Multifractional Poisson process, multistable subordinator and related limit theorems

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Abstract

We introduce a multistable subordinator, which generalizes the stable subordinator to the case of time-varying stability index. This enables us to define a multifractional Poisson process. We study properties of these processes and establish the convergence of a continuous-time random walk to the multifractional Poisson process.

Keywords: multistable process, multifractional process, stable subordinator, fractional Poisson process, continuous-time random walk
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1. Introduction

The multistable processes are generalizations of well-known stable processes. They are locally stable, but the stability index may evolve over time. Various definitions of such processes can be found in Falconer and Liu (2012); Falconer et al. (2009); Falconer and Lévy Véhel (2009); Le Guével and Lévy Véhel (2012); Le Guével et al. (2013).

The literature contains several definitions of the fractional Poisson process, see Beghin and Orsingher (2009); Laskin (2003); Mainardi et al. (2004); Repin and Saichev (2000); Uchaikin et al. (2008). Meerschaert et al. (2011) suggested to define a fractional Poisson process as $N(E(t))$, for a Poisson process $N(t)$ and $E(t)$ being the right-continuous inverse of the standard β-stable subordinator $D(t)$, independent of $N(t)$.

In this paper, we construct a multifractional Poisson process using the multistable subordinator instead of $D(t)$. This process has non-stationary increments, but it retains some important properties of the usual β-stable subordinator such as the stochastic continuity, the independence of the increments and the strictly

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increasing sample paths. We establish that the multistable subordinator is a weak limit of the sums $\sum_{k \leq nt} b_{nk}^{-1} J_{nk}$ where $\{J_{nk}, n \geq 1, k \geq 1\}$ are independent random variables with regularly varying tails and $\{b_{nk}, n \geq 1, k \geq 1\}$ are some normalizing constants. Moreover, we prove that the corresponding right-continuous inverse converges to the right-continuous inverse of the multistable subordinator. This makes possible to construct continuous-time random walks, which converge to the multifractional Poisson process. These results generalize (Meerschaert et al., 2011, Theorem 2.5) and (Meerschaert and Scheffler, 2004, Theorem 3.2) to the multistable case.

The paper is organized as follows. In Section 2 we introduce multistable subordinator, its inverse and multifractional Poisson process, and investigate their properties. In Section 3 we prove various limit theorems for these processes.

2. Multistable subordinator and multifractional Poisson process

2.1. Basic definitions

Let $\beta \colon \mathbb{R}_+ \to (0,1)$ be a continuous function, $\mathbb{R}_+ = [0, \infty)$. Denote

$$\beta^* = \sup_t \beta(t), \quad \beta_* = \inf_t \beta(t).$$

Consider a Poisson point process $\Pi = \{(t_i, x_i)\}$ on $\mathbb{R}_+ \times (0, \infty)$ with intensity measure

$$\nu(dt, dx) = \beta(t) x^{-\beta(t)-1} dt \, dx.$$

Define

$$D(t) = \sum_{\substack{t_i \leq t \\ (t_i, x_i) \in \Pi}} x_i, \quad t \geq 0.$$

Since

$$\int_{[0,t] \times (0,\infty)} \min(1, x) \nu(ds, dx) = \int_0^t \frac{ds}{1 - \beta(s)}, \tag{1}$$

the process $D(t)$ is well defined, if the function $(1 - \beta(s))^{-1}$ is integrable. In particular, this condition holds if $\beta^* < 1$. It follows from Fisz and Varadarajan (1962/1963) that the distribution of $D(t)$ is absolutely continuous for all $t > 0$.

The Laplace transform of $D(t)$ is equal to

$$\begin{aligned}
\mathsf{E} \exp\{-\theta D(t)\} &= \exp \left\{ \int_0^t \int_0^\infty \left(e^{-\theta x} - 1 \right) \beta(s) x^{-\beta(s)-1} dx \, ds \right\} \\
&= \exp \left\{ -\int_0^t \Gamma(1 - \beta(s)) \theta^{\beta(s)} ds \right\},
\end{aligned} \tag{2}$$

see, for instance, (Kingman, 1993, Section 3.2). If $\beta(t) = \beta$ is a constant,

$$\mathsf{E} \exp\{-\theta D(t)\} = \exp \left\{ -\Gamma(1 - \beta) t \theta^\beta \right\},$$

is the Laplace transform of the β-stable subordinator. We will call the process $D(t)$ the *multistable subordinator* with index $\beta(t)$, $t \geq 0$.

Remark 2.1. It is not hard to see that $D(t)$ can be represented as a sum over the stationary Poisson point process Π' on $\mathbb{R}_+ \times (0, \infty)$ (with the Lebesgue intensity measure), namely

$$D(t) \overset{d}{=} \sum_{\substack{t_i \leq t \\ (t_i, x_i) \in \Pi'}} x_i^{-1/\beta(t_i)}.$$

Remark 2.2. Le Guével et al. (2013) study the so-called *independent increments multistable Lévy motion*. It admits the following representation (Le Guével et al., 2013, Proposition 1):

$$L(t) = \sum_{\substack{t_i \leq t \\ (t_i, x_i) \in \Pi''}} C_{\beta(t_i)}^{1/\beta(t_i)} x_i^{<-1/\beta(t_i)>}, \quad t \in \mathbb{R}_+,$$

where Π'' is a stationary Poisson point process on $\mathbb{R}_+ \times \mathbb{R}$, $C_\beta = \Gamma(1-\beta) \cos \frac{\beta \pi}{2}$ and $x_i^{<-1/\beta(t_i)>} = \text{sign}(x_i)|x_i|^{-1/\beta(t_i)}$. We construct a multistable process using a similar approach, but we avoid using factor C_β and consider only positive x's in order to obtain a process with positive and strictly increasing sample paths. Note that the sample paths of $L(t)$ are piecewise-constant functions, see (Le Guével et al., 2013).

Let $E(r)$ be the right-continuous inverse of $D(t)$, i.e.

$$E(r) = \inf\{t \geq 0 : D(t) \geq r\}.$$

Furthermore, let $N(t)$ be a homogeneous Poisson process with intensity λ, independent of Π.

Definition 2.3. We call the process $X(t) = N(E(t))$, $t \geq 0$, *multifractional Poisson process* with index $\beta(t)$, $t \geq 0$.

2.2. Properties of the multistable subordinator

In this subsection we will prove some useful properties of the processes $D(t)$ and $E(t)$.

Proposition 2.4. *Assume that $\beta^* < 1$. Then the process D is continuous in probability. Moreover, for any $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that*

$$\mathsf{P}(D(t+h) - D(t) > \varepsilon) \leq C_\varepsilon h \tag{3}$$

for all $t, h \in \mathbb{R}_+$.

Proof. We have

$$\mathsf{P}(D(t+h) - D(t) > \varepsilon)$$

$$\leq \mathsf{P}\left(\sum_{\substack{t < t_i \leq t+h, x_i \leq 1 \\ (t_i, x_i) \in \Pi}} x_i > \frac{\varepsilon}{2}\right) + \mathsf{P}\left(\sum_{\substack{t < t_i \leq t+h, x_i > 1 \\ (t_i, x_i) \in \Pi}} x_i > \frac{\varepsilon}{2}\right). \tag{4}$$

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By Markov's inequality, we obtain

$$
\mathsf{P}\left(\sum_{\substack{t<t_i\leq t+h, x_i\leq 1 \\ (t_i,x_i)\in\Pi}} x_i > \frac{\varepsilon}{2}\right) \leq \frac{2}{\varepsilon}\mathsf{E}\sum_{\substack{t<t_i\leq t+h, x_i\leq 1 \\ (t_i,x_i)\in\Pi}} x_i
$$

$$
= \frac{2}{\varepsilon}\int_t^{t+h}\int_0^1 \beta(s)x^{-\beta(s)}\,dx\,ds = \frac{2}{\varepsilon}\int_t^{t+h}\frac{\beta(s)}{1-\beta(s)}\,ds \leq \frac{2\beta^*}{\varepsilon(1-\beta^*)}h.
$$

The inequality under the ultimate probability in (4) implies that the Poisson process Π has at least one point in $(t,t+h]\times(1,\infty)$. Therefore,

$$
\mathsf{P}\left(\sum_{\substack{t<t_i\leq t+h, x_i>1 \\ (t_i,x_i)\in\Pi}} x_i > \frac{\varepsilon}{2}\right)
$$

$$
\leq 1 - \exp\left\{-\int_t^{t+h}\int_1^\infty \beta(s)x^{-\beta(s)-1}ds\,dx\right\} = 1 - e^{-h} \leq h.
$$

Thus, (3) holds with $C_\varepsilon = 1 + \frac{2\beta^*}{\varepsilon(1-\beta^*)}$. $\qquad\square$

Proposition 2.5. *The process $D(t)$, $t\geq 0$ has independent increments.*

Proof. The result follows from the Poisson property of Π. $\qquad\square$

Proposition 2.6. *The sample paths of D are a. s. strictly increasing.*

Proof. For all $t\geq 0$ and $h>0$, the event $\{D(t+h)=D(t)\}$ means that the Poisson process Π has no points in $(t,t+h]\times(0,\infty)$. Therefore, $\mathsf{P}(D(t+h)=D(t))=0$. Then D is strictly increasing by considering rational t and h. $\qquad\square$

Corollary 2.7. *The sample paths of E are a. s. continuous and non-decreasing.*

Proof. The statement follows directly from the definition of E and Proposition 2.6. $\qquad\square$

3. Limit theorems

3.1. Convergence of point processes in the scheme of series

Let $\{J_{nk}, n\geq 1, k\geq 1\}$ be non-negative independent random variables such that
$$
\mathsf{P}(J_{nk}>t) = t^{-\beta(k/n)}L\left(t^{\beta(k/n)/\beta^*}\right) \tag{5}
$$

for some slowly varying function $L(t)$, i. e.

$$
L(\lambda t)/L(t) \to 1 \quad \text{as } t\to\infty \quad \text{for any } \lambda > 0. \tag{6}
$$

Define the sequences of normalizing constants $\{a_n, n \geq 1\}$ and $\{b_{nk}, n \geq 1, k \geq 1\}$ by

$$a_n^{-\beta^*} L(a_n) = \frac{1}{n}, \tag{7}$$

$$b_{nk} = a_n^{\beta^*/\beta(k/n)}. \tag{8}$$

Let δ_x be the delta-measure concentrated at x.

Theorem 3.1. *The point process $N_n = \sum_{k=1}^{\infty} \delta_{(kn^{-1}, b_{nk}^{-1} J_{nk})}$ weakly converges to the Poisson point process Π.*

Proof. The proof follows the scheme from (Resnick, 1987, Proposition 3.21). According to (Resnick, 1987, Proposition 3.19), it suffices to prove the convergence of Laplace functionals $\psi_{N_n}(f) \to \psi_{\Pi}(f)$ for arbitrary function f from the family $C_0^+(\mathbb{R}_+ \times (0, \infty))$ of continuous, non-negative functions on $\mathbb{R}_+ \times (0, \infty)$ with compact support. We have

$$\psi_{\Pi}(f) = \exp\left\{-\int_{\mathbb{R}_+ \times (0,\infty)} \left(1 - e^{-f(s,x)}\right) \nu(ds, dx)\right\},$$

see, for example, (Resnick, 1987, Proposition 3.6), and

$$\begin{aligned}
\psi_{N_n}(f) &= \mathsf{E} \exp\left\{-\sum_k f(kn^{-1}, b_{nk}^{-1} J_{nk})\right\} \\
&= \prod_k \left(1 - \int_0^\infty \left(1 - e^{-f(kn^{-1}, b_{nk}^{-1} y)}\right) \mathsf{P}(J_{nk} \in dy)\right) \tag{9} \\
&= \prod_k \left(1 - \int_0^\infty \left(1 - e^{-f(kn^{-1}, x)}\right) \mathsf{P}(b_{nk}^{-1} J_{nk} \in dx)\right).
\end{aligned}$$

Consider the measure $\mu_{nk}(\cdot) = \mathsf{P}(b_{nk}^{-1} J_{nk} \in \cdot)$. We claim that

$$\lim_{n\to\infty} \sup_k \left| n\mu_{nk}([c_1, c_2]) - \left(c_1^{-\beta(k/n)} - c_2^{-\beta(k/n)}\right)\right| = 0 \tag{10}$$

for any $[c_1, c_2] \subset (0, \infty)$. Indeed, using the relations (5), (7) and (8), we can write

$$\begin{aligned}
n\mu_{nk}([c_1, c_2]) &= n(\mathsf{P}(J_{nk} > b_{nk}c_1) - \mathsf{P}(J_{nk} > b_{nk}c_2)) \\
&= \frac{(b_{nk}c_1)^{-\beta(k/n)} L\left((b_{nk}c_1)^{\frac{\beta(k/n)}{\beta^*}}\right) - (b_{nk}c_2)^{-\beta(k/n)} L\left((b_{nk}c_2)^{\frac{\beta(k/n)}{\beta^*}}\right)}{a_n^{-\beta^*} L(a_n)} \\
&= c_1^{-\beta(k/n)} \frac{L\left(a_n c_1^{\frac{\beta(k/n)}{\beta^*}}\right)}{L(a_n)} - c_2^{-\beta(k/n)} \frac{L\left(a_n c_2^{\frac{\beta(k/n)}{\beta^*}}\right)}{L(a_n)}.
\end{aligned}$$
$$\tag{11}$$

It is known (see Seneta, 1976, Theorem 1.1) that for every fixed $[d_1, d_2] \subset (0, \infty)$ the convergence (6) holds uniformly with respect to $\lambda \in [d_1, d_2]$. Therefore, (10) follows from (11), since $a_n \to \infty$ as $n \to \infty$.

Define ν_n by

$$\nu_n(ds, dx) = \sum_k \delta_{kn^{-1}}(ds)\mu_{nk}(dx).$$

Then

$$\nu_n((u, v] \times (c_1, c_2])$$
$$= \frac{1}{n} \sum_k \delta_{kn^{-1}}((u, v]) \left(n\mu_{nk}((c_1, c_2]) - \left(c_1^{-\beta(kn^{-1})} - c_2^{-\beta(kn^{-1})} \right) \right)$$
$$+ \int_{c_1}^{c_2} \left(\frac{1}{n} \sum_k \delta_{kn^{-1}}((u, v]) \beta(kn^{-1}) x^{-\beta(kn^{-1})-1} \right) dx$$

The first term in the right-hand side converges to 0 as $n \to \infty$ by (10). The second one converges to $\nu((u, v] \times (c_1, c_2])$, since the integrand is the Riemann sum for $\int_u^v \beta(s) x^{-\beta(s)-1} ds$. Hence, ν_n converges weakly to ν.

Therefore,

$$\sum_k \int_0^\infty \left(1 - e^{-f(kn^{-1}, x)} \right) \mu_{nk}(dx) = \int_{\mathbb{R}_+ \times (0, \infty)} \left(1 - e^{-f} \right) d\nu_n$$
$$\to \int_{\mathbb{R}_+ \times (0, \infty)} \left(1 - e^{-f(s, x)} \right) \nu(ds, dx) = -\log \psi_\Pi(f) \quad (12)$$

as $n \to \infty$.

If $K \subset \mathbb{R}_+ \times (0, \infty)$ is the compact support of f and $[c_1, c_2] \subset (0, \infty)$ is its projection on the second coordinate, then

$$\sup_k \int_0^\infty \left(1 - e^{-f(kn^{-1}, x)} \right) \mu_{nk}(dx) \leq \sup_k \mu_{nk}([c_1, c_2]) \to 0 \qquad \text{as } n \to \infty$$
$$(13)$$

by (10). Equation (9) implies

$$-\log \psi_{N_n}(f) = -\sum_k \log \left(1 - \int_0^\infty \left(1 - e^{-f(kn^{-1}, x)} \right) \mu_{nk}(dx) \right).$$

The elementary expansion

$$\log(1 + z) = z(1 + \varepsilon(z))$$

for $|\varepsilon(z)| \le |z|$ if $|z| \le 1/2$ yields

$$\left| -\log \psi_{N_n}(f) - \sum_k \int_0^\infty \left(1 - e^{-f(kn^{-1},x)}\right) \mu_{nk}(dx) \right|$$

$$\le \sum_k \left(\int_0^\infty \left(1 - e^{-f(kn^{-1},x)}\right) \mu_{nk}(dx) \right)^2$$

$$\le \left(\sup_k \int_0^\infty \left(1 - e^{-f(kn^{-1},x)}\right) \mu_{nk}(dx) \right)$$

$$\times \sum_k \int_0^\infty \left(1 - e^{-f(kn^{-1},x)}\right) \mu_{nk}(dx) \to 0$$

as $n \to \infty$ by (12) and (13). Together with (12) this implies that

$$-\log \psi_{N_n}(f) \to -\log \psi_\Pi(f) \qquad \text{as } n \to \infty.$$

Hence, $\psi_{N_n}(f) \to \psi_\Pi(f)$ as $n \to \infty$. $\qquad\qquad\qquad\qquad\qquad\square$

3.2. Convergence of sums to the multistable subordinator

Theorem 3.2. *Assume that $\beta_* > 0$ and $\beta^* < 1$. Then $\sum_{k \le nt} b_{nk}^{-1} J_{nk}$ weakly converges to $D(t)$ as $n \to \infty$.*

Proof. By the Skorohod theorem, it is possible to define N_n and Π on the same probability space so that $N_n \to \Pi$ a.s as $n \to \infty$. For all $r \in (0,1)$ we have

$$\left| \sum_{k \le nt} b_{nk}^{-1} J_{nk} - D(t) \right| \le \left| \sum_{\substack{k \le nt \\ b_{nk}^{-1} J_{nk} \le r}} b_{nk}^{-1} J_{nk} \right|$$

$$+ \left| \sum_{\substack{k \le nt \\ b_{nk}^{-1} J_{nk} \ge r}} b_{nk}^{-1} J_{nk} - \sum_{\substack{t_i \le t, x_i \ge r \\ (t_i,x_i) \in \Pi}} x_i \right| + \left| \sum_{\substack{t_i \le t, x_i \le r \\ (t_i,x_i) \in \Pi}} x_i \right|$$

$$= \eta_n(r) + \zeta_n(r) + \gamma(r).$$

Note that the processes N_n and Π have a.s. at most finite number of points in $[0,t] \times [r,\infty)$ for every $t > 0$ and $r > 0$. Therefore the a.s. convergence $N_n \to \Pi$ implies that the second summand $\zeta_n(r) \to 0$ a.s. as $n \to \infty$ for all $r > 0$.

The third summand $\gamma(r) = \int_{[0,t] \times (0,r]} x \, d\Pi \to 0$ a.s. as $r \downarrow 0$, since the integral $\int_{[0,t] \times (0,1]} x \, d\Pi$ is a.s. finite. Indeed, $\int_{[0,t] \times (0,1]} x \, \nu(ds, dx) < \infty$, see (1).

Consider the first summand. If $r < 1$, then

$$\eta_n(r) = \sum_{\substack{k \leq nt \\ b_{nk}^{-1} J_{nk} \leq r}} b_{nk}^{-1} J_{nk} \leq \sum_{\substack{k \leq nt \\ b_{nk}^{-1} J_{nk} \leq r}} \left(b_{nk}^{-1} J_{nk} \right)^{\beta(kn^{-1})/\beta^*}$$

$$= a_n^{-1} \sum_{\substack{k \leq nt \\ b_{nk}^{-1} J_{nk} \leq r}} J_{nk}^{\beta(kn^{-1})/\beta^*},$$

where the last equality follows from (8). Similarly, we can rewrite the inequality $b_{nk}^{-1} J_{nk} \leq r$ in the following form

$$a_n^{-1} J_{nk}^{\beta(kn^{-1})/\beta^*} \leq r^{\beta(kn^{-1})/\beta^*}.$$

Then

$$\eta_n(r) \leq \sum_{\substack{k \leq nt \\ a_n^{-1} J_{nk}^{\beta(kn^{-1})/\beta^*} \leq r^{\beta*/\beta^*}}} a_n^{-1} J_{nk}^{\beta(kn^{-1})/\beta^*} \tag{14}$$

for any $r \in (0, 1)$.

It follows from (5) that

$$\mathsf{P}\left(J_{nk}^{\beta(kn^{-1})/\beta^*} > t \right) = t^{-\beta^*} L(t).$$

This means that $\left\{ J_{nk}^{\beta(kn^{-1})/\beta^*}, n \geq 1, k \geq 1 \right\}$ is a sequence of i. i. d. random variables with regularly varying tails. Binomial point processes generated by i. i. d. sequence of random variables with such tails were considered in Davydov et al. (2008) and the following property was established: if $\xi_1, \xi_2, \ldots$ are i.i.d. random variables with $\mathsf{P}(\xi_1 > t) = t^{-\beta^*} L(t)$ and $\{a_n, n \geq 1\}$ is a corresponding sequence of normalizing constants, then the process $\beta_n = \sum_{k=1}^n \delta_{a_n^{-1} \xi_k}$ satisfies the condition

$$\limsup_n \mathsf{P}\left(\left| \int_{[0,r]} x \beta_n(dx) \right| \geq \varepsilon \right) \to 0 \quad \text{as } r \downarrow 0$$

for each $\varepsilon > 0$ (see the proof of Theorem 4.6 in Davydov et al., 2008). Using this fact, we deduce from (14) that

$$\limsup_n \mathsf{P}(\eta_n(r) \geq \varepsilon) \to 0 \quad \text{as } r \downarrow 0.$$

Thus,

$$\mathsf{P}\left(\left| \sum_{k \leq nt} b_{nk}^{-1} J_{nk} - D(t) \right| \geq \varepsilon \right) \leq \mathsf{P}(\eta_n(r) + \zeta_n(r) + \gamma(r) \geq \varepsilon)$$

$$\leq \mathsf{P}(\eta_n(r) \geq \varepsilon/2) + \mathsf{P}(\zeta_n(r) + \gamma(r) \geq \varepsilon/2).$$

This probability can be made arbitrary small by the choice of n and r. $\qquad \square$

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Corollary 3.3. *Assume that $\beta_* > 0$ and $\beta^* < 1$. Then the finite-dimensional distributions of $E_n(r) = \inf\left\{t : \sum_{k \leq nt} b_{nk}^{-1} J_{nk} \geq r\right\}$ converge to those of $E(r)$ as $n \to \infty$.*

Proof. For arbitrary m, $r_1, \ldots, r_m$ and $u_1, \ldots, u_m$

$$
\mathsf{P}\left(\inf\left\{t : \sum_{k \leq nt} b_{nk}^{-1} J_{nk} \geq r_j\right\} \leq u_j, j = 1, \ldots, m\right)
$$

$$
= \mathsf{P}\left(\sum_{k \leq nu_j} b_{nk}^{-1} J_{nk} \geq r_j, j = 1, \ldots, m\right) \to \mathsf{P}\left(D(u_j) \geq r_j, j = 1, \ldots, m\right)
$$

$$
= \mathsf{P}\left(\inf\{t : D(t) \geq r_j\} \leq u_j, j = 1, \ldots, m\right)
$$

$$
= \mathsf{P}\left(E(r_j) \leq u_j, j = 1, \ldots, m\right)
$$

as $n \to \infty$ by Theorem 3.2. $\square$

Corollary 3.4. *Assume that $\beta_* > 0$ and $\beta^* < 1$. Then E_n weakly converges to E in $D(\mathbb{R}_+, \mathbb{R}_+)$ as $n \to \infty$ in the J_1 topology.*

Proof. Since E_n has non-decreasing sample paths and E is continuous in probability by Corollary 2.7, the convergence under J_1 follows from Corollary 3.3 and (Bingham, 1971, Theorem 3). $\square$

3.3. Convergence of continuous-time random walks to the multifractional Poisson process

Let $Y_i^{(p)}$, $i \geq 1$, be i. i. d. random variables, independent of J_{nk}, $n \geq 1$, $k \geq 1$, such that
$$
\mathsf{P}\left(Y_i^{(p)} = 1\right) = p, \quad \mathsf{P}\left(Y_i^{(p)} = 0\right) = 1 - p.
$$
Denote $S^{(p)}(t) = \sum_{i \leq t} Y_i^{(p)}$.

Theorem 3.5. *Assume that $\beta_* > 0$ and $\beta^* < 1$. If $p_n \downarrow 0$ as $n \to \infty$, then the continuous-time random walk $S^{(p_n)}(\lambda E_n(t)/p_n)$ converges to the multifractional Poisson process $N(E(t))$ as $n \to \infty$ in the M_1 topology on $D(\mathbb{R}_+, \mathbb{R})$.*

Proof. It was established in the proof of (Meerschaert et al., 2011, Theorem 2.5) that $S^{(p)}(\lambda t/p)$ weakly converges to $N(t)$ as $p \to 0$ in the J_1 topology. By Corollary 3.4 and (Billingsley, 1968, Theorem 3.2), $\left(S^{(p_n)}(\lambda t/p_n), E_n(t)\right)$ weakly converges to $(N(t), E(t))$ as $n \to \infty$ in the J_1 topology of the product space $D(\mathbb{R}_+, \mathbb{R} \times \mathbb{R})$. In order to prove the convergence of the compositions in the M_1 topology, we will apply (Whitt, 2002, Theorem 13.2.4). Taking into account the continuity of E (Corollary 2.7), it is sufficient to check that $t = E(r)$ is (almost surely) strictly increasing at r whenever $N(t-) \neq N(t)$. It is easy to see that this condition holds if and only if the processes D and N have (almost surely) no simultaneous jumps. Thus, we need to verify that the probability that both $D(t) - D(t-)$ and $N(t) - N(t-)$ exceed ε for some $t \geq 0$ is equal to zero for

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arbitrary $\varepsilon > 0$. Obviously, it is sufficient to prove this for a finite interval. For the sake of simplicity, consider the interval $[0, 1]$. Then

$$\mathsf{P}(D(t) - D(t-) > \varepsilon, N(t) - N(t-) > \varepsilon \text{ for some } t \in [0, 1])$$

$$\leq \mathsf{P}\left(\bigcap_{j=1}^{\infty} A_{2^j}\right) = \lim_{j \to \infty} \mathsf{P}(A_{2^j}),$$

where $A_j = \bigcup_{k=0}^{j-1} \left\{ D\left(\frac{k+1}{j}\right) - D\left(\frac{k}{j}\right) > \varepsilon, N\left(\frac{k+1}{j}\right) - N\left(\frac{k}{j}\right) > \varepsilon \right\}$. Using the independence of processes, the estimate (3) for D and the stationarity of the increments for N, we obtain

$$\mathsf{P}(A_j) \leq \sum_{k=0}^{j-1} \mathsf{P}\left(D\left(\frac{k+1}{j}\right) - D\left(\frac{k}{j}\right) > \varepsilon\right) \mathsf{P}\left(N\left(\frac{k+1}{j}\right) - N\left(\frac{k}{j}\right) > \varepsilon\right)$$

$$\leq \frac{C_\varepsilon}{j} \sum_{k=0}^{j-1} \mathsf{P}\left(N\left(\frac{k+1}{j}\right) - N\left(\frac{k}{j}\right) > \varepsilon\right) = C_\varepsilon \mathsf{P}(N(1/j) > \varepsilon).$$

Since the Poisson process N is continuous in probability, we have $\mathsf{P}(A_j) \to 0$ as $j \to \infty$. This completes the proof. $\qquad\square$

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