, establishing the first two claims in (2.6). The third claim follows immediately from (2.11).

Finally, to prove the last statement in (2.6) let

$$S = \left\{ \sum_{j=1}^{N} \alpha_j w_j : \alpha_j \in \mathbb{R},\, N \in \mathbb{N} \right\}.$$

Since S is dense in $W_0^{1,p}(\Omega)$, then for any $\phi \in W_0^{1,p}(\Omega)$ there exists a sequence $(\phi_j)_{j=1}^{\infty} \subset S$ such that $\lim_{j\to\infty} \phi_j = \phi$ in $W_0^{1,p}(\Omega)$. Apply identity (2.4) with $w_j = \phi_j$ and take the limit $j \to \infty$ to obtain

$$\begin{aligned}
|\langle u_N''(t), \phi\rangle_p| &= \left| -\left(|\nabla u_N|^{p-2}\nabla u_N, \nabla\phi\right)_\Omega - \left(\nabla u_N', \nabla\phi\right)_\Omega + \left(\mathscr{P}_N(f(u_N)), \phi\right)_\Omega \right| \\
&\le \|\nabla u_N\|_p^{p-1}\,\|\nabla\phi\|_p + \left\|\nabla u_N'\right\|_2 \|\nabla\phi\|_2 + \|f(u_N)\|_2\|\phi\|_2 \\
&\le C(\|\nabla u_N\|_p^{p-1} + \left\|\nabla u_N'\right\|_2 + \|f(u_N)\|_2)\|\phi\|_{W_0^{1,p}(\Omega)} \\
&\le C(\|\nabla u_N\|_p^{p-1} + \left\|\nabla u_N'\right\|_2 + \|f(u_N) - f(0)\|_2 + \|f(0)\|_2)\|\phi\|_{W_0^{1,p}(\Omega)} \\
&\le C(\|\nabla u_N\|_p^{p-1} + \left\|\nabla u_N'\right\|_2 + L\|\nabla u_N\|_p + \|f(0)\|_2)\|\phi\|_{W_0^{1,p}(\Omega)},
\end{aligned} \tag{2.13}$$

where we have appealed to the assumption that $f : W_0^{1,p}(\Omega) \to L^2(\Omega)$ is Lipschitz. Hence, $\left\|u_N''(t)\right\|_{W^{-1,p'}(\Omega)} \le C(\|\nabla u_N(t)\|_p^{p-1} + \left\|\nabla u_N'(t)\right\|_2 + L\|\nabla u_N(t)\|_p + \|f(0)\|_2)$, for all $t \in [0,T]$. Therefore, by using the first three claims in (2.6) it follows that $\left\|u_N''\right\|_{W^{-1,p'}(\Omega)} \in L^2(0,T)$ with the norm bound independent of N. This step completes the proof. □

The following corollary is an immediate consequence from Proposition 2.2 and standard compactness results (see, for instance, Refs. 36 and 38).

Corollary 2.3. Assume the sequence of approximate solutions $(u_N)_N$ satisfies (2.6). Then, there exist a function u and a subsequence of $(u_N)_N$ (again reindexed by N), such that

$$u_N \to u \text{ weakly}^* \text{ in } L^\infty(0,T; W_0^{1,p}(\Omega)), \tag{2.14}$$

$$u_N' \to u' \text{ weakly}^* \text{ in } L^\infty(0,T; L^2(\Omega)), \tag{2.15}$$

$$u_N \to u \text{ strongly in } L^\infty(0,T; W^{1-\epsilon,p}(\Omega)), \tag{2.16}$$

$$u_N' \to u' \text{ strongly in } L^2(0,T; L^2(\Omega)), \tag{2.17}$$

$$u_N' \to u' \text{ weakly in } L^2(0,T; W_0^{1,2}(\Omega)), \tag{2.18}$$

where $\epsilon > 0$ is as defined in Lemma 1.1.

C. Passage to the limit

By integrating (2.4) over $t \in [0,T]$, we obtain

$$\begin{aligned}
(u_N'(t), w_j)_\Omega - (u_N'(0), w_j)_\Omega + \int_0^t \int_\Omega \left(|\nabla u_N|^{p-2}\nabla u_N \cdot \nabla w_j\right) dx d\tau \\
+ \int_0^t \int_\Omega \nabla u_N' \cdot \nabla w_j dx d\tau = \int_0^t \int_\Omega f(u_N) w_j dx d\tau.
\end{aligned} \tag{2.19}$$

In order to pass to the limit in (2.19), we shall need several auxiliary results. The following lemma addresses the last term in (2.19).

Lemma 2.4. If sequence $(u_N)_N$ satisfies (2.6) and $f : W_0^{1,p}(\Omega) \to L^2(\Omega)$ is globally Lipschitz with the Lipschitz constant $L > 0$, then there exists a subsequence $(u_N)_N$ (re-indexed again by N) which satisfies

$$f(u_N) \to f(u) \ weakly^* \ in \ L^\infty(0,T;L^2(\Omega)). \tag{2.20}$$

Proof. Since $f : W_0^{1,p}(\Omega) \to L^2(\Omega)$ is globally Lipschitz continuous and $(u_N)_N$ is bounded in $L^\infty(0,T;W_0^{1,p}(\Omega))$, then for all $t \in [0,T]$

$$\begin{aligned}\|f(u_N(t))\|_2 &= \|f(u_N(t)) - f(0) + f(0)\|_2 \le \|f(u_N(t)) - f(0)\|_2 + \|f(0)\|_2 \\ &\le L\|u_N(t)\|_{W_0^{1,p}(\Omega)} + |f(0)||\Omega|^{\frac{1}{2}} \le LC_T + |f(0)||\Omega|^{\frac{1}{2}} < \infty.\end{aligned} \tag{2.21}$$

Thus, $(f(u_N))_N$ is a bounded sequence in $L^\infty(0,T;L^2(\Omega))$ and thus, there exists $\xi \in L^\infty(0,T;L^2(\Omega))$ such that, for a suitable subsequence

$$f(u_N) \to \xi \text{ weakly}^* \text{ in } L^\infty(0,T;L^2(\Omega)).$$

However, (2.16) implies that there is a subsequence (still denoted as $(u_N)_N$) such that $u_N \to u$ almost everywhere in $\Omega \times [0,T]$. Consequently, by the continuity of f, one has

$$f(u_N) \to f(u) \text{ point-wise a.e. on } \Omega \times [0,T].$$

A standard analysis result implies

$$f(u_N) \to f(u) \text{ weakly in } L^2(\Omega \times [0,T]).$$

Hence, $\xi = f(u)$ almost everywhere in $\Omega \times [0,T]$, completing the proof of Lemma 2.4. □

Remark 2.3. Lemma 2.4 implies that

$$\lim_{N\to\infty} \int_0^t \int_\Omega f(u_N) w_j dx d\tau = \int_0^t \int_\Omega f(u) w_j dx d\tau, \quad \text{for all } \ j \in \mathbb{N}.$$

The next result addresses the passage to the limit in the term containing the p-Laplacian. Indeed, identifying the weak limit of the sequence $(\Delta_p u_N)_N$ with $\Delta_p u$ is a key step in the proof.

Lemma 2.5. Let $X = L^p(0,T;W_0^{1,p}(\Omega))$. If $(u_N)_N$ and u satisfy (2.14)-(2.18), then

$$\Delta_p u_N \to \Delta_p u \ weakly \ in \ X^*, \tag{2.22}$$

where $X^* = L^{p'}(0,T;W^{-1,p'}(\Omega))$ is the dual of X.

Proof. Since $\frac{1}{p} + \frac{1}{p'} = 1$, we note that

$$\int_0^t \int_\Omega \left||\nabla u_N|^{p-2}\nabla u_N\right|^{p'} dx ds = \int_0^t \|\nabla u_N\|_p^p \le C_T, \text{ for all } t \in [0,T]. \tag{2.23}$$

Thus, $(|\nabla u_N|^{p-2}\nabla u_N)_N$ is a bounded sequence in $\left(L^{p'}(0,T;L^{p'}(\Omega))\right)^3$ and so, there exists a (re-indexed) subsequence such that

$$|\nabla u_N|^{p-2}\nabla u_N \to \psi \text{ weakly in } \left(L^{p'}(0,T;L^{p'}(\Omega))\right)^3, \tag{2.24}$$

for some $\psi \in \left(L^{p'}(0,T;L^{p'}(\Omega))\right)^3$.

Let $\phi \in X$ and $\langle \cdot,\cdot \rangle_{(X^*,X)}$ be the duality pairing between X^* and X, then from (2.24), we have, as $N \to \infty$,

$$\langle -\Delta_p u_N, \phi \rangle_{(X^*,X)} = \int_0^T \int_\Omega |\nabla u_N|^{p-2}\nabla u_N \cdot \nabla\phi dx ds \to \int_0^T \int_\Omega \psi \cdot \nabla\phi dx ds.$$

Thus, $(-\Delta_p u_N)_N$ converges weakly* in X^*. By a standard theorem (see Ref. 43, Theorem 7 p.124 for instance), X^* is sequentially weakly complete. Hence, there is $\eta \in X^*$ such that

$$-\Delta_p u_N \to \eta \text{ weakly in } X^*. \tag{2.25}$$

In order to show that η coincides with $-\Delta_p u$ in X^* we will use the fact that the operator $-\Delta_p : X \to X^*$ is maximal monotone (see Ref. 35, for example). It follows from (2.14), (2.25), and [Ref. 6, Lemma 1.3 (p. 49)] that $\eta = -\Delta_p u$ in X^*, provided we demonstrate the inequality

$$\limsup_{N\to\infty} \langle -\Delta_p u_N - \eta,\ u_N - u\rangle_{(X^*,X)} \le 0. \tag{2.26}$$

In order to establish (2.26), we first note that

$$\begin{aligned}&\langle -\Delta_p u_N - \eta, u_N - u\rangle_{(X^*,X)}\\ &\qquad = \langle -\Delta_p u_N, u_N\rangle_{(X^*,X)} - \langle -\Delta_p u_N, u\rangle_{(X^*,X)} - \langle \eta, u_N - u\rangle_{(X^*,X)}.\end{aligned} \tag{2.27}$$

Now recall (2.14), which implies that $u_N \to u$ weakly in X. Hence,

$$\lim_{N\to\infty} \langle \eta, u_N - u\rangle_{(X^*,X)} = 0. \tag{2.28}$$

We also note that (2.25) yields

$$\lim_{N\to\infty} \langle -\Delta_p u_N, u\rangle_{(X^*,X)} = \langle \eta, u\rangle_{(X^*,X)}. \tag{2.29}$$

Now taking $\limsup_{N\to\infty}$ on both sides of (2.27), we obtain

$$\begin{aligned}&\limsup_{N\to\infty} \langle -\Delta_p u_N - \eta,\ u_N - u\rangle_{(X^*,X)}\\ &\qquad = \limsup_{N\to\infty} \langle -\Delta_p u_N,\ u_N\rangle_{(X^*,X)} - \langle \eta, u\rangle_{(X^*,X)}.\end{aligned} \tag{2.30}$$

Thus, (2.26) will follow if we prove

$$\begin{aligned}\limsup_{N\to\infty} \langle -\Delta_p u_N,\ u_N\rangle_{(X^*,X)} &= \limsup_{N\to\infty} \int_0^T \int_\Omega |\nabla u_N|^{p-2}\nabla u_N(\tau)\cdot \nabla u_N(\tau)dxdt\\ &\le \langle \eta, u\rangle_{(X^*,X)}.\end{aligned} \tag{2.31}$$

We will demonstrate (2.31) in the following two steps.

Step 1. We claim that for almost everywhere $t \in [0,T]$ and for an appropriate subsequence of $(u_N)_N$ we have

$$\begin{aligned}&\limsup_{N\to\infty} \int_0^t \int_\Omega |\nabla u_N|^{p-2}\nabla u_N(\tau)\cdot\nabla u_N(\tau)dxd\tau\\ &\qquad \le -(u'(t),u(t))_\Omega + (u'(0),u(0))_\Omega + \int_0^t \|u'(\tau)\|_2^2\, d\tau\\ &\qquad\quad -\frac{1}{2}(\|\nabla u(t)\|_2^2 - \|\nabla u(0)\|_2^2) + \int_0^t \int_\Omega f(u(\tau))u(\tau)dxd\tau.\end{aligned} \tag{2.32}$$

In order to verify (2.32) fix $N \in \mathbb{N}$. Then, multiply (2.4) by $u_{N,j}(t)$ and sum over $j = 1,\ldots,N$, to arrive at

$$\begin{aligned}&(u_N''(t), u_N(t))_\Omega + (|\nabla u_N|^{p-2}\nabla u_N(t), \nabla u_N(t))_\Omega + (\nabla u_N'(t), \nabla u_N(t))_\Omega\\ &\qquad = (f(u_N(t)), u_N(t))_\Omega.\end{aligned} \tag{2.33}$$

Relabel t by τ and integrate identity (2.33) over $\tau \in [0,t]$ to obtain

$$\begin{aligned}&\int_0^t \int_\Omega |\nabla u_N|^{p-2}\nabla u_N(\tau)\cdot\nabla u_N(\tau)dxd\tau\\ &\qquad = -(u_N'(t),u_N(t))_\Omega + (u_N'(0),u_N(0))_\Omega + \int_0^t \|u_N'(\tau)\|_2^2\, d\tau\\ &\qquad\quad -\frac{1}{2}(\|\nabla u_N(t)\|_2^2 - \|\nabla u_N(0)\|_2^2) + \int_0^t \int_\Omega f(u_N(\tau))u_N(\tau)dxd\tau.\end{aligned} \tag{2.34}$$

Next, we handle each term in (2.34) as follows.

The term $(u_N'(t), u_N(t))_\Omega$. From (2.16), we know that there exists a subsequence of $(u_N)_N$, still denoted by $(u_N)_N$, such that $\|u_N - u\|_2 \to 0$ in $L^2(0,T)$. Hence, on a subsequence, $u_N(t) \to u(t)$

strongly in $L^2(\Omega)$ for almost everywhere $t \in [0,T]$. Likewise, it follows from (2.17) that there exists a subsequence of $(u_N')_N$, re-indexed by N, such that $u_N'(t) \to u'(t)$ strongly in $L^2(\Omega)$ for almost everywhere $t \in [0,T]$. Then, from (2.6), (2.16), and (2.17) we conclude that

$$\lim_{N\to\infty} (u_N'(t), u_N(t))_\Omega = (u'(t), u(t))_\Omega \quad \text{for a.e.} \quad t \in [0,T]. \tag{2.35}$$

The term $(u_N'(0),\ u_N(0))_\Omega$. From (2.2) and (2.3), we know that $u_N(0) \to u(0)$ strongly and $u_N'(0) \to u'(0)$ strongly in $L^2(\Omega)$. Therefore,

$$\lim_{N\to\infty} (u_N'(0), u_N(0))_\Omega = (u'(0), u(0))_\Omega. \tag{2.36}$$

The term $\int_0^t \|u_N'(\tau)\|_2^2 d\tau$. It follows immediately from (2.17) that

$$\lim_{N\to\infty} \int_0^t \|u_N'(t)\|_2^2 \, d\tau = \int_0^t \|u'(t)\|_2^2 \, d\tau \quad \text{for all} \quad t \in [0,T]. \tag{2.37}$$

The term $-\|\nabla u_N(t)\|_2^2$. Invoke Proposition A.2 in the Appendix, and results (2.14), (2.16). Conclude that there exists a subsequence of $(u_N)_N$, still denoted as $(u_N)_N$, such that $u_N(t) \to u(t)$ weakly in $H_0^1(\Omega)$ for almost everywhere $t \in [0,T]$. By the weak lower semicontinuity of the L^p norms, we conclude that for almost everywhere $t \in [0,T]$,

$$\limsup_{N\to\infty} -\|\nabla u_N(t)\|_2^2 = -\liminf_{N\to\infty} \|\nabla u_N(t)\|_2^2 \le -\|\nabla u(t)\|_2^2. \tag{2.38}$$

The term $\|\nabla u_N(0)\|_2^2$. An immediate consequence of (2.2) is that

$$\lim_{N\to\infty} \|\nabla u_N(0)\|_2^2 = \|\nabla u(0)\|_2^2. \tag{2.39}$$

The term $\int_0^t \int_\Omega f(u_N(\tau)) u_N(\tau) dx d\tau$. From (2.20) and (2.16), we conclude that

$$\lim_{N\to\infty} \int_0^t \int_\Omega f(u_N(\tau)) u_N(\tau) dx d\tau = \int_0^t \int_\Omega f(u(\tau)) u(\tau) dx d\tau. \tag{2.40}$$

Finally, take lim sup as $N \to \infty$ on both the sides of (2.34) and combine (2.35)-(2.40). Then, (2.32) follows.

Step 2. Here, we show that $\int_0^t \langle \eta,\ u(\tau)\rangle_p d\tau$ is equal to the right-hand side of (2.32) for almost everywhere $t \in [0,T]$,

$$\begin{aligned}\int_0^t \langle \eta, u(\tau)\rangle_p d\tau = &-(u'(t), u(t))_\Omega + (u'(0), u(0))_\Omega + \int_0^t \|u'(\tau)\|_2^2 \, d\tau \\ &- \frac{1}{2}(\|\nabla u'(t)\|_2^2 - \|\nabla u(0)\|_2^2) + \int_0^t (f(u(\tau)), u(\tau))_\Omega d\tau.\end{aligned} \tag{2.41}$$

In order to establish (2.41), fix $j \in \mathbb{N}$, multiply (2.4) by an arbitrary function $\theta \in C^1[0,T]$, and integrate it from 0 to t to obtain

$$\begin{aligned}&(u_N'(t), \theta(t) w_j)_\Omega - (u_N'(0), \theta(0) w_j)_\Omega - \int_0^t (u_N'(\tau), \theta'(\tau) w_j)_\Omega d\tau \\ &\quad + \int_0^t \langle -\Delta_p u_N, \theta(\tau) w_j\rangle_p d\tau + \int_0^t (\nabla u_N'(\tau), \theta(\tau) \nabla w_j)_\Omega d\tau \\ &\quad = \int_0^t (f(u_N(\tau)), \theta(\tau) \nabla w_j)_\Omega d\tau.\end{aligned} \tag{2.42}$$

We now pass to the limit in (2.42) term by term.

- It has already been shown that $u_N'(t) \to u'(t)$ strongly in $L^2(\Omega)$ for almost everywhere $t \in [0,T]$, and $u_N'(0) \to u'(0)$ strongly in $L^2(\Omega)$. Consequently,

$$\lim_{N\to\infty} (u_N'(t), \theta(t) w_j)_\Omega = (u'(t), \theta(t) w_j)_\Omega \quad \text{for a.e.} \quad t \in [0,T], \tag{2.43}$$

$$\lim_{N\to\infty} (u_N'(0), \theta(0) w_j)_\Omega = (u'(0), \theta(0) w_j)_\Omega. \tag{2.44}$$

- It follows directly from (2.17) and (2.18) that

$$\lim_{N\to\infty}\int_0^t (u_N'(\tau),\theta'(\tau)w_j)_\Omega d\tau = \int_0^t (u'(\tau),\theta'(\tau)w_j)_\Omega d\tau \quad \text{for all} \quad t\in[0,T], \tag{2.45}$$

$$\lim_{N\to\infty}\int_0^t (\nabla u_N'(\tau),\theta(\tau)\nabla w_j)_\Omega d\tau = \int_0^t (\nabla u'(\tau),\theta(\tau)\nabla w_j)_\Omega d\tau. \tag{2.46}$$

- From (2.25) and (2.20), we obtain

$$\lim_{N\to\infty}\int_0^t \langle -\Delta_p u_N,\theta(\tau)w_j\rangle_p d\tau = \int_0^t \langle \eta,\theta(\tau)w_j\rangle_p d\tau, \tag{2.47}$$

$$\lim_{N\to\infty}\int_0^t (f(u_N(\tau)),\theta(\tau)w_j)_\Omega d\tau = \int_0^t (f(u(\tau)),\theta(\tau)w_j)_\Omega d\tau. \tag{2.48}$$

Take the limit as $N\to\infty$ on both sides of (2.42) and combine (2.43)–(2.48), to derive

$$\begin{aligned}\int_0^t \langle \eta,\theta(\tau)w_j\rangle_p d\tau &= -(u'(t),\theta(t)w_j)_\Omega + (u'(0),\theta(0)w_j)_\Omega \\ &\quad + \int_0^t (u'(\tau),\theta'(\tau)w_j)_\Omega d\tau - \int_0^t (\nabla u'(\tau),\theta(\tau)\nabla w_j)_\Omega d\tau \\ &\quad + \int_0^t (f(u(\tau)),\theta(\tau)w_j)_\Omega d\tau \quad \text{for all } j\in\mathbb{N},\ a.e.\ t\in[0,T].\end{aligned} \tag{2.49}$$

Since (2.49) holds for each $j\in\mathbb{N}$, we may replace $\theta(\tau)w_j$ by $u_N(\tau)$ to get

$$\begin{aligned}\int_0^t \langle \eta,u_N(\tau)\rangle_p d\tau &= -(u'(t),u_N(t))_\Omega + (u'(0),u_N(0))_\Omega \\ &\quad + \int_0^t (u'(\tau),u_N'(\tau))_\Omega d\tau - \int_0^t (\nabla u'(\tau),\nabla u_N(\tau))_\Omega d\tau \\ &\quad + \int_0^t (f(u(\tau)),u_N(\tau))_\Omega d\tau \quad \text{for a.e. } t\in[0,T].\end{aligned} \tag{2.50}$$

Let us note here that (2.14) implies that $\nabla u_N \to \nabla u$ weakly in $L^2(0,T;L^2(\Omega))$. Whence,

$$\begin{aligned}\lim_{N\to\infty}\int_0^t (\nabla u'(\tau),\nabla u_N(\tau))_\Omega d\tau &= \int_0^t (\nabla u'(\tau),\nabla u(\tau))_\Omega d\tau \\ &= \frac{1}{2}(\|\nabla u(t)\|_2^2 - \|\nabla u(0)\|_2^2), \quad \text{for all} \quad t\in[0,T].\end{aligned} \tag{2.51}$$

By recalling the already established properties of (a subsequence of) $(u_N)_N$: (2.2), (2.14), (2.16), (2.17), and (2.51), we can pass to the limit in (2.50) to verify (2.41).

Now, it follows from (2.32), (2.41) and Ref. 6, Lemma 1.3 (p. 49) that

$$\eta = -\Delta_p u \quad \text{in} \quad L^{p'}(0,t;W^{-1,p'}(\Omega))$$

for almost everywhere $t\in[0,T]$, which completes the proof of Lemma 2.5. □

1. *Verification that the limit is a weak solution*

Here, we show that u satisfies equality (1.19) in Definition 1.2 of weak solutions. More precisely, for almost everywhere $t\in[0,T]$ and for all $\phi\in W_0^{1,p}(\Omega)$, u must satisfy

$$\begin{aligned}(u'(t),\phi)_\Omega - (u'(0),\phi)_\Omega &= -\int_0^t \left(|\nabla u|^{p-2}\nabla u,\nabla\phi\right)_\Omega d\tau \\ &\quad - \int_0^t (\nabla u_t,\nabla\phi)_\Omega d\tau + \int_0^t (f(u(\tau)),\phi)_\Omega d\tau.\end{aligned} \tag{2.52}$$

From the fact that $\eta = -\Delta_p u$ (provided in Lemma 2.5), we know that taking $\theta(t) = 1$ in (2.49) gives

$$\begin{aligned}(u'(t), w_j)_\Omega - (u'(0), w_j)_\Omega + \int_0^t \langle -\Delta_p u, w_j\rangle_p d\tau + \int_0^t (\nabla u'(\tau), \nabla w_j)_\Omega d\tau \\ = \int_0^t (f(u(\tau)), w_j)_\Omega d\tau \quad \text{for all} \quad j \in \mathbb{N}, \text{ and for a.e. } t \in [0,T].\end{aligned} \tag{2.53}$$

Since $S = \left\{ \sum_{j=1}^N \alpha_j w_j \ : \ \alpha_j \in \mathbb{R}, N \in \mathbb{N}\right\}$ is dense in $W_0^{1,p}(\Omega)$, then for any $\phi \in W_0^{1,p}(\Omega)$ there exists a sequence $(\phi_n)_{n=1}^\infty \subset S$ such that $\lim_{n\to\infty} \phi_n = \phi$ in $W_0^{1,p}(\Omega)$. Thus, by replacing w_j by ϕ_n in (2.53), we have

$$\begin{aligned}(u'(t), \phi_n)_\Omega - (u'(0), \phi_n)_\Omega + \int_0^t \langle -\Delta_p u, \phi_n\rangle_p d\tau + \int_0^t (\nabla u'(\tau), \nabla \phi_n)_\Omega d\tau \\ = \int_0^t (f(u(\tau)), \phi_n)_\Omega d\tau.\end{aligned} \tag{2.54}$$

Take the limit as $n \to \infty$ on both sides of (2.54) to verify that (2.52) holds for arbitrary $\phi \in W_0^{1,p}(\Omega)$ and almost everywhere $t \in [0,T]$.

D. Additional regularity in time

At this point, recall that the constructed weak solution u of (1.1) satisfies $u \in L^\infty(0,T; W_0^{1,p}(\Omega))$, $u_t \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; W_0^{1,2}(\Omega))$, and u satisfies equality (1.19). We still need to check that $u \in C_w([0,T], W_0^{1,p}(\Omega))$, and $u_t \in C_w([0,T], L^2(\Omega))$. In order to do so, the following lemma addressing the regularity of u'' will be needed.

Lemma 2.6. Assume that $u \in L^\infty(0,T; W_0^{1,p}(\Omega))$, $u' \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; W_0^{1,2}(\Omega))$, and u satisfies (1.19). Then,

$$\Delta_p u \in L^\infty(0,T; W^{-1,p'}(\Omega)), \quad \Delta u' \in L^2(0,T; W^{-1,p'}(\Omega))$$

and

$$u'' \in L^2(0,T; W^{-1,p'}(\Omega)),$$

where $W^{-1,p'}(\Omega)$ is the dual space of $W_0^{1,p}(\Omega)$.

Proof. Let $\langle \cdot,\cdot\rangle$ denote the duality pairing between $W^{-1,p'}(\Omega)$ and $W_0^{1,p}(\Omega)$ or the duality pairing between $W^{-1,2}(\Omega)$ and $W_0^{1,2}(\Omega)$. Now, the fact that $\Delta_p u \in L^\infty(0,T; W^{-1,p'}(\Omega))$ is trivial, since $u \in L^\infty(0,T; W_0^{1,p}(\Omega))$ and

$$\begin{aligned}\left|\langle \Delta_p u(t), \phi\rangle\right| &= \left|\left(|\nabla u|^{p-2}\nabla u, \nabla\phi\right)_\Omega\right| \le C\|u(t)\|^{p-1}_{W_0^{1,p}(\Omega)} \|\phi\|_{W_0^{1,p}(\Omega)} \\ &\le C_T \|\phi\|_{W_0^{1,p}(\Omega)}, \quad \text{for every } \phi \in W_0^{1,p}(\Omega) \text{ and all } t \in [0,T].\end{aligned} \tag{2.55}$$

Similarly, because $\nabla u' \in L^2(0,T; L^2(\Omega))$, one can show that

$$|\langle \Delta u'(t), \phi\rangle| = |(\nabla u'(t), \nabla\phi(t))_\Omega| \le \|\nabla u'\|_2 \|\nabla \phi\|_2 \le C\|\nabla u'\|_2 \|\phi\|_{W_0^{1,p}(\Omega)}, \tag{2.56}$$

for every $\phi \in W_0^{1,p}(\Omega)$ and all $t \in [0,T]$.

It follows from (2.52) and (2.21) that for every $\phi \in W_0^{1,p}(\Omega)$,

$$\begin{aligned}&|\langle u''(t), \phi\rangle| = \left|\frac{d}{dt}\langle u'(t), \phi\rangle\right| = \left|\frac{d}{dt}(u'(t), \phi)_\Omega\right| \\ &\le \left|\left(|\nabla u|^{p-2}\nabla u, \nabla\phi\right)_\Omega\right| + |(\nabla u', \nabla \phi)_\Omega| + |(f(u), \phi)_\Omega| \\ &\le C\|u\|_{W_0^{1,p}(\Omega)} \|\phi\|_{W_0^{1,p}(\Omega)} + C\|u'\|_{W_0^{1,2}(\Omega)} \|\phi\|_{W_0^{1,p}(\Omega)} + (LC_T + \|f(0)\|_2)\|\phi\|_2.\end{aligned} \tag{2.57}$$

Estimates (2.55) and (2.56) imply $u'' \in L^2(0,T; W^{-1,p'}(\Omega))$. ◻

Thus, so far, we have proven

$$u \in L^\infty(0,T;W_0^{1,p}(\Omega)),\ \ u_t \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;W_0^{1,2}(\Omega)),$$

In addition, u verifies equality (1.19) and satisfies the conclusion of Lemma 2.6. It now follows from standard results Ref. 27, Lemmas 8.1 and 8.2, pp. 275–276 (after possibly a modification on a set of measure zero) that

$$u \in C_w([0,T],W_0^{1,p}(\Omega)) \text{ and } u_t \in C_w([0,T],L^2(\Omega)) \tag{2.58}$$

as required by Definition 1.2.

E. Proof of the energy inequality

We show here that any weak solution u to (1.1) described by Proposition 2.1 satisfies energy inequality (1.20) and equality (1.22).

Proof. Multiply (2.4) by $u'_{N,j}(t)$, sum over $j = 1,2,\dots,N$, and integrate on $[0,t]$,

$$\int_0^t \int_\Omega u''_N(\tau)u'_N(\tau)dxd\tau + \int_0^t \int_\Omega |\nabla u_N|^{p-2}\nabla u_N \cdot \nabla u'_N(\tau)dxd\tau$$
$$+ \int_0^t \|\nabla u'_N(\tau)\|_2^2\, d\tau = \int_0^t \int_\Omega f(u_N(\tau))u'_N(\tau)dxd\tau. \tag{2.59}$$

Since u_N is regular, then

$$\frac{d}{dt}\left(\frac{1}{2}\|u'_N(\tau)\|_2^2\right) = \int_\Omega u''_N(\tau)u'_N(t)dx, \tag{2.60}$$

$$\frac{d}{dt}\left(\frac{1}{p}\|\nabla u_N(\tau)\|_p^p\right) = \int_\Omega |\nabla u_N|^{p-2}\nabla u_N \cdot \nabla u'_N(t)dx, \tag{2.61}$$

$$\frac{d}{dt}\int_\Omega F(u_N(t))dx = \int_\Omega f(u_N(t))u'_N(t)dx. \tag{2.62}$$

Define

$$E_N(t) = \frac{1}{2}\|u'_N(\tau)\|_2^2 + \frac{1}{p}\|\nabla u_N(\tau)\|_p^p - \int_\Omega F(u_N(t))dx,$$

then it follows from (2.59)–(2.62) that

$$E_N(t) + \int_0^t \|\nabla u'_N(\tau)\|_2^2\, d\tau = E_N(0). \tag{2.63}$$

Next, we pass to the limit in (2.63). First, by mean value theorem, we have

$$\left|\int_\Omega F(u_N(t)) - F(u(t))dx\right| \le \int_\Omega |f(\xi)|\,|u_N(t) - u(t)|\,dx$$
$$\le \int_\Omega c(|u_N(t)|^r + |u(t)|^r + 1)\,|u_N(t) - u(t)|\,dx, \tag{2.64}$$

where r is as defined in (1.6), and $\xi = \lambda u_N(t) + (1-\lambda)u(t)$ for some $\lambda \in (0,1)$. All the terms on the right-hand side of (2.63) can be estimated in the same manner. In particular, we can find $1 < \delta$, $\delta' < \frac{3p}{3-p}$ with $\frac{1}{\delta} + \frac{1}{\delta'} = 1$ such that

$$\int_\Omega |u_N(t) - u(t)|\,|u_N|^r dx \le \|u_N(t) - u(t)\|_\delta\, \|u_N(t)\|_{r\delta'}^r. \tag{2.65}$$

Specifically, we can pick $\delta = \frac{3p(1-\epsilon)}{3-p}$ and $\delta' = \frac{3p(1-\epsilon)}{4p-3-3p\epsilon}$ for some small enough $\epsilon > 0$; indeed, for any $0 < \epsilon < \frac{2(4p-3)}{p} - 5 \le 1$, we have $\delta' < \frac{6}{5}$ and $r\delta' < \frac{6}{5}\cdot\frac{5p}{2(3-p)} = \frac{3p}{3-p}$.

Since $(u_N)_N$ is bounded in $L^\infty(0,T;W_0^{1,p}(\Omega))$ and the (3D) embedding : $W_0^{1,p}(\Omega) \hookrightarrow L^s(\Omega)$ is compact for all $1 \le s < \frac{3p}{3-p}$, then for a subsequence, still denoted as $(u_N)_N$, we have $u_N \to u$

strongly in $L^\infty(0,T;L^\delta(\Omega))$. In addition, $(u_N)_N$ is bounded in $L^\infty(0,T;L^{r\delta'}(\Omega))$. Therefore, it follows from (2.65) that

$$\int_\Omega |u_N(t)-u(t)|\,|u_N|^r dx \le \|u_N(t)-u(t)\|_\delta\,\|u_N(t)\|^r_{r\delta'} \to 0, \tag{2.66}$$

as $N\to\infty$, for almost everywhere $t\in[0,T]$. Similarly, one can prove that

$$\int_\Omega \Big(|u_N(t)-u(t)|\,|u|^r + |u_N(t)-u(t)|\Big)dx \to 0, \tag{2.67}$$

as $N\to\infty$, for almost everywhere $t\in[0,T]$.

Combine (2.64)–(2.67) to conclude

$$\lim_{N\to\infty}\int_\Omega F(u_N(t))dx = \int_\Omega F(u(t))dx. \tag{2.68}$$

Also, since $u_N(0)\to u(0)$ strongly in $W^{1,p}_0(\Omega)$, we have

$$\lim_{N\to\infty}\int_\Omega F(u_N(0))dx = \int_\Omega F(u(0))dx. \tag{2.69}$$

Next, recall (2.14), (2.18), (2.63), (2.68), and appeal to the weak lower semicontinuity of the L^q norms. Then, one has for almost everywhere $t\in[0,T]$,

$$E(t) + \int_0^t \|\nabla u'(\tau)\|_2^2\,d\tau \le \liminf_{N\to\infty}\big(E_N(t) + \int_0^t \|\nabla u_N'(\tau)\|_2^2\,d\tau\big) = \liminf_{N\to\infty} E_N(0). \tag{2.70}$$

Since $(u_N(0),u_N(0)')\to(u(0),u'(0))$ strongly in $W^{1,p}_0(\Omega)\times L^2(\Omega)$ then with the help of (2.69) we have

$$\lim_{N\to\infty} E_N(0) = E(0). \tag{2.71}$$

Therefore, energy inequality (1.20) follows.

Now let us prove another version of energy inequality which will be needed in Section III,

$$\mathscr{E}(t) + \int_0^t \|\nabla u_t(\tau)\|_2^2\,d\tau \le \mathscr{E}(0) + \int_0^t\int_\Omega f(u(\tau))u_t(\tau)dxd\tau \tag{2.72}$$

for all $t\in[0,T]$, where $\mathscr{E}(t)=\frac{1}{2}\|u_t(t)\|_2^2+\frac{1}{p}\|\nabla u(t)\|_p^p$. To verify (2.72), we need to show equality (1.22): for almost everywhere $t\in[0,T]$

$$\int_\Omega F(u(t))dx - \int_\Omega F(u(0))dx = \int_0^t\int_\Omega f(u(\tau))u_t(\tau)dxd\tau. \tag{2.73}$$

It follows from (2.62),(2.68), and (2.69) that

$$\begin{aligned}\int_\Omega F(u(t))dx - \int_\Omega F(u(0))dx &= \lim_{N\to\infty}\Big(\int_\Omega F(u_N(t))dx - \int_\Omega F(u_N(0))dx\Big)\\ &= \lim_{N\to\infty}\int_0^t\int_\Omega f(u_N(\tau))u_N'(\tau)dxd\tau.\end{aligned}$$

Therefore, (2.73) holds provided we show that

$$\lim_{N\to\infty}\int_0^t\int_\Omega f(u_N(\tau))u_N'(\tau)dxd\tau = \int_0^t\int_\Omega f(u(\tau))u'(\tau)dxd\tau, \text{ a.e. } t\in[0,T]. \tag{2.74}$$

From Lemma 1.1, we know that $f: W^{1,p}_0(\Omega)\to L^{\frac{6}{5}(1+\delta)}(\Omega)$ is locally Lipschitz continuous, and also $f: W^{1-\epsilon,p}(\Omega)\to L^{\frac{6}{5}}(\Omega)$ is locally Lipschitz continuous. Then, by Hölder's inequality, we have

$$\begin{aligned}&\left|\int_0^t\int_\Omega \Big(f(u_N(\tau))u_N'(\tau) - f(u(\tau))u'(\tau)\Big)dxd\tau\right|\\ &\le \int_0^t\int_\Omega |f(u_N)(u_N'-u')|\,dxd\tau + \int_0^t\int_\Omega |f(u_N)-f(u)|\,|u'(\tau)|\,dxd\tau\end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^t \|f(u_N)\|_{\frac{6(1+\delta)}{5}} \|u_N' - u'\|_{\frac{6(1+\delta)}{1+6\delta}} d\tau + \int_0^t \|f(u_N) - f(u)\|_{\frac{6}{5}} \|u'(\tau)\|_6 d\tau \\
&\leq C_f \Big(\int_0^t (\|u_N\|_{W_0^{1,p}(\Omega)} + 1) \|u_N' - u'\|_{\frac{6(1+\delta)}{1+6\delta}} d\tau + \int_0^t \|u_N - u\|_{W^{1-\epsilon,p}(\Omega)} \|u'(\tau)\|_6 d\tau \Big) \\
&\leq C_{f,T} \Big((\|u_N\|_{L^\infty(0,T;W_0^{1,p}(\Omega))} + 1) \|u_N' - u'\|_{L^2(0,T;L^\theta(\Omega))} \\
&\qquad\qquad + \|u_N - u\|_{L^\infty(0,T;W^{1-\epsilon,p}(\Omega))} \|u'(\tau)\|_{L^2(0,T;L^6(\Omega))} \Big),
\end{aligned} \tag{2.75}
$$

where $\theta = \frac{6(1+\delta)}{1+6\delta} < 6$. Since $u' \in L^2(0,T;W^{1,2}(\Omega))$, then the Sobolev embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$ gives $u' \in L^2(0,T;L^6(\Omega))$. So, by (2.16) we conclude that

$$
\lim_{N\to\infty} \|u_N - u\|_{L^\infty(0,T;W^{1-\epsilon,p}(\Omega))} \|u'(\tau)\|_{L^2(0,T;L^6(\Omega))} = 0. \tag{2.76}
$$

Since the embedding $W_0^{1,2}(\Omega) \hookrightarrow L^\theta(\Omega)$ is compact and $(u_N'')_N$ is bounded in the space $L^2(0,T;W^{-1,p'}(\Omega))$, then by Aubin's compactness theorem, there exists a subsequence of $(u_N')_N$, still denoted as $(u_N')_N$, such that $u_N' \to u'$ strongly in $L^2(0,T;L^\theta(\Omega))$. It follows from (2.6) that

$$
\lim_{N\to\infty} \big(\|u_N\|_{L^\infty(0,T;W_0^{1,p}(\Omega))} + 1\big) \|u_N' - u'\|_{L^2(0,T;L^\theta(\Omega))} = 0. \tag{2.77}
$$

Combine (2.75), (2.76), and (2.77) to get (2.74). The proof of Proposition 2.1 is now complete. □

III. LOCAL SOLUTION FOR LOCALLY LIPSCHITZ SOURCES

In this subsection, we relax the conditions on the source term and allow f to be *locally* Lipschitz from $W_0^{1,p}(\Omega)$ into $L^2(\Omega)$.

Proposition 3.1. In addition to Assumption 1.1, assume that $f : W_0^{1,p}(\Omega) \to L^2(\Omega)$ is locally Lipschitz. Then, system (1.1) *has a local weak solution u, in the sense of Definition 1.2, on $[0,T_0]$ for some $T_0 > 0$ dependent on initial data $u_0, u_1, f(0)$, and the appropriate local Lipschitz constant of the mapping $f : W_0^{1,p}(\Omega) \to L^{\frac{6}{5}}(\Omega)$. Moreover, u satisfies energy inequality* (1.20) *and equality* (1.22).

Remark 3.1. By assumption, the mapping $f : W_0^{1,p}(\Omega) \to L^{\frac{6}{5}}(\Omega)$ is a fortiori locally Lipschitz. However, it is essential to note here that the local existence time T in Proposition 3.1 **does not** depend on the local Lipschitz constant of f as a map from $W_0^{1,p}(\Omega)$ to $L^2(\Omega)$.

Proof. We use a standard truncation of the sources (for instance, Refs. 14 and 15). Let $\mathscr{E}(t) = \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{p}\|\nabla u(t)\|_p^p$ denote the **positive energy** and put

$$
f_K(u) = \begin{cases} f(u), & \text{if } \|\nabla u\|_p \leq K, \\ f\left(\dfrac{Ku}{\|\nabla u\|_p}\right), & \text{if } \|\nabla u\|_p > K, \end{cases}
$$

where K is a positive constant such that $K^2 > 4\mathscr{E}(0)$.

With the truncation of the source above, we consider the following "(K)" problem:

$$
(K) \quad \begin{cases} u_{tt} - \Delta_p u - \Delta u_t = f_K(u) \ \text{ in } \Omega \times (0,T), \\ \big(u(0), u_t(0)\big) \in W_0^{1,p}(\Omega) \times L^2(\Omega), \\ u = 0, \ \text{ on } \Gamma \times (0,T). \end{cases} \tag{3.1}
$$

We note here that for each such K, the operators $f_K : W_0^{1,p}(\Omega) \to L^2(\Omega)$ are globally Lipschitz continuous (see Ref. 15). Therefore, by Proposition 2.1, the (K) problem has a local weak solution u^K defined on $[0,T]$ where T depends on u_0, u_1, and f_K. Since f_K is also globally Lipschitz continuous from $W_0^{1,p}(\Omega) \to L^{\frac{6}{5}}(\Omega)$, there exists a constant $L_f(K) > 0$ such that

$$
\|f_K(u) - f_K(v)\|_{\frac{6}{5}} \leq L_f(K) \|\nabla(u - v)\|_p, \quad \text{for all} \quad u, v \in W_0^{1,p}(\Omega).
$$

In what follows, we shall for brevity denote $u^K(t)$ by $u(t)$. According to Proposition 2.1, u^K satisfies the following energy inequality:

$$\mathscr{E}(t) + \int_0^t \|\nabla u_t(\tau)\|_2^2\, d\tau \leq \mathscr{E}(0) + \int_0^t \int_\Omega f_K(u(\tau))u_t(\tau)dxd\tau. \tag{3.2}$$

We now estimate the terms on the right-hand side of (3.2) with the help of Hölder's and Young's inequalities,

$$\begin{aligned}\int_\Omega f_K(u(\tau))u_t(\tau)dx &\leq C_\epsilon\big(\|f_K(u(\tau)) - f_K(0)\|_{\frac{6}{5}}^2 + \|f_K(0)\|_{\frac{6}{5}}^2\big) + \epsilon\,\|u_t(\tau)\|_6^2 \\ &\leq C_\epsilon\big(L_f(K)\big)^2\,\|\nabla u(\tau)\|_p^2 + C_f + \epsilon\,\|u_t(\tau)\|_6^2,\end{aligned} \tag{3.3}$$

where $C_f = C_\epsilon\,\|f_K(0)\|_{\frac{6}{5}}^2$. Thus,

$$\int_\Omega f_K(u(\tau))u_t(\tau)dx \leq C_K\mathscr{E}(\tau)^{\frac{2}{p}} + C_f + \epsilon C\,\|\nabla u_t(\tau)\|_2^2, \tag{3.4}$$

where $C_K = C_\epsilon\big(L_f(K)\big)^2$. It follows from (3.2), (3.4), and the fact $\frac{2}{p} \leq 1$ that

$$\begin{aligned}&\mathscr{E}(t) + \int_0^t \|\nabla u_t(\tau)\|_2^2\, d\tau \\ &\qquad \leq \mathscr{E}(0) + C_K\int_0^t \mathscr{E}(\tau)d\tau + C_{K,f}T_0 + \epsilon C\int_0^t \|\nabla u_t(\tau)\|_2^2\, d\tau,\end{aligned} \tag{3.5}$$

for all $0 \leq t \leq T_0$, where $T_0 > 0$ will be chosen below and $C_{K,f} = C_K + C_f$. By choosing $\epsilon > 0$ sufficiently small, we obtain

$$\mathscr{E}(t) + c_\epsilon\int_0^t \|\nabla u_t(\tau)\|_2^2\, d\tau \leq \mathscr{E}(0) + C_{K,f}T_0 + C_K\int_0^t \mathscr{E}(\tau)d\tau, \tag{3.6}$$

for all $0 \leq t \leq T_0$. Gronwall's inequality gives

$$\mathscr{E}(t) \leq (\mathscr{E}(0) + C_{K,f}T_0)e^{C_K t}, \quad \text{for all } t \in [0,T_0]. \tag{3.7}$$

Now we recall that $K^2 > 4\mathscr{E}(0)$ and choose

$$T_0 = \min\Big\{\frac{K^2 - 4\mathscr{E}(0)}{4C_{K,f}}, T, \frac{1}{C_K}\ln 2\Big\}. \tag{3.8}$$

We then have

$$\mathscr{E}(t) \leq (\mathscr{E}(0) + C_{K,f}T_0)e^{C_K t} \leq \frac{K^2}{4}e^{C_K t} \leq \frac{K^2}{2}, \quad \text{for all } t \in [0,T_0]. \tag{3.9}$$

Therefore, the definition of $\mathscr{E}(t)$, (3.9), and the condition $2 < p < 3$ imply that $\|\nabla u(t)\|_p \leq K$, $t \in [0,T_0]$. Thus, $f_K(u(t)) = f(u(t))$ on the interval $[0,T_0]$, and so the considered solution of (K) problem (3.1) is, in fact, a solution u of original problem (1.1) on $[0,T_0]$. The fact that u satisfies energy inequality (1.20) and equality (1.22) follows trivially from Proposition 2.1, completing the proof. □

IV. LOCAL SOLUTION FOR MORE GENERAL SOURCES

Now, we further relax the conditions on the source. Specifically, we will allow for $f \in C^1(\mathbb{R})$ with the following growth restrictions:

$$|f(s)| \leq c_0|s|^r, \quad |f'(s)| \leq c_1|s|^{r-1}, \quad \text{for} \quad |s| \geq 1,$$

for some constants $c_0, c_1 > 0$. Throughout this section, the exponent of the source satisfies

$$1 \leq r < \frac{5p}{2(3-p)}.$$

Before completing the proof of Theorem 1.3, additional preparation will be needed. Recall Lemma 1.1 which established, under Assumption 1.1, that $f : W^{1-\epsilon,p}(\Omega) \to L^{\frac{6}{5}}(\Omega)$ is locally Lipschitz for some $\epsilon > 0$. However, since f is not in general locally Lipschitz from $W_0^{1,p}(\Omega)$ into $L^2(\Omega)$, we shall construct Lipschitz approximations of f. Consider a sequence of smooth cut-off functions η_n, as introduced in Ref. 32. More precisely, we choose a sequence $\eta_n \in C_0^\infty(\mathbb{R})$ that satisfies

$$0 \le \eta_n \le 1, \quad |\eta_n'(u)| \le \frac{C}{n}, \quad \text{and} \quad \begin{cases} \eta_n(u) = 1, & |u| \le n, \\ \eta_n(u) = 0, & |u| > 2n, \end{cases} \tag{4.1}$$

for some constant C (independent of n). Define

$$f_n(u) := f(u)\eta_n(u). \tag{4.2}$$

Lemma 4.1. Under Assumption 1.1, for each $n \in \mathbb{N}$, the function f_n has the following properties:

- *$f_n : W_0^{1,p}(\Omega) \to L^2(\Omega)$ is globally Lipschitz continuous with Lipschitz constant possibly dependent on n.*
- *$f_n : W^{1-\epsilon,p}(\Omega) \to L^{\frac{6}{5}}(\Omega)$ is locally Lipschitz continuous. Furthermore, on any bounded set the local Lipschitz constant **does not** depend n. Here, the parameter ϵ is as defined in Lemma 1.1.*

Proof. The proof is very similar to Ref. 35, Lemma 2.3, and thus it is omitted. □

A. Approximate solutions and passage to the limit

In order to prove the existence statement in Theorem 1.3, we approximate the original problem (1.1) by using the cut-off functions η_n introduced in (4.1). In particular, consider the nth problem given by

$$\begin{cases} u_{tt}^n - \Delta_p u^n - \Delta u_t^n = f_n(u^n) \ \text{ in } \Omega \times (0,T), \\ (u^n(0), u_t^n(0)) = (u_{n,0}, u_{n,1}) \in W_0^{1,p}(\Omega) \times L^2(\Omega), \\ u^n = 0, \quad \text{on } \Gamma \times (0,T), \end{cases} \tag{4.3}$$

where $f_n = f\eta_n$ as defined in (4.2), and $(u_{n,0}, u_{n,1}) \to (u_0, u_1)$ in $W_0^{1,p}(\Omega) \times L^2(\Omega)$, as $n \to \infty$, with $\mathscr{E}_n(0) \le \mathscr{E}(0) + 1$ for all $n \in \mathbb{N}$. Recall that $\mathscr{E}_n(0) = \frac{1}{2}\|u_{n,1}\|_2^2 + \frac{1}{p}\|\nabla u_{n,0}\|_p^p$, and $\mathscr{E}(0) = \frac{1}{2}\|u_1\|_2^2 + \frac{1}{p}\|\nabla u_0\|_p^p$.

We would like to apply Proposition 3.1 to nth problem (4.3). In order to do so, we recall the second statement in Lemma 4.1 which guarantees that on any bounded set the local Lipschitz constants of $f_n : W_0^{1,p}(\Omega) \to L^{\frac{6}{5}}(\Omega)$ are independent of n. Hence, by the proof of Proposition 3.1, the local existence time depends on the choice $K^2 > 4\mathscr{E}_n(0)$. Moreover, by choosing $K^2 > 4(\mathscr{E}(0) + 1)$ in the proof of Proposition 3.1, we have one K that properly bounds the norms of the initial data for each $n \in \mathbb{N}$. Therefore, it follows from Proposition 3.1 that for each $n \in \mathbb{N}$, nth problem (4.3) has a local weak solution u^n on $[0,T]$ for some $T > 0$ (independent of n), and u^n satisfies the energy inequality

$$\mathscr{E}_n(t) + \int_0^t \|\nabla u_t^n(\tau)\|_2^2 \, d\tau \le \mathscr{E}_n(0) + \int_0^t \int_\Omega f_n(u^n(\tau)) u_t^n(\tau) \, dx d\tau \tag{4.4}$$

for all $t \in [0,T]$. By the same analysis used to obtain (3.6) and (3.7), we conclude that there exists $C_T > 0$ independent of n such that

$$\mathscr{E}_n(t) + \int_0^t \|\nabla u_t^n(\tau)\|_2^2 \, d\tau \le C_T \quad \text{for all} \quad t \in [0,T]. \tag{4.5}$$

Now, by employing standard compactness theorem (see, for instance, Ref. 38), there exist function u and subsequence of $(u^n)_n$, which we still denote by $(u^n)_n$, such that

$$(u^n)_n \quad \text{is a bounded sequence in } L^\infty\left(0,T;W_0^{1,p}(\Omega)\right), \tag{4.6}$$

$$(u_t^n)_n \quad \text{is a bounded sequence in } L^\infty\left(0,T;L^2(\Omega)\right), \tag{4.7}$$

$$(u_t^n)_n \quad \text{is a bounded sequence in } L^2\left(0,T;W_0^{1,2}(\Omega)\right). \tag{4.8}$$

Moreover, for each $n \in \mathbb{N}$, the function u_n satisfies

$$\begin{aligned}(u_t^n(t),\phi)_\Omega - (u_t^n(0),\phi)_\Omega + \int_0^t\int_\Omega \left(|\nabla u^n|^{p-2}\nabla u^n\cdot\nabla\phi\right)dxd\tau \\ + \int_0^t\int_\Omega \nabla u_t^n\cdot\nabla\phi dxd\tau = \int_0^t\int_\Omega f(u^n)\phi dxd\tau,\end{aligned} \tag{4.9}$$

for all $\phi \in W_0^{1,p}(\Omega)$. We know that

$$\begin{aligned}\left|\langle u_{tt}^n(t),\phi\rangle\right| &= \left|\frac{d}{dt}\langle u_t^n(t),\phi\rangle\right| = \left|\frac{d}{dt}(u_t^n(t),\phi)_\Omega\right| \\ &\le \left|(|\nabla u^n|^{p-2}\nabla u^n,\nabla\phi)_\Omega\right| + \left|(\nabla u_t^n,\nabla\phi)_\Omega\right| + \left|(f(u^n),\phi)_\Omega\right| \\ &\le \|\nabla u^n\|_p\|\nabla\phi\|_p + \left\|\nabla u_t^n\right\|_2\|\nabla\phi\|_2 + \|f(u^n)\|_{\frac{6}{5}}\|\phi\|_6 \\ &\le (\|\nabla u^n\|_p + c\left\|\nabla u_t^n\right\|_2)\|\nabla\phi\|_p + (\|f(u^n)-f(0)\|_{\frac{6}{5}} + \|f(0)\|_{\frac{6}{5}})\|\phi\|_6 \\ &\le (\|\nabla u^n\|_p + c\left\|\nabla u_t^n\right\|_2)\|\nabla\phi\|_p + (C_f\|u^n\|_{W_0^{1,p}(\Omega)} + \|f(0)\|_{\frac{6}{5}})\|\phi\|_6,\end{aligned} \tag{4.10}$$

wherein we have invoked the second statement of Lemma 4.1 for the last inequality. Since Lemma 4.1 assures us that the local Lipschitz constant of $f_n : W_0^{1,p}(\Omega) \to L^{\frac{6}{5}}(\Omega)$ is independent of n (only on the norm of u^n which is, in turn, uniformly bounded in n), we conclude that

$$(u_{tt}^n)_n \quad \text{is a bounded sequence in } L^2\left(0,T;W_0^{-1,p'}(\Omega)\right). \tag{4.11}$$

From Corollary 2.3, we know

$$u^n \to u \ \text{ weakly}^* \text{ in } L^\infty(0,T;W_0^{1,p}(\Omega)), \tag{4.12}$$

$$u_t^n \to u_t \ \text{ weakly}^* \text{ in } L^\infty(0,T;L^2(\Omega)), \tag{4.13}$$

$$u^n \to u \ \text{ strongly in } L^\infty(0,T;W^{1-\epsilon,p}(\Omega)), \tag{4.14}$$

$$u_t^n \to u_t \ \text{ strongly in } L^2(0,T;L^2(\Omega)), \tag{4.15}$$

$$u_t^n \to u_t \ \text{ weakly in } L^2(0,T;W_0^{1,2}(\Omega)), \tag{4.16}$$

where ϵ is as defined as in (1.8).

In order to pass to the limit in (4.9) and show that u actually solves (1.1), we need the following lemma.

Lemma 4.2. If $(u_N)_N$ *and* u *satisfy* (4.14)*, then* $f_N(u_N) \to f(u)$ weakly in Y^**, where* $Y = L^6(0,T,W^{1-\epsilon,p}(\Omega))$ *and* Y^* *is its dual space, and where* $\epsilon > 0$ *is sufficiently small as defined in Lemma 1.1.*

Proof. We first pick a proper $\epsilon > 0$. Recall that Lemma 1.1 requires ϵ to obey $0 < \epsilon < \frac{5}{2r} - \frac{3-p}{p}$. Here, we choose

$$0 < \epsilon < \min\left\{\frac{5}{2r} - \frac{3-p}{p},\ \frac{3p-6}{2p}\right\}. \tag{4.17}$$

This choice of ϵ implies that $6 < \frac{3p}{3-(1-\epsilon)p}$. It follows from the embedding $W^{1-\epsilon,p}(\Omega) \hookrightarrow L^{\frac{3p}{3-(1-\epsilon)p}}(\Omega)$ that

$$\|\phi\|_6 \le C\|\phi\|_{W^{1-\epsilon,p}(\Omega)}, \quad \text{for all} \quad \phi \in W^{1-\epsilon,p}(\Omega). \tag{4.18}$$

By Hölder's inequality and by (4.18), we have

$$\begin{aligned}\left|\int_0^t\int_\Omega (f_N(u_N(\tau))-f(u(\tau)))\phi dx d\tau\right| &\le \int_0^t \|f_N(u_N(\tau))-f(u(\tau))\|_{\frac{6}{5}}\|\phi\|_6 d\tau\\ &\le \underbrace{C\int_0^T \|f_N(u_N(\tau))-f_N(u(\tau))\|_{\frac{6}{5}}\|\phi\|_{W^{1-\epsilon,p}(\Omega)}d\tau}_{I}\\ &\quad+\underbrace{C\int_0^T \|f_N(u(\tau))-f(u(\tau))\|_{\frac{6}{5}}\|\phi\|_{W^{1-\epsilon,p}(\Omega)}d\tau}_{II},\end{aligned} \tag{4.19}$$

for all $\phi \in Y$ and all $t \in [0,T]$.

By the second part of the statement of Lemma 4.1 and by convergence result (4.14), we have

$$\begin{aligned} I &\le C_R\int_0^T \|u_N-u\|_{W^{1-\epsilon,p}(\Omega)}\|\phi\|_{W^{1-\epsilon,p}(\Omega)}d\tau\\ &\le C_R\|u_N-u\|_{L^\infty(0,T,W^{1-\epsilon,p}(\Omega))}\|\phi\|_{L^6(0,T,W^{1-\epsilon,p}(\Omega))}\\ &\to 0,\ \text{as } N\to\infty. \end{aligned} \tag{4.20}$$

From Hölder's inequality, we obtain

$$\begin{aligned} II &\le C\left[\int_0^T \|f_N(u(\tau))-f(u(\tau))\|_{\frac{6}{5}}^{\frac{6}{5}}d\tau\right]^{\frac{5}{6}}\|\phi\|_{L^6(0,T,W^{1-\epsilon,p}(\Omega))}\\ &= C\|f_N(u(\tau))-f(u(\tau))\|_{L^{\frac{6}{5}}(\Omega\times[0,T])}\|\phi\|_{L^6(0,T,W^{1-\epsilon,p}(\Omega))}. \end{aligned} \tag{4.21}$$

Since $\eta_N \to 1$ from below as $N\to\infty$, then $|f_N(u)-f(u)|\to 0$ almost everywhere on $\Omega\times[0,T]$. In addition, by (4.17) we have $\frac{6r}{5} < \frac{3p}{3-(1-\epsilon)p}$ (see (1.9) in the proof of Lemma 1.1). Hence, it follows from the regularity $u\in L^\infty(0,T;W^{1-\epsilon,p}(\Omega))$ and the 3D embedding: $W^{1-\epsilon,p}(\Omega)\hookrightarrow L^{\frac{3p}{3-(1-\epsilon)p}}(\Omega)$ that

$$|f_N(u)-f(u)|^{\frac{6}{5}} \le 2|f(u)|^{\frac{6}{5}} \le C(|u|^{\frac{6r}{5}}+1)\in L^1(\Omega\times[0,T]).$$

Now Lebesgue dominated convergence theorem gives us

$$\int_0^T \|f_N(u(\tau))-f(u(\tau))\|_{6/5}^{6/5}d\tau \to 0 \quad\text{as}\quad N\to\infty. \tag{4.22}$$

Finally, combine (4.19)–(4.22) to complete the proof of Lemma 4.2. □

Now using the same argument as in Sections II C–II E, the proof of Theorem 1.3 can be easily completed.

V. GLOBAL EXISTENCE

This section is devoted to the existence of global solutions, as stated in Theorem 1.4. Here, we appeal to a standard continuation procedure in order to conclude that either the weak solution u is global or there exists $0<T<\infty$ such that

$$\limsup_{t\to T^-}\mathscr{E}(t)=\infty, \tag{5.1}$$

where $\mathscr{E}(t)$ is the positive energy defined by

$$\mathscr{E}(t) := \frac{1}{2}\|u_t(t)\|_2^2+\frac{1}{p}\|\nabla u(t)\|_p^p. \tag{5.2}$$

We note here that in view of the 3D embedding $W_0^{1,p}(\Omega)\hookrightarrow L^{\frac{3p}{3-p}}(\Omega)$ and $r+1<\frac{5p}{2(3-p)}+1<\frac{3p}{3-p}$, the energy $\mathscr{E}(t)$ is well defined for all $t\in[0,T]$.

We aim at proving that (5.1) cannot happen under the assumption of Theorem 1.4.

Proposition 5.1. Let u be a weak solution of (1.1) *on* $[0,T]$ *as furnished by Theorem 1.3. Then, we have the following.*

- *If $r \le p/2$, then for all $t \in [0,T]$, u satisfies*

$$\mathscr{E}(t) + \int_0^t \|\nabla u_t(\tau)\|_2^2 \, d\tau \le C(T, \mathscr{E}(0)), \tag{5.3}$$

where $T > 0$ is being arbitrary.

- *If $r > p/2$, then the bound in (5.3) holds for $0 \le t < T'$, for some $T' > 0$ dependent on $\mathscr{E}(0)$ and T.*

Proof. First, revisit energy inequality (2.72),

$$\mathscr{E}(t) + \int_0^t \|\nabla u_t(\tau)\|_2^2 \, d\tau \le \mathscr{E}(0) + \int_0^t \int_\Omega f(u(\tau)) u_t(\tau) dx d\tau. \tag{5.4}$$

Now we proceed to estimate the last term on the right-hand side of (5.4). Recall the assumption $|f(s)| \le c_0 |s|^r$ for $|s| \ge 1$ and define $Q_t := \Omega \times (0,t)$,

$$Q_t' := \{(x,\tau) \in Q_t : |u(x,\tau)| \le 1\}, \quad Q_t'' := \{(x,\tau) \in Q_t : |u(x,\tau)| > 1\}.$$

By Young's inequality

$$\begin{aligned} \Big| \int_0^t \int_\Omega f(u(\tau)) \, u_t(\tau) dx d\tau \Big| &\le \int_{Q_t'} |f(u(\tau)) u_t(\tau)| \, dx d\tau + \int_{Q_t''} |f(u(\tau)) u_t(\tau)| \, dx d\tau \\ &\le C \int_{Q_t'} |u_t(\tau)| \, dx d\tau + C \int_{Q_t''} |u(\tau)|^r \, |u_t(\tau)| \, dx d\tau \\ &\le C \int_0^t \mathscr{E}(\tau) d\tau + C \, |Q_T| + C \int_0^t \int_\Omega |u(\tau)|^r \, |u_t(\tau)| \, dx d\tau, \end{aligned} \tag{5.5}$$

for all $t \in [0,T]$, where $|Q_T|$ denotes the lebesgue measure of Q_T. Thus, it follows from (5.4) and (5.5) that

$$\begin{aligned} \mathscr{E}(t) + \int_0^t \|\nabla u_t(\tau)\|_2^2 \, d\tau \le \mathscr{E}(0) + C \, |Q_T| + C \int_0^t \mathscr{E}(\tau) d\tau \\ + C \int_0^t \int_\Omega |u_t(\tau)| \, |u(\tau)|^r dx d\tau, \end{aligned} \tag{5.6}$$

for $t \in [0,T]$. By Hölder's inequality along with Sobolev embedding result (1.5) (with $\frac{6r}{5} < \frac{3p}{3-p}$), we obtain

$$\int_0^t \int_\Omega |u(\tau)|^r \, |u_t(\tau)| \, dx d\tau \le \int_0^t \|u_t(\tau)\|_6 \, \|u(\tau)\|_{\frac{6r}{5}}^r \, d\tau \tag{5.7}$$

$$\le C_1 \int_0^t \|\nabla u_t(\tau)\|_2 \, \|\nabla u(\tau)\|_p^r \, d\tau. \tag{5.8}$$

Again by Young's inequality, we have

$$\int_0^t \int_\Omega |u(\tau)|^r \, |u_t(\tau)| \, dx d\tau \le \epsilon \int_0^t \|\nabla u_t(\tau)\|_2^2 \, d\tau + C_\epsilon \int_0^t \|\nabla u(\tau)\|_p^{2r} \, d\tau. \tag{5.9}$$

Case 1. If $r \le p/2$, then

$$\begin{aligned} \int_0^t \int_\Omega |u(\tau)|^r \, |u_t(\tau)| \, dx d\tau &\le \epsilon \int_0^t \|\nabla u_t(\tau)\|_2^2 \, d\tau + C_{\epsilon,T} \int_0^t \|\nabla u(\tau)\|_p^p \, d\tau + C_{\epsilon,T} \\ &\le \epsilon \int_0^t \|\nabla u_t(\tau)\|_2^2 \, d\tau + C_{\epsilon,T} \int_0^t \mathscr{E}(\tau) d\tau + C_{\epsilon,T}. \end{aligned} \tag{5.10}$$

It now follows from (5.6) and (5.10) that for $t \in [0,T]$,

$$\begin{aligned} \mathscr{E}(t) + \int_0^t \|\nabla u_t(\tau)\|_2^2 \, d\tau \le \mathscr{E}(0) + C \, |Q_T| + C \int_0^t \mathscr{E}(\tau) d\tau \\ + C \cdot \epsilon \int_0^t \|\nabla u_t(\tau)\|_2^2 \, d\tau + C_{\epsilon,T} \int_0^t \mathscr{E}(\tau) d\tau + C_{\epsilon,T}. \end{aligned} \tag{5.11}$$

Choose $0 < \epsilon < \frac{1}{2C}$, then (5.11) gives us

$$\mathscr{E}(t) + \frac{1}{2}\int_0^t \|\nabla u_t(\tau)\|_2^2\, d\tau \le C_2 + C_0 \int_0^t \mathscr{E}(\tau) d\tau, \tag{5.12}$$

where $C_2 = \mathscr{E}(0) + C\,|Q_T| + C_{\epsilon,T}$ and $C_0 = C + C_{\epsilon,T}$. By Gronwall's inequality,

$$\mathscr{E}(t) + \frac{1}{2}\int_0^t \|\nabla u_t(\tau)\|_2^2\, d\tau \le C_2 e^{C_0 T}, \tag{5.13}$$

for $t \in [0,T]$. Hence, (5.3) follows.

Case 2. If $r > p/2$ appeal to (5.9) and obtain

$$\begin{aligned}\int_0^t \int_\Omega |u(\tau)|^r\, |u_t(\tau)|\, dx d\tau &\le \epsilon \int_0^t \|\nabla u_t(\tau)\|_2^2\, d\tau + C_\epsilon \int_0^t \|\nabla u(\tau)\|_p^{2r}\, d\tau \\ &\le \epsilon \int_0^t \|\nabla u_t(\tau)\|_2^2\, d\tau + C_\epsilon \int_0^t \mathscr{E}(\tau)^{\frac{2r}{p}}\, d\tau. \end{aligned}\tag{5.14}$$

Thanks to (5.6) and (5.14), we have

$$\begin{aligned}\mathscr{E}(t) + \int_0^t \|\nabla u_t(\tau)\|_2^2\, d\tau &\le \mathscr{E}(0) + C\,|Q_T| + C\int_0^t \mathscr{E}(\tau) d\tau \\ &+ C\epsilon \int_0^t \|\nabla u_t(\tau)\|_2^2\, d\tau + C_\epsilon \int_0^t \mathscr{E}(\tau)^{\frac{2r}{p}}\, d\tau. \end{aligned}\tag{5.15}$$

Choose $0 < \epsilon < \frac{1}{2C}$, then (5.15) gives

$$\mathscr{E}(t) + \frac{1}{2}\int_0^t \|\nabla u_t(\tau)\|_2^2\, d\tau \le C_3 + C_4 \int_0^t (\mathscr{E}(\tau) + \mathscr{E}(\tau)^{\frac{2r}{p}}) d\tau, \tag{5.16}$$

where $C_3 = \mathscr{E}(0) + C\,|Q_T|$ and $C_4 = C + C_\epsilon$. Now, put

$$Y(t) = 1 + \mathscr{E}(t),$$

and since $\frac{2r}{p} > 1$, then (5.16) implies

$$Y(t) + \frac{1}{2}\int_0^t \|\nabla u_t(\tau)\|_2^2\, d\tau \le C_3 + 2C_4 \int_0^t Y(\tau)^{\frac{2r}{p}}\, d\tau. \tag{5.17}$$

In particular,

$$Y(t) \le C_3 + 2C_4 \int_0^t Y(\tau)^\sigma d\tau \;\text{ for }\; t \in [0,T_0], \tag{5.18}$$

where $\sigma = \frac{2r}{p} > 1$. By using a standard comparison theorem (see Ref. 24, for instance), then (5.18) guarantees that $\mathscr{E}(t) \le Y(t) \le z(t)$, where $z(t) = [C_3^{1-\sigma} - 2C_4(\sigma-1)t]^{-\frac{1}{\sigma-1}}$ is the solution of the Volterra integral equation

$$z(t) = C_3 + 2C_4 \int_0^t z(s)^\sigma ds.$$

Since $\sigma > 1$, then clearly $z(t)$ blows up at the finite time $T_1 = \frac{C_3^{1-\sigma}}{2C_4(\sigma-1)}$, i.e., $z(t) \to \infty$, as $t \to T_1^-$. Note that T_1 depends only on the initial energy $\mathscr{E}(0)$ and the original existence time T. Nonetheless, whenever $0 < T' < \min\{T, T_1/2\}$, we have $\mathscr{E}(t) \le Y(t) \le z(t) \le C(T', \mathscr{E}(0))$ for all $t \in [0,T']$. Hence, the proof of the proposition is complete. □

VI. BLOW-UP OF SOLUTIONS

In this section, we provide the proof of Theorem 1.6. Throughout the proof, we shall adopt Assumptions 1.1 and 1.5 with $r > p-1$. We define the lifespan T of the solution to be the supremum of all $T^* > 0$ such that u is a solution to (1.1) in the sense of Definition 1.2 on $[0,T^*]$, as furnished by Theorem 1.3. Our goal is to show that T is necessarily finite and obtain an upper bound for T.

As in Refs. 3 and 12, for $t \in [0,T]$, we introduce

$$G(t) = \int_0^t \|\nabla u_t(\tau)\|_2^2 \, d\tau - E(0), \quad N(t) = \|u(t)\|_2^2, \quad S(t) = \int_\Omega F(u(t))dx,$$

where the "total energy" is

$$E(t) = \frac{1}{2}\|u'(t)\|_2^2 + \frac{1}{p}\|\nabla u(t)\|_p^p - \int_\Omega F(u(t))dx.$$

Claim 1: G is absolutely continuous, increasing, and positive function on $[0,T]$. It follows from the regularity of $u_t \in L^2(0,T; W_0^{1,2}(\Omega))$ in Definition 1.2 that G is absolutely continuous on $[0,T]$ and $G'(t) = \|\nabla u_t(t)\|_2^2 \geq 0$ on almost everywhere on $[0,T]$. Since $G(0) = -E(0) > 0$, then the claim follows.

Claim 2: We show that

$$0 < -E(0) \leq G(t) \leq -E(t) \leq S(t) \leq c_1 \|u(t)\|_{r+1}^{r+1}, \ t \in [0,T]. \tag{6.1}$$

Indeed, by energy inequality (1.20) and the definition of $G(t)$, we have $E(t) + G(t) \leq 0$ which proves the first three inequalities in (6.1). Also, it follows from the definition of total energy $E(t)$ and $S(t)$ that $E(t) + S(t) = \frac{1}{2}\|u'(t)\|_2^2 + \frac{1}{p}\|\nabla u(t)\|_p^p \geq 0$. Hence, the third inequality in (6.1) is true. The last inequality follows directly from Assumption 1.5.

In order to show our blow-up result, we introduce a parameter a and function $Y(t)$ such that

$$0 < a < \min\left\{\frac{p-2}{r+1}, \frac{p-2}{2p}\right\}, \tag{6.2}$$

$$Y(t) := G(t)^{1-a} + \epsilon N'(t), \tag{6.3}$$

where $\epsilon > 0$. It is clear that $0 < a < 1/2$ and later in the proof we will further restrict the condition on ϵ. Our aim is to show that there exists $\Gamma > 0$ and $\xi > 0$ such that

$$Y'(t) \geq \Gamma\left[G(t) + \|\nabla u(t)\|_p^p + \|u'(t)\|_2^2\right] \geq \xi (Y(t))^{\frac{1}{1-a}}. \tag{6.4}$$

Step 1. Here, we demonstrate that

$$Y'(t) = (1-a)G(t)^{-a}G'(t) + \epsilon N''(t), \tag{6.5}$$

where

$$N''(t) = 2\|u'(t)\|_2^2 - 2\|\nabla u(t)\|_p^p - 2(\nabla u_t(t), \nabla u(t))_\Omega + 2(r+1)\int_\Omega F(u(t))dx. \tag{6.6}$$

The regularity of u implies that

$$N'(t) = 2(u(t), u'(t))_\Omega. \tag{6.7}$$

In addition, by recalling (1.19), we obtain

$$\begin{aligned} \langle u''(t), \phi\rangle_p &= \frac{d}{dt}\langle u'(t), \phi\rangle_p = \frac{d}{dt}(u'(t), \phi)_\Omega \\ &= -(|\nabla u(t)|^{p-2}\nabla u(t), \nabla\phi)_\Omega - (\nabla u_t(t), \nabla\phi)_\Omega + (f(u(t)), \phi)_\Omega, \end{aligned} \tag{6.8}$$

for all $\phi \in W_0^{1,p}(\Omega)$ and almost everywhere on $[0,T]$. It follows now from Proposition A.1 that

$$\frac{1}{2}\frac{d}{dt}N'(t) = \frac{d}{dt}(u'(t), u(t))_\Omega = \|u'(t)\|_2^2 + \langle u''(t), u(t)\rangle_p. \tag{6.9}$$

The regularity of u allows us to replace ϕ in (6.8) with $u(t)$, and so, by using (6.9) and the property $sf(s) = (r+1)F(s)$ asserted in Assumption 1.5, we obtain (6.6). Therefore, we have proven (6.5), i.e.,

$$\begin{aligned} Y'(t) = &(1-a)G(t)^{-a}G'(t) + 2\epsilon\|u'(t)\|_2^2 \\ &- 2\epsilon\|\nabla u(t)\|_p^p - 2\epsilon(\nabla u_t(t), \nabla u(t))_\Omega + 2\epsilon(r+1)S(t). \end{aligned} \tag{6.10}$$

Step 2. Let us verify the first inequality in (6.4). Recalling (6.1) that

$$G(t) \le -E(t) = S(t) - \frac{1}{2}\|u'(t)\|_2^2 - \frac{1}{p}\|\nabla u(t)\|_p^p,$$

we have

$$S(t) \ge G(t) + \frac{1}{2}\|u'(t)\|_2^2 + \frac{1}{p}\|\nabla u(t)\|_p^p > 0. \tag{6.11}$$

By combining (6.11) with (6.10), we obtain

$$\begin{aligned} Y'(t) &\ge (1-a)G(t)^{-a}G'(t) + \epsilon(r+3)\|u'(t)\|_2^2 + 2\epsilon(r+1)G(t) \\ &+ 2\epsilon\left(\frac{r+1}{p} - 1\right)\|\nabla u(t)\|_p^p - 2\epsilon(\nabla u_t(t), \nabla u(t))_\Omega. \end{aligned} \tag{6.12}$$

Now estimate the last term on the right-hand side of (6.12) with the help of Hölder's and Young's inequalities,

$$(\nabla u_t(t), \nabla u(t))_\Omega \le \|\nabla u_t(t)\|_2\|\nabla u(t)\|_2 \le \frac{1}{2\lambda}\|\nabla u_t(t)\|_2^2 + \frac{\lambda}{2}\|\nabla u(t)\|_2^2. \tag{6.13}$$

Let $\lambda = KG(t)^{-a}$. Now, (6.13) and the fact that $G'(t) = \|\nabla u_t(t)\|_2^2$ imply

$$2\epsilon(\nabla u_t(t), \nabla u(t))_\Omega \le \epsilon K G(t)^{-a}G'(t) + \epsilon K^{-1}G(t)^a\|\nabla u(t)\|_2^2. \tag{6.14}$$

Use inequality (6.1), Assumption 1.5, and embedding result (1.5) to arrive at

$$G(t) \le S(t) \le c_1\|u(t)\|_{r+1}^{r+1} \le C\|\nabla u(t)\|_p^{r+1}. \tag{6.15}$$

Now Hölder's inequality and (6.15) give

$$G(t)^a\|\nabla u(t)\|_2^2 \le C\|\nabla u(t)\|_p^{a(r+1)}\|\nabla u(t)\|_p^2 = C(\|\nabla u(t)\|_p^p)^{\frac{a(r+1)+2}{p}}. \tag{6.16}$$

From condition (6.2) that $0 < a \le \frac{p-2}{r+1}$ we know $\frac{a(r+1)+2}{p} \le 1$. Next, since

$$z^\nu \le z + 1 \le \left(1 + \frac{1}{\alpha}\right)(z+\alpha), \quad \text{for all} \quad 0 \le z, \quad 0 < \nu \le 1, \quad 0 < \alpha, \tag{6.17}$$

then replacing z with $\|\nabla u(t)\|_p^p$ and ν with $\frac{a(r+1)+2}{p}$ in (6.17) gives

$$C(\|\nabla u(t)\|_p^p)^{\frac{a(r+1)+2}{p}} \le d(\|\nabla u(t)\|_p^p + G(0)) \le d(\|\nabla u(t)\|_p^p + G(t)), \tag{6.18}$$

where $d = C(1 + 1/G(0))$; we have also used the fact that G is increasing and positive function on $[0, T]$. Therefore, by combining (6.12), (6.14), and (6.18), one has

$$\begin{aligned} Y'(t) \ge& (1 - a - \epsilon K)G(t)^{-a}G'(t) + \epsilon(r+3)\|u'(t)\|_2^2 + \epsilon\left(2(r+1) - \frac{d}{K}\right)G(t) \\ &+ \epsilon\left(2\left(\frac{r+1}{p} - 1\right) - \frac{d}{K}\right)\|\nabla u(t)\|_p^p. \end{aligned} \tag{6.19}$$

Since $r > p - 1$, then by choosing a large enough K, and a small enough ϵ, we have

$$1 - a - \epsilon K > 0, \quad 2(r+1) - \frac{d}{K} > 0, \quad \text{and} \quad 2\left(\frac{r+1}{p} - 1\right) - \frac{d}{K} > 0.$$

Therefore, we can define

$$\Gamma = \min\left\{\epsilon(r+3),\ \epsilon\left(2(r+1) - \frac{d}{K}\right),\ \epsilon\left(2\left(\frac{r+1}{p} - 1\right) - \frac{d}{K}\right)\right\} > 0$$

to conclude

$$Y'(t) \ge \Gamma\left(\|u'(t)\|_2^2 + G(t) + \|\nabla u(t)\|_p^p\right) \ge 0, \quad \text{for all} \quad t \in [0, T] \tag{6.20}$$

completing the first inequality in (6.4).

Step 3. Here, we further adjust the value of ϵ to guarantee that

$$Y(t) > \frac{1}{2}G(0)^{1-a} > 0 \quad \text{for all} \quad t \in [0,T]. \tag{6.21}$$

We have two cases to consider. If $N'(0) \geq 0$, then no further adjustment is necessary. However, if $N'(0) < 0$, then we impose an additional condition $0 < \epsilon \leq -\frac{G(0)^{1-a}}{2N'(0)}$. Thus, inequality (6.21) holds for any value of $N'(0)$.

Step 4. We will now verify the second inequality in (6.4) and hence prove

$$Y'(t) \geq \xi Y(t)^{\frac{1}{1-a}}, \quad \text{for all} \quad t \in [0,T]. \tag{6.22}$$

Case a. If $N'(t) \leq 0$ for some $t \in [0,T]$, then for such values of t we have

$$Y(t)^{\frac{1}{1-a}} = \left[G(t)^{1-a} + \epsilon N'(t)\right]^{\frac{1}{1-a}} \leq G(t), \tag{6.23}$$

and in this case, (6.20) and (6.23) yield

$$Y'(t) > \Gamma G(t) \geq \Gamma Y(t)^{\frac{1}{1-a}}. \tag{6.24}$$

Hence, (6.22) holds for all $t \in [0,T]$ such that $N'(t) \leq 0$.

Case b. If $N'(t) > 0$ for some $t \in [0,T]$, we need to do more work to show that (6.22) still holds. Since $0 < a < 1/2$, we have

$$Y(t)^{\frac{1}{1-a}} \leq 2^{\frac{1}{1-a}}\left[G(t) + \epsilon^{\frac{1}{1-a}}(N'(t))^{\frac{1}{1-a}}\right]. \tag{6.25}$$

It follows from (6.7) and Young's inequality with $\frac{1}{\mu} + \frac{1}{\theta} = 1$ that

$$(N'(t))^{\frac{1}{1-a}} \leq 2^{\frac{1}{1-a}} \|u(t)\|_2^{\frac{1}{1-a}} \|u'(t)\|_2^{\frac{1}{1-a}} \leq C_1\left(\|u(t)\|_2^{\frac{\mu}{1-a}} + \|u'(t)\|_2^{\frac{\theta}{1-a}}\right). \tag{6.26}$$

By taking $\theta = 2(1-a) > 1$ (because $0 < a < 1/2$), $\mu = \frac{2(1-a)}{1-2a}$, and by appealing to the embedding in (1.5), we have

$$(N'(t))^{\frac{1}{1-a}} \leq C_1\left(\|\nabla u(t)\|_p^{\frac{2}{1-2a}} + \|u'(t)\|_2^2\right) = C_1\big((\|\nabla u(t)\|_p^p)^{\frac{2}{p(1-2a)}} + \|u'(t)\|_2^2\big). \tag{6.27}$$

From the assumption on a in (6.2), we know $0 < a < \frac{p-2}{2p}$ which implies $0 < \frac{2}{p(1-2a)} \leq 1$. Hence, replacing z with $\|\nabla u(t)\|_p^p$ and v with $\frac{2}{p(1-2a)}$ in (6.17) gives

$$C_1(\|\nabla u(t)\|_p^p)^{\frac{2}{p(1-2a)}} \leq d(\|\nabla u(t)\|_p^p + G(0)) \leq d(\|\nabla u(t)\|_p^p + G(t)), \tag{6.28}$$

where $d = C_1\big(1 + \frac{1}{G(0)}\big)$. Combine (6.25), (6.27), and (6.28) to conclude

$$\begin{aligned} Y(t)^{\frac{1}{1-a}} &\leq 2^{\frac{1}{1-a}}\left[G(t) + \epsilon^{\frac{1}{1-a}}\left(d\|\nabla u(t)\|_p^p + dG(t) + C_1\|u'(t)\|_2^2\right)\right] \\ &\leq C_2\left(G(t) + \|\nabla u(t)\|_p^p + \|u'(t)\|_2^2\right), \end{aligned} \tag{6.29}$$

where $C_2 = 2^{\frac{1}{1-a}}\left(1 + d\epsilon^{\frac{1}{1-a}} + C_1\epsilon^{\frac{1}{1-a}}\right) > 0$. Let $\xi = \frac{\Gamma}{C_2}$. Then, our desired inequality (6.22) follows from (6.20) and (6.29) for all values of $t \in [0,T]$ such that $N'(t) > 0$.

Combining the above cases **a** and **b** we conclude for all $t \in [0,T]$ that

$$Y'(t) \geq \xi Y^{\eta}(t), \quad \text{for} \quad \xi > 0, \quad \text{and} \quad 1 < \eta = \frac{1}{1-a} < 2. \tag{6.30}$$

Hence, by a simple calculation, it follows from (6.30) and (6.21) that T is necessarily finite and

$$T < \frac{Y(0)^{\frac{-a}{1-a}}}{\xi} < \frac{\frac{1}{2}^{\frac{-a}{1-a}}G(0)^{-a}}{\xi} < \infty. \tag{6.31}$$

This completes the proof of Theorem 1.6.

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APPENDIX: WEAK CONVERGENCE IN $H_0^1(\Omega)$ ALMOST EVERYWHERE ON $[0, T]$

The following auxiliary results were invoked in the proof of the main theorem. We could not find exactly matching statements in the literature, but these helpful technical propositions can be easily verified either directly, or, by following the proofs of the indicated existing theorems.

Proposition A.1 (Similar to the proof of Ref. 35, Proposition 3.2). Let H be a Hilbert space and X be a Banach space with its dual X^* such that $X \subset H \subset X^*$ where the injections are continuous and each space is dense in the following one. Assume that $f \in L^2(0,T,H)$, $g \in L^2(0,T,X)$, $f' \in L^2(0,T,X^*)$, and $g' \in L^2(0,T,H)$. Then, $\psi(t) = (f(t), g(t))_H$ coincides with an absolutely continuous function almost everywhere on $[0,T]$ and

$$\frac{d}{dt}(f(t),g(t))_H = \langle f'(t), g(t)\rangle + (f(t), g'(t))_H, \quad a.e.\ [0,T], \tag{A1}$$

where $\langle\cdot,\cdot\rangle$ is the duality pairing between X^* and X.

Proposition A.2. Let H be a Hilbert space and X a Banach space with its dual X^* such that $X \subset H \subset X^*$ where the injections are continuous and each space is dense in the following one. Suppose X^* is separable and $(u_N)_N$ is a bounded sequence in $L^1(0,T;X)$ satisfying

$$u_N \to u \quad weakly\ in \quad L^1(0,T;X), \tag{A2}$$

$$u_N \to u \quad strongly\ in \quad L^1(0,T;H), \tag{A3}$$

as $N \to \infty$. Then, there exists a subsequence of $(u_N)_N$ (again reindexed by N) such that

$$u_N(t) \to u(t)\ weakly\ in\ X \text{ a.e. } [0,T] \quad as \quad N \to \infty. \tag{A4}$$

Proof. Let $\langle\cdot,\cdot\rangle$ denote the standard duality pairing between X^* and X while $(\cdot,\cdot)$ will stand for the inner product in H. It follows from the Cauchy-Schwarz inequality and (A3) that for every $\sigma \in H$

$$\begin{aligned}\int_0^T |(\sigma, u_N(t) - u(t))|\, dt &\le \int_0^T \|\sigma\|_H \|u_N(t) - u(t)\|_H dt \\ &\le \|\sigma\|_H \|u_N - u\|_{L^1(0,T;H)} \to 0, \text{ as } N \to \infty. \end{aligned}\tag{A5}$$

That is, for every $\sigma \in H$,

$$(\sigma, u_N(t) - u(t)) \to 0 \text{ in } L^1[0,T]. \tag{A6}$$

Next, we show that the convergence in (A6) also holds for any $\phi \in X^*$, i.e.,

$$\langle \phi, u_N(t) - u(t)\rangle \to 0 \text{ in } L^1[0,T], \text{ for every } \phi \in X^*. \tag{A7}$$

Since H is dense in X^*, then for a given $\epsilon > 0$, there exists $\sigma \in H$ such that

$$\|\phi - \sigma\|_{X^*} < \epsilon. \tag{A8}$$

It follows from (A5) that, for the fixed $\epsilon > 0$ as given above, there is $N_0 \in \mathbb{N}$ such that

$$\int_0^T |(\sigma, u_N(t) - u(t))|\, dt < \epsilon, \quad \text{for all} \quad N \ge N_0. \tag{A9}$$

By (A8) and (A9), we conclude for any fixed $\epsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that

$$\begin{aligned}
&\int_0^T |\langle \phi, u_N(t) - u(t)\rangle|\,dt \\
&\quad \le \int_0^T \|\phi - \sigma\|_{X^*}\|u_N(t) - u(t)\|_X dt + \int_0^T |(\sigma, u_N(t) - u(t))|\,dt \\
&\quad \le \epsilon \int_0^T (\|u_N(t)\|_X + \|u(t)\|_X)\,dt + \epsilon \le C\epsilon, \text{ for all } N \ge N_0.
\end{aligned} \tag{A10}$$

Hence, (A7) follows.

Since X^* is separable, there is a countable set $\{\phi_m\}$ which is dense in X^*. We now apply (A7) to the sequence $\{\phi_m\}$ as follows. Denote $u_N(t) - u(t)$ by $U_N(t)$. For ϕ_1, we have

$$\langle \phi_1, U_N(t)\rangle \to 0 \text{ in } L^1[0,T].$$

Then, there exists a (reindexed) subsequence $(U_{N,1})_N$ of $(U_N)_N$ such that

$$\langle \phi_1, U_{N,1}(t)\rangle \to 0 \text{ a.e. } t \in [0,T].$$

Similarly, we have

$$\langle \phi_2, U_{N,1}(t)\rangle \to 0 \text{ in } L^1[0,T].$$

Then, there exists a subsequence $(U_{N,2})_N$ of $(U_{N,1})_N$ such that

$$\langle \phi_2, U_{N,2}(t)\rangle \to 0 \text{ a.e. } t \in [0,T].$$

Continuing in this manner we obtain, for any integer $j \ge 1$, a subsequence $(U_{N,j+1})_N$ of $(U_{N,j})_N$ such that

$$\langle \phi_{j+1}, U_{N,j+1}(t)\rangle \to 0 \text{ a.e. } t \in [0,T], \text{ as } N \to \infty. \tag{A11}$$

Since $\{\phi_m\}$ is dense for X^*, then by a density argument, the following convergence holds for all $\phi \in X^*$:

$$\langle \phi, U_{N,N}(t)\rangle \to 0 \text{ a.e. } t \in [0,T]. \tag{A12}$$

That is, there exist a subsequence (reindexed again by N) of the original sequence $(u_N)_N$ such that

$$u_N(t) \to u(t) \text{ weakly in } X \text{ a.e. } [0,T], \text{ as } N \to \infty, \tag{A13}$$

which completes the proof. □

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