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We define and study parafermion stabilizer codes, which can be viewed as generalizations of Kitaev's one-dimensional (1D) model of unpaired Majorana fermions. Parafermion stabilizer codes can protect against low-weight errors acting on a small subset of parafermion modes in analogy to qudit stabilizer codes. Examples of several smallest parafermion stabilizer codes are given. A locality-preserving embedding of qudit operators into parafermion operators is established that allows one to map known qudit stabilizer codes to parafermion codes. We also present a local 2D parafermion construction that combines topological protection of Kitaev's toric code with additional protection relying on parity conservation.

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I. INTRODUCTION

Topologically protected systems are potentially useful for realizations of fault-tolerant elements in a quantum computer [1,2]. The zero-temperature stability of such systems leads to exponential suppression of decoherence induced by local environmental perturbations. On the other hand, the manipulation of the degenerate ground state can be achieved by braiding operations with non-Abelian anyons [3,4].

The Kitaev chain provides an enlightening example of how interactions can result in non-Abelian quasiparticles [5]. Networks of one-dimensional realizations of such quasiparticles can be employed for realizations of quantum gates via braiding operations [6,7]. However, only a nonuniversal set of quantum gates can be realized with Majorana zero modes. A generalization of the Kitaev chain model has been proposed recently in which quasiparticles obey parafermion $\mathbb{Z}_D$ algebra as opposed to $\mathbb{Z}_2$ algebra for Majorana zero modes [8]. Many recent publications address possible realizations of parafermion zero modes [9–29]. The braiding properties of parafermion systems have some advantages over the Majorana modes, while still remaining nonuniversal [10,11,16]. However, parafermion systems can be used for obtaining quasiparticles that permit universal quantum computations [19].

The presence of finite temperature introduces inevitable errors and in principle requires continuous error correction [30]. "Self-correcting" quantum memories are stable at finite temperatures [31,32]; however, they cannot be realized in two dimensions with local interactions [33,34]. Parafermion stabilizer codes considered here can protect against low-weight fermionic errors, i.e., errors that act on a small subset of parafermion modes. The measurement and manipulation schemes required for code implementations have been formulated for Majorana zero modes [35–37] and should in principle generalize to parafermion zero modes [11].

In this paper, we address the possibility of active error correction in systems containing a set of parafermion modes as opposed to typical systems containing qubits or qudits. Earlier works on quantum error correction usually addressed the qubit case with a Hilbert space dimension $D = 2$ [30,38–40]. Error correction on qudits with $D > 2$ has also been considered, and qudit stabilizer codes have been introduced [41–47]. The formalism is usually applied to situations in which D is prime or a prime power [42,48,49], while generalizations to composite D are also possible [50].

Parafermion codes can also be interpreted in terms of termwise commuting Hamiltonians of interacting parafermion zero modes, thus generalizing Kitaev's one-dimensional (1D) model of unpaired Majorana fermions to the $D > 2$ case and to arbitrary interactions preserving the commutativity of terms in the Hamiltonian. Of particular interest are the Hamiltonians corresponding to geometrically local interactions on a d-dimensional lattice. Thus, one can ask similar questions to those posed in Ref. [51] in relation to Majorana codes, i.e., what is the role of superselection rules in the finite-temperature stability of topological order defined by interacting parafermion modes. Such superselection rules are characteristic of fermionic systems when only interactions with bosonic environments are present. On the other hand, the superselection rule prohibiting parity-violating error operators is not likely to always hold, for instance, when the environment supports gapless fermionic modes that can couple to the system [52,53]. Parafermion stabilizer codes can help in such situations by providing protection associated with the code distance of parity-violating logical operators.

The paper is organized as follows. In Sec. II, we introduce notations and provide background on the theory of qudit stabilizer codes. Here we also discuss the Jordan-Wigner transformation, which leads us to the introduction of parafermion operators. In Sec. III, we give a formal definition of parafermion stabilizer codes and establish their basic properties. We also discuss the commutativity condition on stabilizer generators, define the code distance, and prove basic results on the dimension of the code space. In Sec. IV, we present several examples of the smallest parafermion stabilizer codes. In Sec. V, we construct mappings between qudit stabilizer codes and parafermion stabilizer codes. By employing such mappings, we are able to construct parafermion toric code with an adjustable degree of protection against the parity-violating errors. Finally, we give our conclusions in Sec. VI.

II. BACKGROUND

A. Qudits

Qudits are D-dimensional generalizations of qubits, and are generally implemented using D-level physical systems.

One of the well-known generating sets for qudit operations is constructed by the generators of the finite *discrete Weyl group* $\mathcal{W}_D$ that obey the defining relations [54,55]

$$X^D = Z^D = \mathbb{1}, \quad ZX = \omega XZ. \tag{1}$$

This group is sometimes referred to as the discrete Heisenberg group [42], and the generators are sometimes referred to as generalized Pauli matrices [50]. By diagonalizing one of these operators, say Z, one obtains the D-dimensional representation

$$X = \sum_{j=0}^{D-1} |j+1\rangle\langle j|, \quad Z = \sum_{j=0}^{D-1} \omega^j |j\rangle\langle j|, \tag{2}$$

where $\omega = e^{2\pi i/D}$ and the addition $j+1$ is in mod D. Above and throughout the paper, $\mathbb{1}$ denotes the identity operator with proper dimensions. Products of X and Z span the Lie algebra $\mathfrak{su}(D)$, hence their linear combinations can generate universal SU(D) operations. Operations on multiple qudits are tensor products of the single-qudit operators, hence operators acting on distinct qudits commute. We will denote an X operator acting on the jth site as X_j, which is equivalent to an X operator at the jth slot of the tensor product padded with identity operators: $X_j = \mathbb{1} \otimes \cdots \otimes X \otimes \cdots \otimes \mathbb{1}$ (and similar for Z_j).

B. Stabilizer codes for qudits

Stabilizer codes are an important class of quantum error-correcting codes [30,56], which, under appropriate mapping, can be also thought of as additive classical codes [57]. Stabilizer codes utilize a set of commuting operators, called the stabilizer group, for defining the code space. In this section, we review the stabilizer formalism for qudits (see, e.g., [50]). Let $\mathcal{S}$ be a maximal Abelian subgroup of $\mathcal{W}_D^{\otimes n}$ that does not contain $\omega^j \mathbb{1}$ ($j \in \mathbb{Z}_D$ and $j \neq 0$) and $C_\mathcal{S}$ be the code subspace of the Hilbert space stabilized by all the elements of $\mathcal{S}$, i.e., $S_i|\psi\rangle = |\psi\rangle \forall\ S_i \in \mathcal{S}$ and $|\psi\rangle \in C_\mathcal{S}$, then $\mathcal{S}$ is called the stabilizer group and it is generally denoted by its generating set $\mathcal{S} = \langle S_1, S_2, \ldots, S_k\rangle$.

Since the stabilizer group $\mathcal{S}$ is an Abelian group, its elements must commute with each other by definition. The commutativity condition of its generators depends upon the particular case of $\mathcal{W}_D^{\otimes n}$ at hand. Two arbitrary elements of $\mathcal{W}_D^{\otimes n}$, $G = \omega^\lambda X^{\boldsymbol{u}} Z^{\boldsymbol{v}}$ and $G' = \omega^{\lambda'} X^{\boldsymbol{u}'} Z^{\boldsymbol{v}'}$, where $X^{\boldsymbol{u}} = X_1^{u_1} X_2^{u_2} \cdots X_n^{u_n}$, $Z^{\boldsymbol{v}} = Z_1^{v_1} Z_2^{v_2} \cdots Z_n^{v_n}$ (and similarly for G'), will commute iff

$$\boldsymbol{u} \cdot \boldsymbol{v}' = \boldsymbol{v} \cdot \boldsymbol{u}' \mod D \tag{3}$$

is satisfied [50].

The support of a Weyl operator $w \in \mathcal{W}_D^{\otimes n}$, denoted as Supp($w$), is defined as the set of qudits on which it acts nontrivially. The cardinality of the support, $|\mathrm{Supp}(w)|$, is called the weight of the operator w, also denoted as $|w|$. The set of all Weyl operators in $\mathcal{W}_D^{\otimes n}$ that commute with all the elements of $\mathcal{S}$ is called the *centralizer* of $\mathcal{S}$ and is denoted as $\mathcal{C}(S)$.

For prime D, a stabilizer group with $n-k$ independent generators implies that the corresponding centralizer is generated by $n+k$ generators. The logical operators $\{\bar{X}, \bar{Z}\}$ of a stabilizer code $\mathcal{S}$ are the elements of $\mathcal{C}(\mathcal{S})$ that are not in $\mathcal{S}$.

The robustness of a quantum code can be measured by how far two encoded states are apart, which is quantified through the notion of *distance*. The weight of the logical operators implies the separation of the encoded states. Therefore, the distance of a stabilizer code is defined as

$$d = \min_{L_i \in \mathcal{C}(\mathcal{S}) \setminus \mathcal{S}} |L_i|. \tag{4}$$

The longer the code distance is, the better protection the code provides. A code of distance d can detect any error of weight up to $d-1$, and correct up to $\lfloor d/2 \rfloor$. A quantum error-correcting code that encodes n physical qudits into k logical qudits with distance d is denoted as $[[n,k,d]]_D$.

C. Parafermion operators

Parafermion operators can be obtained by the Jordan-Wigner transformation of the D-state spin operators $\{X_j, Z_j\} \in \mathcal{W}_D^{\otimes n}$ as

$$\gamma_{2j-1} = \left(\prod_{k=1}^{j-1} X_k\right) Z_j,$$
$$\gamma_{2j} = \omega^{(d-1)/2} \left(\prod_{k=1}^{j-1} X_k\right) Z_j X_j, \tag{5}$$

which is a mapping of n *local* spin operators into $2n$ *nonlocal* parafermion operators, therefore the total number of parafermion modes is always even. Parafermion operators γ_j obey the following relations:

$$\gamma_j^d = \mathbb{1}, \quad \gamma_j\gamma_k = \omega\gamma_k\gamma_j \quad (j < k, \quad \omega = e^{i2\pi/D}). \tag{6}$$

A special case with $D = 2$ gives us the anticommuting self-adjoint Majorana fermions.

Realizations of parafermion zero modes corresponding to Eq. (6) have been suggested. In such realizations, the localized state is described by a parafermion operator that commutes with the corresponding Hamiltonian and changes the parity of $\mathbb{Z}_D$ charge by 1 [8]. They are non-Abelian anyons and can be used for realizations of fault-tolerant topological quantum gates.

There are recent proposals to construct solid-state systems that accommodate parafermion zero modes. Realizations employing exotic fractional quantum Hall (FQH) states and quantum nanowires have been proposed [9–21].

III. PARAFERMION STABILIZER CODES

A. The group PF(D,2n)

We shall call the group generated by the single-mode operators γ_j given in Eq. (6) the parafermion group PF(D,$2n$). Arbitrary elements of PF(D,$2n$) can be written as $\omega^\lambda \gamma^{\boldsymbol{\alpha}}$, where $\lambda \in \mathbb{Z}_D$ and

$$\gamma^{\boldsymbol{\alpha}} = \gamma_1^{\alpha_1} \cdots \gamma_{2n}^{\alpha_{2n}} \tag{7}$$

with $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_{2n}) \in \mathbb{Z}_D^{2n}$, and by convention the terms are arranged in increasing order in their indices. The ordered set of nonzero elements in $\boldsymbol{\alpha}$ is called the support of $\gamma^{\boldsymbol{\alpha}}$, or Supp($\gamma^{\boldsymbol{\alpha}}$). We define the weight of $\gamma^{\boldsymbol{\alpha}}$ as the number of nonzero entries in $\boldsymbol{\alpha}$, denoted as $|\mathrm{Supp}(\gamma^{\boldsymbol{\alpha}})|$ or simply $|\gamma^{\boldsymbol{\alpha}}|$.

A parafermion operator $\omega^{\lambda}\gamma^{\boldsymbol{\alpha}} \in \mathrm{PF}(D,2n)$ will preserve parity iff

$$\sum_{i=1}^{2n} \alpha_i = 0 \mod D. \tag{8}$$

One can generalize Eq. (6) to obtain $\gamma_i^m \gamma_j^n = \omega^{mn} \gamma_j^n \gamma_i^m$ for $i < j$. Using this, it can be shown that two parafermion operators $\gamma^{\boldsymbol{\alpha}}$ and $\gamma^{\boldsymbol{\beta}}$ commute iff

$$\boldsymbol{\alpha} \Lambda \boldsymbol{\beta}^T = 0 \mod D \tag{9}$$

is satisfied, where Λ is a $2n \times 2n$ antisymmetric matrix $\Lambda_{ij} = \mathrm{sgn}(j-i)$ or explicitly

$$\Lambda = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ -1 & 0 & 1 & \cdots & 1 \\ -1 & -1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ -1 & -1 & -1 & \cdots & 0 \end{pmatrix}. \tag{10}$$

In particular, when the index of the last nonzero entry in $\boldsymbol{\alpha}$ is smaller than the index of the first nonzero entry in $\boldsymbol{\beta}$, the commutativity condition Eq. (9) is reduced to

$$\left(\sum_j \alpha_j\right)\left(\sum_j \beta_j\right) = 0 \mod D. \tag{11}$$

The parity-conservation condition for a parafermion operator can also be expressed in terms of the $\mathbb{Z}_D$ charge operator

$$Q = \prod_{j=1}^{n} \gamma_{2j-1}^{\dagger}\gamma_{2j}. \tag{12}$$

For any $\gamma^{\boldsymbol{\alpha}} \subset \mathrm{PF}(D,2n)$,

$$\gamma^{\boldsymbol{\alpha}} Q = \omega^p Q \gamma^{\boldsymbol{\alpha}}, \quad p = \sum_{i=1}^{2n} \alpha_i \mod D, \tag{13}$$

where p is the $\mathbb{Z}_D$ charge of $\gamma^{\boldsymbol{\alpha}}$, thus the parity-conservation condition can also be written as $[\gamma^{\boldsymbol{\alpha}}, Q] = 0$.

Since Majorana zero modes correspond to the $D = 2$ case, evidently we have $\mathrm{PF}(2,2n) \cong \mathrm{Maj}(2n)$.

B. Stabilizer groups in PF(D,2n)

It is not generally possible to map parafermion operators in $\mathrm{PF}(D,2n)$ onto qudit operators in $\mathcal{W}_D^{\otimes k}$ due to the nonlocality of parafermion operators. The tensor product structure of k-qudit operators in $\mathcal{W}_D^{\otimes k}$ guarantees that operators acting on different sites commute, whereas parafermion operators fail to commute for all distinct sites. Nevertheless, even though a one-to-one mapping between a single-mode parafermion operator and a qudit operator is impossible, it is indeed possible to map multiple parafermion modes onto multiple qudits at once (see Sec. IV B) or to map multiple parafermion modes onto a local single-qudit in a consistent way (see Sec. V). Indeed, as we observe in the next section, $\mathrm{PF}(D,2n)$ proves to be rich group with many nontrivial Abelian subgroups.

Definition. Parafermion stabilizer codes $C_{\mathcal{S}_{\mathrm{PF}}}$, similar to qudit stabilizer codes, are completely determined by their corresponding stabilizer group, which in our case is $\mathcal{S}_{\mathrm{PF}} \subseteq \mathrm{PF}(D,2n)$. We list the defining properties of parafermion stabilizer codes as follows:

(i) Elements of $\mathcal{S}_{\mathrm{PF}}$ are parity-preserving operators.

(ii) $\mathcal{S}_{\mathrm{PF}}$ is an Abelian group not containing $\omega^j\mathbb{1}$, where $j \in \mathbb{Z}_D$ and $j \neq 0$.

Whether these conditions hold for a given parafermion stabilizer code or not can be verified using Eqs. (8) and (9), respectively.

The set of all parafermion operators in $\mathrm{PF}(D,2n)$ that commute with all the elements of $\mathcal{S}_{\mathrm{PF}}$ is called the *centralizer* of $\mathcal{S}_{\mathrm{PF}}$ and is denoted as $\mathcal{C}(\mathcal{S}_{\mathrm{PF}})$. The set of logical operators $\mathcal{L}(\mathcal{S}_{\mathrm{PF}})$ encoding k qudits of a parafermion code $\mathcal{S}_{\mathrm{PF}}$ are the elements of $\mathcal{C}(\mathcal{S}_{\mathrm{PF}})$ that are not in $\mathcal{S}_{\mathrm{PF}}$, that is, $\mathcal{L}(\mathcal{S}_{\mathrm{PF}}) = \mathcal{C}(\mathcal{S}_{\mathrm{PF}}) \setminus \mathcal{S}_{\mathrm{PF}}$. When D is a prime number, the order of the generating set (excluding the identity operator) of $\mathcal{S}_{\mathrm{PF}}$ is $n-k$ and the centralizer is generated by $n+k$ generators.

When writing the generating sets explicitly, we will omit the phase factors ω^l ($l \in \mathbb{Z}_D$) for all generators for brevity throughout the paper, however one should keep in mind that such phase factors are in general required in order to satisfy the second defining property of parafermion codes listed above.

The code space of a parafermion stabilizer code $\mathcal{S}_{\mathrm{PF}}$ is the subspace that is invariant under the action of all the elements of $\mathcal{S}_{\mathrm{PF}}$. The distance d of a parafermion code is given by the minimum weight of its logical operators,

$$d = \min_{\gamma^{\boldsymbol{\alpha}} \in \mathcal{L}(\mathcal{S}_{\mathrm{PF}})} |\gamma^{\boldsymbol{\alpha}}|. \tag{14}$$

We denote a parafermion stabilizer code that encodes $2n$ parafermion modes into k logical qudits with distance d as $[[2n,k,d]]_D$. A parafermion stabilizer code of distance d can detect any parafermion error of weight up to $d-1$, and it can correct up to $\lfloor d/2 \rfloor$ in analogy to qudit codes. However, it should be noted that similar to Majorana fermion codes [51], the robustness of parafermion codes is not solely determined by the code distance d: when some of the logical operators have nonzero parity, the conservation of parafermion parity will offer additional protection, that is, a subspace of the code space will be protected against such errors. Following Ref. [51], we introduce an additional parameter l_{con} defined as the minimum diameter of a region that can support a parity-conserving logical operator:

$$l_{\mathrm{con}} = \min_{\substack{\gamma^{\boldsymbol{\alpha}} \in \mathcal{L}(\mathcal{S}_{\mathrm{PF}}) \\ \sum_i \alpha_i = 0 \mod D}} \mathrm{diam}[\mathrm{Supp}(\gamma^{\boldsymbol{\alpha}})], \tag{15}$$

which can be used in order to measure the degree of protection relying on the superselection rules.

What can be said about the order of $\mathcal{S}_{\mathrm{PF}}$? Below, we adapt the theorem and proof given by Gheorghiu [50] to parafermion stabilizer codes.

Theorem. Let $\mathcal{S}_{\mathrm{PF}}$ be a parafermion stabilizer code in $\mathrm{PF}(D,2n)$, where D is allowed to be composite, let $|\mathcal{S}_{\mathrm{PF}}|$ denote the order of $\mathcal{S}_{\mathrm{PF}}$, and let $|C_{\mathcal{S}_{\mathrm{PF}}}|$ be the dimension of code space. Then the following equation holds:

$$|C_{\mathcal{S}_{\mathrm{PF}}}||\mathcal{S}_{\mathrm{PF}}| = D^n. \tag{16}$$

Proof. The operator

$$P = \frac{1}{|\mathcal{S}_{\mathrm{PF}}|}\sum_{j=1}^{|\mathcal{S}_{\mathrm{PF}}|} S_j \tag{17}$$

is a projection operator satisfying $P^2 = P = P^\dagger$. Clearly, for any $|\psi_j\rangle \in C_{\mathcal{S}_{\rm PF}}$, $P|\psi_j\rangle = |\psi_j\rangle$ holds. Thus the subspace $\mathcal{W}$ that P projects onto includes $C_{\mathcal{S}_{\rm PF}}$, or $C_{\mathcal{S}_{\rm PF}} \subseteq \mathcal{W}$.

Next we show that this relation holds the other way around. Let $|\phi\rangle$ be an arbitrary element of $\mathcal{W}$ (thus $P|\phi\rangle = |\phi\rangle$) and S_k be an arbitrary element of $\mathcal{S}_{\rm PF}$. Since $S_k P = P$ for all k, we obtain $S_k(P|\phi\rangle) = P|\phi\rangle$, meaning all $|\phi\rangle \in \mathcal{W}$ is stabilized by $\mathcal{S}_{\rm PF}$ or $\mathcal{W} \subseteq C_{\mathcal{S}_{\rm PF}}$, leading us to the conclusion that $\mathcal{W} = C_{\mathcal{S}_{\rm PF}}$. The dimension of the code space is then given as tr(P). Since $\mathcal{S}_{\rm PF}$ is an Abelian group and the trace condition $\mathrm{tr}(\gamma^{\boldsymbol\alpha}) = 0$ when $\gamma^{\boldsymbol\alpha} \neq \mathbb{1}$ and $\mathrm{tr}(\mathbb{1}) = D^n$ for $\gamma^{\boldsymbol\alpha},\mathbb{1} \in \mathrm{PF}(D,2n)$, holds, we arrive at the result

$$\mathrm{tr}(P) = |C_{\mathcal{S}_{\rm PF}}| = \frac{1}{|\mathcal{S}_{\rm PF}|}D^n. \tag{18}$$

Corollary. When D is a prime power p^l, $|C_{\mathcal{S}_{\rm PF}}| = p^{lk}$ and $|\mathcal{S}_{\rm PF}| = p^r$ with $r = l(n-k)$ (we refer the reader to [42] for a detailed derivation).

In later sections, we will also use a matrix form of the stabilizer code $\mathcal{S}_{\rm PF} = \langle S_1,\ldots,S_l\rangle = \langle \gamma^{\boldsymbol\alpha_1},\ldots,\gamma^{\boldsymbol\alpha_l}\rangle$, whose rows are given by $\boldsymbol\alpha_i$, that is,

$$S_{\rm PF} = \begin{pmatrix} \boldsymbol\alpha_1 \\ \vdots \\ \boldsymbol\alpha_l \end{pmatrix}. \tag{19}$$

The same construction is also extended for the logical operators, yielding the matrix $L_{\rm PF}$. Since $\mathcal{S}_{\rm PF}$ is an Abelian group, due to Eq. (9), we have $S_{\rm PF}\Lambda S_{\rm PF}^T = 0 \mod D$. The logical operator matrix $L_{\rm PF}$, on the other hand, obeys the relations $L_{\rm PF}\Lambda S_{\rm PF}^T = 0$ and $L_{\rm PF}\Lambda L_{\rm PF}^T \neq 0$ in $\mod D$.

IV. EXAMPLES OF PARAFERMION STABILIZER CODES

A. Three-state quantum clock model

We present a simple example of a parafermion code starting from a three-state quantum clock model Hamiltonian (for $h = 0$):

$$H_3 = -J\sum_{j=1}^{n-1}(Z_j^\dagger Z_{j+1} + Z_{j+1}^\dagger Z_j). \tag{20}$$

By employing the Jordan-Wigner transformation, this Hamiltonian can be rewritten in terms of parafermion operators in the following form:

$$H = iJ\sum_{j=1}^{n-1}(\gamma_{2j}^\dagger\gamma_{2j+1} - \gamma_{2j+1}^\dagger\gamma_{2j}), \tag{21}$$

which is known as the Fendley [8] generalization of the Kitaev chain model. For $D = 2$, Eq. (20) reduces to the familiar Ising model with $h = 0$.

We form the corresponding stabilizer group taking individual terms of the Hamiltonian for each value of j as

$$\langle i\gamma_2^\dagger\gamma_3, -i\gamma_3^\dagger\gamma_2, \ldots, i\gamma_{2n-2}^\dagger\gamma_{2n-1}, -i\gamma_{2n-1}^\dagger\gamma_{2n-2}\rangle. \tag{22}$$

Logical operators of the code can be chosen as $\bar{Z} = \gamma_1$ and $\bar{X} = \gamma_{2n}$. Then the distance of the code is $d = 1$. But these logical operators are not parity-preserving, we can combine them as $\gamma_1^\dagger\gamma_{2n}$ and $\gamma_1\gamma_{2n}^\dagger$ to obtain parity-preserving logical operators. Even though this code does not provide protection against parity-violating errors, in the absence of such errors the code protection can be described by the diameter of even logical operators, i.e., $l_{\rm con} = 2n$.

B. Minimal parafermion stabilizer codes

Quantum error-correcting schemes come at the expense of introducing additional qudits in order to protect information encoded into quantum states. The ratio of the number of encoded qudits k (whose state can be restored after decoherence) to the number of underlying physical qudits n is called encoding rate $r = k/n$. The relative distance is defined as $\delta = d/n$. Codes with higher encoding rate r and relative distance are preferable, and it is known that both δ and r can be finite for a particular code family [58]. In this section, we discuss the minimal stabilizer codes encoding the $k = 1$ qudit and try to find codes with the best encoding rate r for the minimal nontrivial distance $d = 3$ for prime D.

Using an exhaustive search, we find that for $D = 3$ the smallest nontrivial code requires eight parafermion modes and results in an $[[8,1,3]]_3$ parafermion stabilizer code:

$$\begin{aligned}\mathcal{S}_{\rm PF} &= \langle \gamma_1^\dagger\gamma_2\gamma_4^\dagger\gamma_6, \gamma_2^\dagger\gamma_3\gamma_5^\dagger\gamma_7, \gamma_3^\dagger\gamma_4\gamma_6^\dagger\gamma_8\rangle,\\ \mathcal{L}(\mathcal{S}_{\rm PF}) &= \langle \gamma_1^\dagger\gamma_2\gamma_3\gamma_7, \gamma_2^\dagger\gamma_3^\dagger\gamma_6\rangle.\end{aligned} \tag{23}$$

The logical operators generate $\mathcal{W}_3$, encoding eight parafermion modes into a single logical qutrit.

Realizations of $D = 6$ parafermion zero modes have been proposed recently [11], making this case particularly interesting. Because $D = 6$ is not a prime or prime power, the original construction for qudit stabilizer codes [42] is not directly applicable. We will instead "double" the $D = 3$ code given above by squaring all the generators. However, this is a mapping onto a larger space and we need to take care of the additional operators that commute with the new stabilizer generators. The full set of generators for $D = 6$ thus becomes

$$\begin{aligned}\mathcal{S}_{\rm PF} = \big\langle &\gamma_1^3\gamma_2^3, \gamma_3^3\gamma_4^3, \gamma_5^3\gamma_6^3, \gamma_7^3\gamma_8^3\\ &(\gamma_1^\dagger\gamma_2\gamma_4^\dagger\gamma_6)^2, (\gamma_2^\dagger\gamma_3\gamma_5^\dagger\gamma_7)^2, (\gamma_3^\dagger\gamma_4\gamma_6^\dagger\gamma_8)^2\big\rangle,\\ \mathcal{L}(\mathcal{S}_{\rm PF}) = \langle &(\gamma_1^\dagger\gamma_2\gamma_3\gamma_7)^2, (\gamma_2^\dagger\gamma_3^\dagger\gamma_6)^2\rangle.\end{aligned} \tag{24}$$

Since these logical operators behave like X^2 and Z^2 for $D = 6$ qudits, the code above essentially encodes a qutrit using $2n = 8$ parafermion zero modes. We also note that this code may not have the best encoding rate for $D = 6$.

However, the minimal number of modes depends on D. For the case of $D = 7$, there exists a $[[6,1,3]]_7$ code that requires only six modes,

$$\begin{aligned}\mathcal{S}_{\rm PF} &= \big\langle \gamma_1\gamma_2\gamma_5^5, \gamma_1\gamma_4^5\gamma_6\big\rangle,\\ \mathcal{L}(\mathcal{S}_{\rm PF}) &= \big\langle \gamma_1^3\gamma_2^6\gamma_6, \gamma_1^2\gamma_2^5\gamma_3\big\rangle.\end{aligned} \tag{25}$$

This indicates that there is a minimal D for which the encoding rate is optimal [59].

V. MAPPINGS BETWEEN QUDITS AND PARAFERMION MODES

A. Mappings to and from parafermion codes

There is an established literature on stabilizer codes for qudits when D is prime or a prime power [60,61]. Recently, some properties of qudit stabilizer codes for the nonprime case have been discussed in [50]. An isomorphism between multi-qudit and multi-parafermion mode operators will let us construct parafermion stabilizer codes based on qudit codes. In this section, we establish such an isomorphism by mapping four parafermion modes to a single qudit.

Remark. Let $\tilde{X}_j$ and $\tilde{Z}_j$ ($j = 1,\ldots,k$) denote the generating operators of $\mathcal{W}_D^{\otimes k}$ embedded into PF$(D,2n)$, encoding k qudits into $2n$ parafermion modes. Such an embedding has the following properties:

(i) Logical qudit operators $\{\tilde{X}_j,\tilde{Z}_j\}$ obey Eq. (1), that is, they generate the embedded Weyl group $\mathcal{W}_D^{\otimes k} \subseteq$ PF$(D,2n)$.

(ii) Logical qudit operators for different sites commute ($[\tilde{X}_i,\tilde{X}_j] = [\tilde{Z}_i,\tilde{Z}_j] = [\tilde{X}_i,\tilde{Z}_j] = 0$ when $i \neq j$).

(iii) The embedding of $\mathcal{W}_D^{\otimes k}$ into the larger group PF$(D,2n)$ may require additional parafermion operators $\{\tilde{Q}_j^{(i)}\}$ that commute with the original qudit stabilizer group $\mathcal{S}$ or its corresponding logical operators $\mathcal{L}(\mathcal{S}_{\rm PF})$. Such operators must be included in the parafermion stabilizer group $\mathcal{S}_{\rm PF}$ and hence must preserve parity [an example is given in Eq. (26) below].

In turns out that the minimum number of parafermion modes required for such an embedding is four, that is, four parafermion modes will map to a single qudit. This mapping leads to the following lemma.

Lemma. Every $[[n,k,d]]_D$ stabilizer code can be mapped onto a $[[4n,k,2d]]_D$ parafermion stabilizer code, encoding four parafermion modes into a single qudit.

Proof. Let us define the operators

$$\begin{aligned}\tilde{Z}_{j+1} &= \gamma_{1+4j}^{\dagger}\gamma_{2+4j}, \quad \tilde{X}_{j+1} = \gamma_{1+4j}^{\dagger}\gamma_{3+4j},\\ \tilde{Q}_{j+1} &= \gamma_{1+4j}^{\dagger}\gamma_{2+4j}\gamma_{3+4j}^{\dagger}\gamma_{4+4j}.\end{aligned} \tag{26}$$

It is straightforward to show that $\langle\tilde{X}_j,\tilde{Z}_j\rangle$ generate the embedded Weyl group $\mathcal{W}_D^{\otimes k} \subseteq$ PF$(D,2n)$ (that is, $\tilde{Z}_i\tilde{X}_j = \omega\tilde{X}_j\tilde{Z}_i\delta_{ij}$ and $\tilde{X}_j^D = \tilde{Z}_j^D = \mathbb{1}$) and are parity-preserving. We can treat $\mathcal{L}(\mathcal{S}_{\rm PF}) = \langle\tilde{X}_j,\tilde{Z}_j\rangle$ as the logical operators of a stabilizer group $\mathcal{S}_{\rm PF} = \langle\tilde{Q}_j\rangle$. This makes the purpose of the additional fourth mode (which does not appear in the logical operators) clear: without it, the stabilizer group would include a non-parity-preserving operator. Finally, since every Weyl operator is mapped to a parafermion operator with two modes, the distance of the new code is $2d$.

This mapping allows us to construct families of parafermion stabilizer codes from known families of qudit stabilizer codes. In particular, one can map the qudit toric codes [60] (and their generalizations [62,63]) to the corresponding parafermion code. The advantage of this mapping is that a local stabilizer generator in a d-dimensional lattice will map to a local parafermion operator. The disadvantage is that all logical operators preserve parity, thus there is no additional protection associated with the presence of parity-violating logical operators.

It turns out that we can do a similar mapping in the opposite direction, albeit without preserving the locality of stabilizer generators.

Lemma. Any parafermion stabilizer code with parameters $[[2n,k,d]]_D$ and stabilizer group $\mathcal{S}_{\rm PF}$ can generate a $[[2n,2k,d']]_D$ qudit CSS code.

Proof. Consider the check matrix

$$S_{\rm CSS} = \begin{pmatrix} S_{\rm PF}\Lambda & 0 \\ 0 & S_{\rm PF}\end{pmatrix}. \tag{27}$$

For a parafermion code, $k = n - \text{rank}(S_{\rm PF})$, whereas for the CSS code, $k' = 2n - 2\times\text{rank}(S_{\rm PF}) = 2k$ (Λ is a full-rank matrix). Hence $S_{\rm CSS}$ is the check matrix of a $[[2n,2k,d']]_D$ CSS code. The corresponding logical operator matrices $L_{\rm PF}$ and $L_{\rm PF}\Lambda$ behave like X- and Z-type logical qudit operators.

We note that this construction is a proper generalization of the doubling lemma described in [51], which maps a Majorana fermion code to a weakly self-dual CSS code. Unfortunately, for $D > 2$ this mapping becomes nonlocal, i.e., a local qudit operator will generally map to a nonlocal parafermion operator.

B. Parafermion toric code with parity-violating logical operators

In this section, we construct a parafermion analog of Kitaev's toric code [1] for qudits [60]. The toric code is a stabilizer code defined on an $a \times b$ lattice on the surface of a torus. A portion of the lattice is depicted in Fig. 1, where each dot represents a single qudit (hence, there are $2ab$ qudits overall).

Let $D = p^{2l}$, where p is a prime number and $l \in \mathbb{Z}^+$. The operators

$$\begin{aligned}\tilde{Z}_{j+1} &= \gamma_{1+4j}^{p^l-1}\gamma_{2+4j}, \quad \tilde{X}_{j+1} = \gamma_{1+4j}^{p^l-1}\gamma_{3+4j},\\ \tilde{Q}_{j+1} &= \gamma_{1+4j}^{\dagger}\gamma_{2+4j}^{\dagger}\gamma_{3+4j}\gamma_{4+4j}\end{aligned} \tag{28}$$

define a mapping of four parafermion modes onto a single qudit via the one-qudit stabilizer group $\mathcal{S}_{\rm PF} = \langle\tilde{Q}_j\rangle$ and its corresponding logical operators $\mathcal{L}(\mathcal{S}_{\rm PF}) = \langle\tilde{X}_j,\tilde{Z}_j\rangle$.

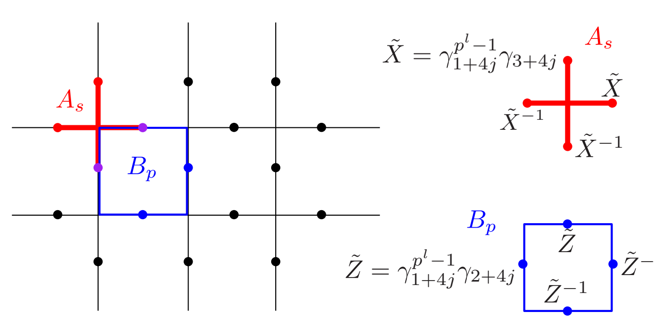

FIG. 1. (Color online) A portion of the lattice place on a torus, where each dot represents four parafermion modes (the index $j \geqslant 0$ uniquely denotes the lattice point). On the right, parafermion star and plaquette operators A_s and B_p are given in detail [p^l is a prime power; further details are given in Eq. (28)].

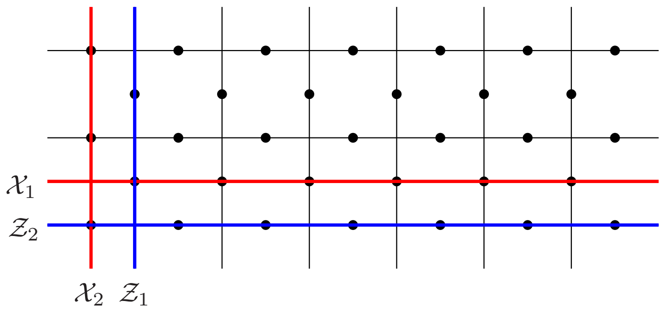

FIG. 2. (Color online) Loops corresponding to the logical operators of the toric code.

Consider the operators defined on a star-shaped and plaquette-shaped portion of the lattice:

$$A_s = \prod_{j\in \text{star}(s)} \tilde{X}_j^{a_j}, \quad B_p = \prod_{j\in \text{plaquette}(p)} \tilde{Z}_j^{b_j}, \tag{29}$$

where a_j and b_j are ± 1, specified on the right side of Fig. 1. In general, A_s and B_p either do not overlap or overlap at two sites. One can easily verify that the construction given in Eq. (29) ensures that the commutator $[A_s, B_p]$ vanishes in both cases. We also note that both A_s and B_p are parity-conserving operators. The set of all A_s and B_p forms a stabilizer group.

Due to the fact that the lattice is defined on the surface of a torus, the lattice is periodic in both dimensions, leading to the result

$$\prod_s A_s = \mathbb{1}, \quad \prod_p B_p = \mathbb{1}. \tag{30}$$

This implies $|\mathcal{S}| = 2(ab-1)$, and using Eq. (16), we find that $k = 2$. The logical operators $\mathcal{X}_l, \mathcal{Z}_l$ ($l = 1,2$) are horizontal and vertical loops along the lattice, as given in Fig. 2. Since these loops go all the way through the torus, they commute with the stabilizer generators A_s and B_p at all sites.

We note that the parity (charge) associated with operators is $p^l \neq 0 \mod D$ [64]. Hence, the parity of the horizontal (vertical) logical operators of the parafermion toric code is $a \times p^l$ ($b \times p^l$) mod D. By tuning a and b, we can ensure that one of the logical operators will violate parity (that is, p^l divides a but does not divide b). The choice of the smallest b would correspond to the absence of parity-violating errors. In general, b can be tuned depending on the probability of parity-violating errors. Therefore, this code construction combines topological protection of Kitaev's toric code with additional protection relying on suppression of parity-violating errors.

VI. CONCLUSION

We have introduced stabilizer codes in which parafermion zero modes represent the constructing blocks as opposed to qudit stabilizer codes. Our work generalizes earlier constructions based on Majorana zero modes [51]. While it is possible in general to start with a stabilizer code for qudits and use it with parafermion zero modes through the mapping given in Eq. (26), which utilizes the embedding $\mathcal{W}_D^{\otimes n} \subset \text{PF}(D,4n)$, we find that there are more efficient codes in PF$(D,2n)$ requiring fewer parafermion modes, as we have exemplified in Sec. IV B. These results also show that the parafermions can achieve a better encoding rate than Majorana fermions. We have also shown that by using a similar embedding with a qudit toric code, it is possible to construct a code protecting parafermion modes against parity-violating errors where the degree of protection (i.e., distance) can be adjusted. A similar construction has been introduced for color codes using Majorana zero modes [51].

Parafermion stabilizer codes can be used for constructing Hamiltonians in which commuting terms correspond to stabilizer generators. Parafermion stabilizer codes thus lead to a multitude of models generalizing Kitaev's 1D chain of unpaired Majorana zero modes to higher dimensions ($D > 2$) and to arbitrary interactions defined by the choice of stabilizer generators. An important question arising here is related to finite-temperature stability of topological order in such systems. In general, a 2D lattice with local interactions cannot lead to stable topological order at finite temperature. Thus, it could be plausible to assume that by requiring some of the logical operators to be parity-violating operators, one can add additional protection to topological order where this additional protection relies on superselection rules. Whether such constructions can lead to the absence of parity-conserving stringlike logical operators (e.g., stringlike logical operators are absent in Haah's code [65]) is an open problem.

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