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Loop formulation of the supersymmetric nonlinear O(N) sigma model

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We derive the fermion loop formulation for the supersymmetric nonlinear O(N) sigma model by performing a hopping expansion using Wilson fermions. In this formulation the fermionic contribution to the partition function becomes a sum over all possible closed non-oriented fermion loop configurations. The interaction between the bosonic and fermionic degrees of freedom is encoded in the constraints arising from the supersymmetry and induces flavour changing fermion loops. For $N \geq 3$ this leads to fermion loops which are no longer self-avoiding and hence to a potential sign problem. Since we use Wilson fermions the bare mass needs to be tuned to the chiral point. For $N = 2$ we determine the critical point and present boson and fermion masses in the critical regime.

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1. Motivation

Despite the fact that nature has so far not revealed any trace of supersymmetry in its elementary particle spectrum, supersymmetric quantum field theories remain to be fascinating objects to study *per se*. It is for example most interesting to examine various discretisation schemes for supersymmetric field theories regularised on the lattice, e.g. using twisted supersymmetry or orbifolding techniques, in order to understand how the supersymmetry is realised in the continuum limit. Moreover, recent developments in simulating supersymmetric field theories in low dimensions efficiently and without critical slowing down [1, 2] brings the non-perturbative study of these theories to a new, unprecedented level of accuracy, allowing for example precise investigations of spontaneous supersymmetry breaking phase transitions [3, 4]. In these proceedings we report on our ongoing study to apply such a programme to the supersymmetric nonlinear O(N) sigma model regularised on the lattice. This model has already been investigated numerically using a variety of different discretisations [5, 6, 7]. Here we concentrate on reformulating the model in terms of fermion loops in order to make use of the efficient simulation algorithms and related methods to control the fermion sign problem accompanying any spontaneous supersymmetry breaking [8].

2. Continuum model

The Lagrangian density of the supersymmetric nonlinear O(N) sigma model in two-dimensional Euclidean spacetime, originally derived in [9, 10], can be written as

$$\mathscr{L} = \frac{1}{2g^2}\left(\partial_\mu\phi\partial^\mu\phi + i\overline{\psi}\partial\!\!\!/\psi + \frac{1}{4}(\overline{\psi}\psi)^2\right) \tag{2.1}$$

where ϕ is a N-tuple of real scalar fields, ψ a N-tuple of real Majorana fields with $\overline{\psi} = \psi^T\mathscr{C}$ and $\mathscr{C}$ the charge conjugation matrix. For the action to be O(N)-invariant and supersymmetric the fields must fulfill the constraints

$$\phi^2 = 1 \qquad \text{and} \qquad \phi\psi = 0\,. \tag{2.2}$$

The model described in eq.(2.1) and the constraints in eq.(2.2) are both invariant under the $\mathscr{N}=1$ supersymmetry transformations

$$\delta\phi = i\bar{\varepsilon}\psi \quad \text{and} \quad \delta\psi = (\partial\!\!\!/ + \frac{i}{2}\overline{\psi}\psi)\phi\varepsilon$$

where ε is a constant Majorana spinor. There is an additional $\mathbb{Z}_2$ chiral symmetry realised by $\psi \to i\gamma_5\psi$ with $\gamma_5 = i\gamma_0\gamma_1$. As shown in [11] the one-loop β-function coincides with the one calculated for the model without SUSY, so it is asymptotically free for $N \geq 3$. As pointed out in [12] there exists an $\mathscr{N}=2$ extension for supersymmetric nonlinear sigma models which have a Kähler target manifold. This is the case for $N=3$ and hence an additional SUSY can be worked out [7]. The constraints in eq.(2.2) are implemented in the partition function by inserting adequate Dirac delta functions in the integration measure,

$$\begin{aligned} Z &= \int \mathscr{D}\phi\, \delta(\phi^2-1) \int \mathscr{D}\psi\, \delta(\phi\psi) e^{-S(\phi,\psi)} \\ &= \int \mathscr{D}\phi\, \delta(\phi^2-1) e^{-S_B(\phi)} \int \mathscr{D}\psi \left(\sum_{i\leq j}^{N} \phi_i\phi_j\overline{\psi}_i\psi_j\right) e^{-S_F(\psi)}\,, \end{aligned}$$

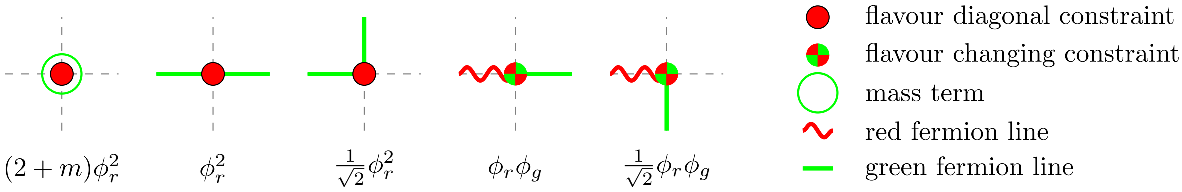

Figure 1: All possible vertices for $N = 2$ up to rotations and exchange of the two flavours denoted as r (red) and g (green) with the corresponding weights.

where the fermionic measure is rewritten in the second line using the Grassmann integration rules for Dirac delta functions. In this form it is explicit that the fermionic constraint $\phi\psi = 0$ induces flavour-diagonal ($i = j$) and flavour-changing ($i \neq j$) interactions between the bosonic and fermionic degrees of freedom.

3. Loop formulation and sign problem

When regularising the model on a discrete space-time lattice we follow the strategy in [7] and use the Wilson derivative for both the fermionic and the bosonic action. This strategy is well supported by various theoretical and numerical arguments [3, 4, 8, 13, 14, 15, 16, 17, 18]. We can then perform a hopping expansion to all orders for the fermionic and the bosonic variables in order to obtain an exact reformulation of the partition function in terms of non-oriented, closed fermion loops and constrained bosonic bond configurations (boson loops) [8]. These loops can in principle be simulated effectively by enlarging the configuration space by an open fermionic string [1, 2] and bosonic worms [19], respectively. The fermion loop formulation in particular solves the fermion sign problem since it naturally decomposes the contributions to the partition function into bosonic and fermionic ones, each with a definite sign depending on the employed boundary conditions [8]. The control of these signs is particularly important in the context of spontaneous supersymmetry breaking when the partition function for periodic boundary conditions, i.e. the Witten index, vanishes due to the exact cancellation of the bosonic and fermionic contributions.

Unfortunately, the model as discretised above nevertheless suffers from a sign problem. Firstly, the fermionic constraint $\phi\psi = 0$ requires on each lattice site exactly one term of the form $\phi_i\phi_j\overline{\psi}_i\psi_j$ inducing the interactions between fermions and bosons. Since the flavour-nondiagonal constraint with $i \neq j$ changes the flavour content within a fermion loop, a loop can be self-intersecting for $N \geq 3$ and hence looses the crucial property of having a definite sign depending only on its winding topology. Secondly, the Wilson term introduced in the bosonic sector generates a next-to-nearest neighbour diagonal hopping term with the wrong sign leading to an overall sign which fluctuates under local changes of the bosonic bond configuration. In order to avoid these complications we restrict ourselves in the following to $N = 2$ for which the fermion loops remain self-avoiding and in addition treat the bosonic degrees of freedom in the standard way, i.e., without employing a hopping expansion. The partition function for a system with V lattice sites can then be written as

$$Z = \left(\frac{1}{g^2}\right)^V \int \mathscr{D}\phi\, \delta(\phi^2 - 1)\, e^{-S_B(\phi)} \sum_{\{l\}} w(l, \phi)$$

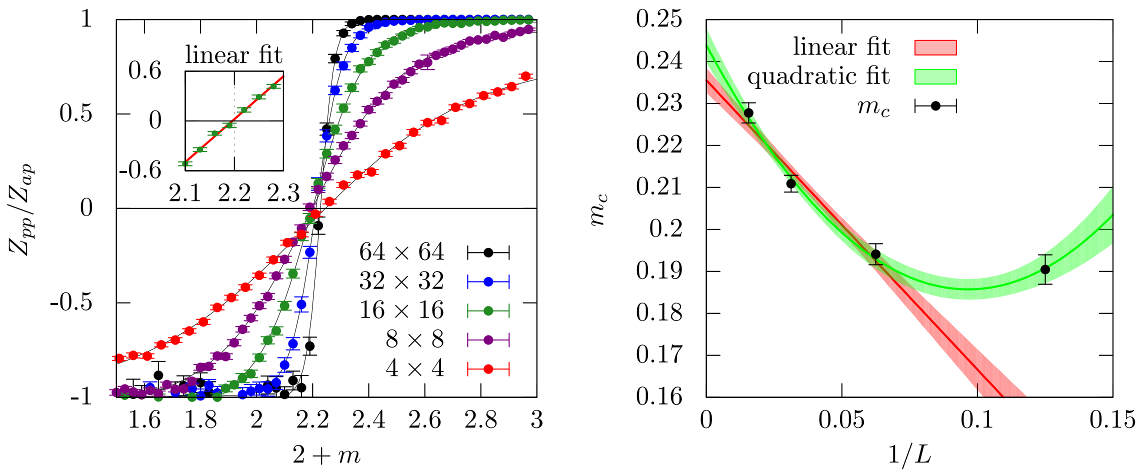

Figure 2: *Left plot*: Z_{pp}/Z_{ap} as a function of the bare mass m at $g = 0.5$ for various lattice extents. The small inset shows a linear fit used to obtain the critical mass defined by $Z_{pp}(m_c) = 0$. *Right plot*: Critical masses $m_c(L)$ at $g = 0.5$ for various lattice extents together with an extrapolation to the thermodynamic limit.

where the weight $w(l,\phi)$ for a given loop configuration l is determined solely by the geometry of the loops and, through the constraints in eq.(2.2), depends on the bosonic field ϕ. As an interesting side remark we note that the coupling g has no influence on the fermion loops except indirectly through the bosonic fields. In Figure 1 we show all allowed vertices up to rotations and exchange of the two flavours denoted as r (red) and g (green). The additional factors are $1/\sqrt{2}$ from the Dirac algebra structure for each corner in the loop [20] and $(2+m)$ from the fermionic monomer term (with m being the bare fermion mass). Note that the weights corresponding to the flavour changing constraints are still not guaranteed to be positive definite since the bosonic field components appear linearly in those. For certain values of the parameters m and g this leads to a sign fluctuating under local changes of the bosonic field and this is why we consider the phase quenched model with $w(l,\phi) \to |w(l,\phi)|$ in the following. The correct model can then be recovered by reweighting with the sign $\sigma(w)$ and this seems to work well in some of the interesting parameter regimes.

4. Results for $N = 2$

The supersymmetric nonlinear O(N) sigma model is defined in the chiral limit where both the fermions and bosons are massless. However, since the Wilson term explicitly breaks chiral symmetry even at zero bare fermion mass, the chiral limit of the regularised theory is not defined simply by the vanishing of that mass. Instead, it needs to be tuned to the critical value m_c where the correlation length of the fermion diverges and the fermion develops a zero mode yielding $Z_{pp}(m_c) = 0$ for the partition function with periodic boundary conditions in both directions [20, 21].

The behaviour of Z_{pp}/Z_{ap} when varying the bare mass is illustrated in the left panel of fig. 2 for various lattice extents L at $g = 0.5$. The inset illustrates for the 16×16 lattice how the values of $m_c(L)$ are extracted by performing linear fits to Z_{pp}/Z_{ap} in the region close to zero. It is evident that this definition of the critical mass allows rather accurate determinations of m_c and hence leads to precise extrapolations to the thermodynamic limit. In the right panel of fig. 2 we show such an

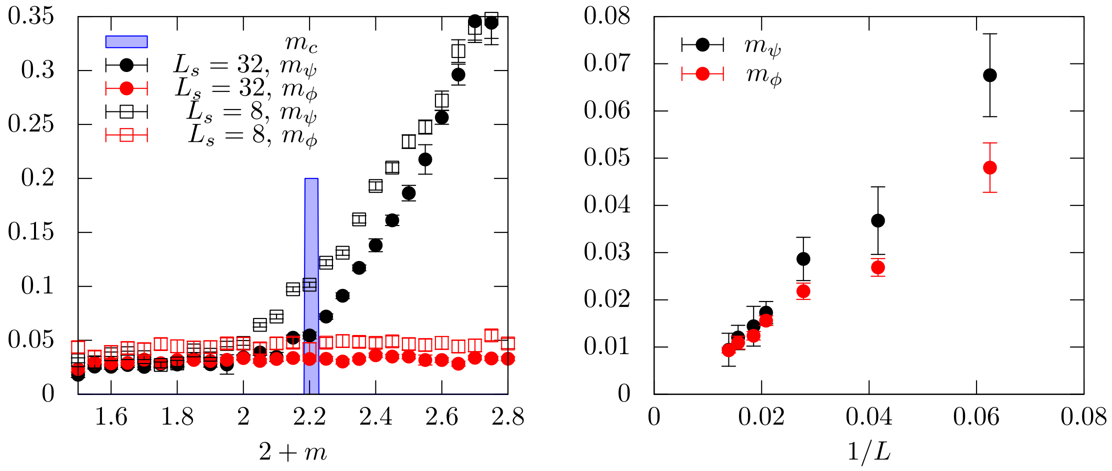

Figure 3: *Left plot*: Boson and fermion masses at coupling $g = 0.5$ as a function of the bare mass for two different lattice volumes. *Right plot*: Boson and fermion masses for $g = 0.5$ at the critical mass m_c versus the inverse lattice extent.

extrapolation at fixed coupling $g = 0.5$. Eventually, this procedure can be repeated for a range of couplings in order to obtain $m_c(g)$ in the thermodynamic limit. We note however that the behaviour of the system changes significantly for $g \gtrsim 0.8$ where we expect the occurrence of a Kosterlitz-Thouless phase transition.

Next we investigate the boson and fermion mass spectrum of the theory. The fermion correlator can be measured with high accuracy using the open fermionic string update introduced in [1]. In the left panel of fig. 3 we show the observed masses at fixed coupling $g = 0.5$ for the two volumes with spatial extent $L_s = 8$ and 32 and temporal extent $L_t = 32$. The boson mass m_ϕ depends noticeably on the volume, but is essentially independent of the bare fermion mass and remains constant across the phase transition. The fermion mass m_ψ on the other hand drops when the bare mass is decreased towards its critical value and becomes degenerate with the boson mass in the regime close to the critical point m_c (denoted by the blue bar). The mass degeneracy is further examined in the right panel of fig. 3 where the two masses m_ϕ and m_ψ at the critical mass m_c for $g = 0.5$ are plotted versus the inverse lattice extent. The data indicate that the mass degeneracy between the boson and fermion mass at the critical point survives the continuum limit where the model becomes chirally invariant.

We emphasise again that the above calculations are made in the phase quenched model. The only sources of negative contributions are the flavour changing constraints $\phi_r\phi_g\overline{\psi}_r\psi_g$. We are investigating their occurrence in the left panel of fig. 4 where the flavour changing constraint density is plotted versus the bare mass for various couplings g on a lattice of size 8×8. For large g the number of flavour changing interactions grows for decreasing $m < m_c$ while for smaller couplings the number remains rather small. On the other hand, for large values of $m > m_c$ the density of flavour changing interactions vanishes for any coupling. In this region there are predominantly flavour diagonal constraints contributing positively with ϕ_r^2 or ϕ_g^2. As a consequence the average sign $\langle\sigma\rangle$ plotted in the right panel of fig. 4 appears to behave reasonably well and this suggests that approaching the chiral limit from $m > m_c$ is under good control.

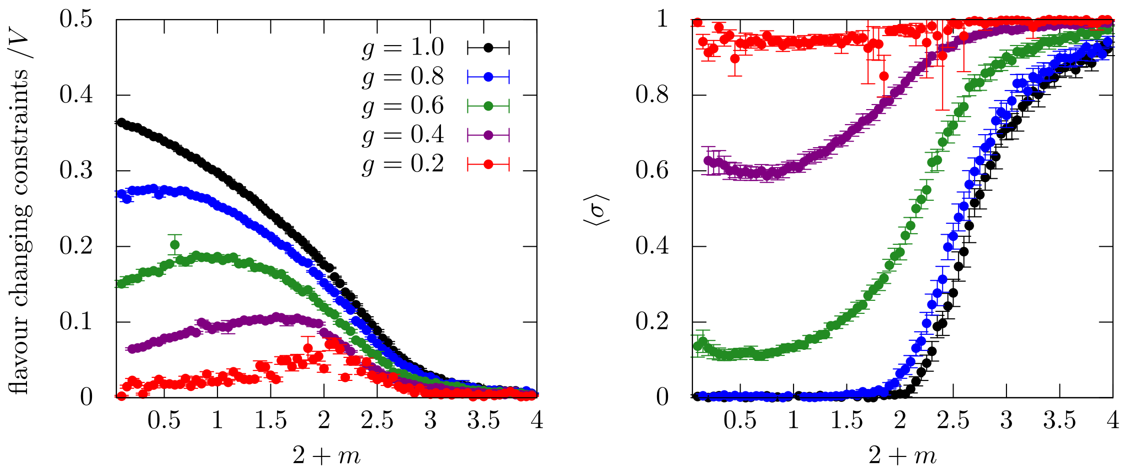

Figure 4: *Left plot*: Density of flavour changing constraints as a function of the bare mass for a range of couplings. *Right plot*: Same but for the average sign $\langle\sigma\rangle$.

5. Summary and outlook

We constructed a fermion loop formulation of the supersymmetric nonlinear O(N) sigma model on an Euclidean space-time lattice using the Wilson derivative for both the bosonic and fermionic fields. This formulation maintains both the O(N) symmetry and the constraints arising from the supersymmetry and in principle allows a straightforward simulation of the model. For $N = 2$ we determined the critical mass, defining the massless model in the continuum limit, for a range of couplings and showed how the fermion and boson masses become degenerate at this point. A careful analysis of Ward identities is necessary in order to clarify whether or not the supersymmetry is fully restored in the continuum limit, but in principle it is now possible to determine the full phase diagram of the model, i.e., the range of parameters for which the supersymmetry is spontaneously broken, if at all.

One obstacle towards this goal is the fluctuating sign stemming from the bosonic degrees of freedom entering in the constraints $\phi\psi = 0$. However, it is not clear whether this sign problem survives the continuum limit since it is a pure artefact of the particular regularisation chosen. In fact, the sign problem can be completely avoided by introducing the Wilson term only in the fermionic sector and performing the hopping expansion also for the bosonic degrees of freedom as outlined in section 3, but of course it remains to be seen whether supersymmetry is restored in the continuum also for such a regularisation.

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