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Invariance properties of random vectors and stochastic processes based on the zonoid concept

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Two integrable random vectors ξ and ξ^* in $\mathbb{R}^d$ are said to be zonoid equivalent if, for each $u \in \mathbb{R}^d$, the scalar products $\langle \xi, u\rangle$ and $\langle \xi^*, u\rangle$ have the same first absolute moments. The paper analyses stochastic processes whose finite-dimensional distributions are zonoid equivalent with respect to time shift (zonoid stationarity) and permutation of its components (swap invariance). While the first concept is weaker than the stationarity, the second one is a weakening of the exchangeability property. It is shown that nonetheless the ergodic theorem holds for swap-invariant sequences and the limits are characterised.

Keywords: ergodic theorem; exchangeability; isometry; swap-invariance; zonoid

1. Introduction

The first absolute moments $\mathbf{E}|\langle \xi, u\rangle|$, $u \in \mathbb{R}^d$, for the scalar product of an integrable random vector ξ in $\mathbb{R}^d$ and u, admit a straightforward geometric interpretation as the support function of a zonoid of ξ, see [29]. Zonoids form an important family of convex bodies (i.e., convex compact sets) in the Euclidean space $\mathbb{R}^d$, see [37]. *Zonoids* are obtained as limits of zonotopes in the Hausdorff metric, while zonotopes are Minkowski (elementwise) sums of a finite number of segments.

The sums of segments and the limits of sums can be interpreted as expectations of random segments. By translation, it is possible to assume that all segments are centred and so are of the form $[-\xi, \xi]$ for a random vector $\xi \in \mathbb{R}^d$. Recall that the *support function* of a set K in $\mathbb{R}^d$ is given by

$$h_K(u) = \sup\{\langle u, x\rangle : x \in K\}, \qquad u \in \mathbb{R}^d,$$

where $\langle u, x\rangle$ denotes the scalar product. The expectation of $[-\xi, \xi]$ is the convex set Z_ξ^o identified by its support function, which is equal to the expected support function of the segment (see [23], Section 2.1), that is,

$$h_{Z_\xi^o}(u) = \mathbf{E}\bigl|\langle u, \xi\rangle\bigr|, \qquad u \in \mathbb{R}^d.$$

If ξ is integrable, Z^o_ξ is an origin symmetric *convex body* (compact convex set). For instance, if ξ is discrete in $\mathbb{R}^2$ with only two possible values, then Z^o_ξ is a parallelogram; if ξ is isotropic, then Z^o_ξ is a ball.

A slightly different construction of zonoids associated with random vectors was suggested by Koshevoy and Mosler, see [21] and [29]. Namely, the zonoid Z_ξ of ξ is the expectation of $[0, \xi]$ and so the support function of Z_ξ is given by

$$h_{Z_\xi}(u) = \mathbf{E}\langle u, \xi\rangle_+, \qquad u \in \mathbb{R}^d,$$

where $x_+ = \max(x, 0)$. In order to stress the difference between the two variants of zonoids, we call Z^o_ξ the *centred zonoid* of ξ, see Section 5 for the comparison of the two concepts. Note that Z_ξ is also well defined for some non-integrable ξ. Nonetheless from now on we always assume that all mentioned random variables and random vectors are integrable and not identically zero.

It is well known that the zonoid of ξ does not uniquely characterise its distribution. For instance, on the line, Z_ξ is the segment with end-points determined by the expectations of the positive and negative parts of ξ, while Z^o_ξ is the segment with end-points $\pm\mathbf{E}|\xi|$. Thus, all random variables with the same first absolute moment are not distinguishable in terms of their centred zonoids.

The concept of zonoid is useful in multivariate statistics to define trimming and data depth, see [6,29]. In case of (non-centred) zonoids, the expectations $h(k, u) = \mathbf{E}(k + \langle u, \xi\rangle)_+$ for $k \in \mathbb{R}$ and $u \in \mathbb{R}^d$ uniquely determine the distribution of ξ, and determine the support function of a convex body in $\mathbb{R}^{d+1}$ called the *lift zonoid* of ξ, see [21,29]. In finance, $\mathbf{E}(k + \langle u, \xi\rangle)_+$ becomes the non-discounted price of a basket call option with strike $-k$ for $k \le 0$ (if the expectation is taken with respect to a chosen martingale measure). The well-known result of Breeden and Litzenberger [2] saying that the prices of all call options determine the distribution of ξ now becomes a corollary of a general uniqueness result for lift zonoids, see [29], Theorem 2.21, and [26].

Definition 1. *Two integrable random vectors ξ and ξ^* in $\mathbb{R}^d$ are called* zonoid equivalent *if their centred zonoids coincide*, *that is*,

$$\mathbf{E}\bigl|\langle u, \xi\rangle\bigr| = \mathbf{E}\bigl|\langle u, \xi^*\rangle\bigr|$$

for all $u \in \mathbb{R}^d$. Two families of integrable random variables $\{\xi_t, t \in T\}$ and $\{\xi^*_t, t \in T\}$ are called zonoid equivalent *if all their finite-dimensional distributions are zonoid equivalent.*

The concept of zonoid equivalence is closely related to spectral representations of symmetric stable ($S\alpha S$) and max-stable processes. For instance, each $S\alpha S$ process with $\alpha \in (0, 2)$ admits the spectral representation

$$X_t \overset{d}{\sim} \int_E f_t(z) M_\alpha(\mathrm{d}z), \qquad t \in T, \tag{1.1}$$

where the equality is understood in the sense of all finite-dimensional distributions, $\{f_t, t \in T\}$ is a family of functions from $L^\alpha(E, \mathcal{E}, \mu)$ for a measurable space $(E, \mathcal{E}, \mu)$ and M_α is an $S\alpha S$ random measure with control measure μ, see [33]. If X_t admits another spectral representation on a measurable space $(G, \mathcal{G}, \nu)$ with functions $\{g_t\}$, then the collections of functions $\{f_t\}$ and $\{g_t\}$ satisfy

$$\int_E \left| \sum_{i=1}^n u_i f_{t_i} \right|^\alpha \mathrm{d}\mu = \int_G \left| \sum_{i=1}^n u_i g_{t_i} \right|^\alpha \mathrm{d}\nu \tag{1.2}$$

for all $n \geq 1$, $u_1, \dots, u_n \in \mathbb{R}$ and $t_1, \dots, t_n \in T$. This is easily seen by computing the characteristic function of the spectral representations, see [33], Section 3.2. If $\alpha = 1$ and both μ and ν are probability measures, then (1.2) can be interpreted as the zonoid equivalence of stochastic processes $\{f_t\}$ and $\{g_t\}$.

Fairly similar facts hold for max-stable processes, see [15,18,39,40]. This close relationship between stable processes and zonoid equivalence makes it possible to figure out a number of properties of stochastic processes in relation to their zonoid equivalence.

The paper starts with the analysis of the main implication of the zonoid equivalence. Namely, in Section 2 we show that the zonoid equivalence yields the equality of the expected values for all even one-homogeneous function of the random vectors. Stochastic processes whose finite-dimensional distributions remain zonoid equivalent for time shifts are discussed in Section 3. This zonoid stationarity property is brought in relationship to the stationarity of related stable and max-stable processes through their LePage representations.

A result of Hardin ([13], Theorem 1.1) implies that the distribution of an integrable random vector ξ is uniquely determined by $\mathbf{E}|1 + \langle u, \xi \rangle|$ for all $u \in \mathbb{R}^d$, equivalently by the centred zonoid of $(1, \xi)$. In Theorem 8, we show that, if ξ is symmetric, it is possible to replace 1 by any random variable taking values ± 1.

Section 4 introduces the swap-invariance property for a random sequence that amounts to the zonoid equivalence of each permutation of all its finite subsequences, which is weaker than the exchangeability property. We prove the ergodic theorem for swap-invariant sequences and characterise the limits, thereby generalising the classical results for exchangeable sequences. Zonoid equivalence of positive random vectors with respect to permutation of two their components has been investigated in [27] and for all possible permutations in [28] in view of financial applications.

Section 5 discusses relationships between centred and non-centred zonoids and also another symmetry property being stronger than the exchangeability. In this relation, consider

$$\mathbf{E}|u_0 + u_1\xi_1 + \cdots + u_d\xi_d|$$

as function $f(u_0, u_1, \dots, u_d)$ of $(d+1)$ real arguments. The swap invariance means exactly that f is invariant for permutations of $u_1, \dots, u_d$ with $u_0 = 0$; the exchangeability corresponds to the permutation invariance of $u_1, \dots, u_d$ for any (and then all) $u_0 \neq 0$. Assuming the full permutation invariance for all $u_0, u_1, \dots, u_d$ imposes a property (called *lift swap-invariance*), which is stronger than the exchangeability of $\xi_1, \dots, \xi_d$. A variant of this property for non-centred zonoids has been considered in [26] and [28] motivated by applications in finance.

Finally, Section 6 collects a number of relevant results concerning zonoids of particular distributions. It is shown that zonoids identify uniquely distributions from location-scale families under rather mild conditions. The special case of random vectors with positive coordinates is also analysed, in particular log-infinitely divisible laws being important in financial applications.

The consideration of (non-centred) zonoids makes it possible to study possibly non-integrable random vectors, which is left for a future work. The same relates to L^p-zonoids considered in [25]. A number of results of this paper can be generalised for random elements in Banach spaces along the lines of [1].

2. Expectations of homogeneous functions

Let $\mathcal{H}$ (resp., $\mathcal{H}_e$) denote the family of all (resp., even) measurable homogeneous functions $\mathbb{R}^d \mapsto \mathbb{R}_+$, so that $f(cx) = cf(x)$ for all $x \in \mathbb{R}^d$ and $c \geq 0$.

Theorem 2. *Two random vectors ξ and ξ^* are zonoid equivalent if and only if $\mathbf{E}f(\xi) = \mathbf{E}f(\xi^*)$ for all $f \in \mathcal{H}_e$.*

Proof. *Sufficiency* is immediate, since $f(x) = |\langle u, x\rangle|$ belongs to $\mathcal{H}_e$.

Necessity. First, show that $\mathbf{E}\|\xi\| = \mathbf{E}\|\xi^*\|$. The integral of the support function of a convex body K over the unit sphere is $\frac{1}{2}d\kappa_d b(K)$, where $b(K)$ is called the mean width of K and κ_d is the volume of the unit ball in $\mathbb{R}^d$. By changing the order of integral and expectation, it is easy to see that the mean width of Z^o_ξ equals the expected mean width of the segment $[-\xi, \xi]$. The mean width of this segment can be found from the Steiner formula ([37], Equation (4.1.1)), see also [37], page 210, as $b([-\xi, \xi]) = 4\|\xi\|\kappa_{d-1}/(d\kappa_d)$. Thus, $\mathbf{E}\|\xi\| = b(Z^o_\xi)d\kappa_d/(4\kappa_{d-1})$ is uniquely determined by Z^o_ξ.

Denote the common value of $\mathbf{E}\|\xi\|$ and $\mathbf{E}\|\xi^*\|$ by c, and define probability measure $\mathbf{Q}$ with density

$$\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}} = \frac{\|\xi\|}{c}$$

and another measure $\mathbf{Q}^*$ generated by ξ^* in the same way. Denote by $\mathbf{E}_\mathbf{Q}$ the expectation with respect to $\mathbf{Q}$ (and, resp., with respect to $\mathbf{Q}^*$). Then for all $u \in \mathbb{R}^d$

$$\frac{1}{c}\mathbf{E}\big|\langle u, \xi\rangle\big| = \frac{1}{c}\mathbf{E}\big|\langle u, \xi\rangle\big|\mathbb{1}_{\{\|\xi\|\neq 0\}} = \mathbf{E}_\mathbf{Q}\left|\left\langle u, \frac{\xi}{\|\xi\|}\right\rangle\right|\mathbb{1}_{\{\|\xi\|\neq 0\}} = \mathbf{E}_\mathbf{Q}\left|\left\langle u, \frac{\xi}{\|\xi\|}\right\rangle\right|$$

and similarly $c^{-1}\mathbf{E}|\langle u, \xi^*\rangle| = \mathbf{E}_{\mathbf{Q}^*}|\langle u, \xi^*/\|\xi^*\|\rangle|$. Therefore, $\xi/\|\xi\|$ under $\mathbf{Q}$ and $\xi^*/\|\xi^*\|$ under $\mathbf{Q}^*$ share the same zonoid. Define measure μ on the unit Euclidean sphere by setting $\mu(A) = \mathbf{Q}(\xi/\|\xi\| \in A)$ and correspondingly μ^*. The convex body Z^o_μ with the support function

$$h_{Z^o_\mu}(u) = \int_{\mathbb{S}^{d-1}} \big|\langle u, x\rangle\big|\mu(\mathrm{d}x) = \mathbf{E}_\mathbf{Q}\left|\left\langle u, \frac{\xi}{\|\xi\|}\right\rangle\right|$$

is termed the zonoid of μ, see [37], Section 3.5. It is well known that an even finite measure on the unit sphere is uniquely determined by its zonoid, see [37], Theorem 3.5.3. Since μ and μ^* share the same zonoid, the integrals of any even and integrable function with respect to them coincide.

For $f \in \mathcal{H}_e$, we have $f(0) = 0$ and therefore

$$\mathbf{E} f(\xi) = \mathbf{E}\big[f(\xi)\mathbb{1}_{\|\xi\| \neq 0}\big] = \mathbf{E}_{\mathbf{Q}} f\big(\xi/\|\xi\|\big) = \int_{\mathbb{S}^{d-1}} f(u)\mu(\mathrm{d}u).$$

Hence, $\mathbf{E} f(\xi) = \mathbf{E} f(\xi^*)$ for each $f \in \mathcal{H}_e$. A short calculation shows that integrability of $f(\xi/\|\xi\|)$ under $\mathbf{Q}$ implies integrability of $f(\xi^*/\|\xi^*\|)$ under $\mathbf{Q}^*$ and vice versa. □

If ξ and ξ^* are zonoid equivalent, then $f(\xi)$ and $f(\xi^*)$ are two zonoid equivalent random variables for all $f \in \mathcal{H}_e$. The following result is easily derived by observing that $\mathbf{E} f(\xi) = \mathbf{E}\frac{1}{2}(f(\xi) + f(-\xi))$ for symmetric ξ.

Corollary 3. *Two symmetric random vectors ξ and ξ^* are zonoid equivalent if and only if $\mathbf{E} f(\xi) = \mathbf{E} f(\xi^*)$ for all $f \in \mathcal{H}$. In particular, $\mathbf{E} h_K(\xi) = \mathbf{E} h_K(\xi^*)$ for each convex body K.*

Corollary 4. *Let $f_1, \dots, f_k \in \mathcal{H}_e$. If ξ and ξ^* are zonoid equivalent, then the vectors $(f_1(\xi), \dots, f_k(\xi))$ and $(f_1(\xi^*), \dots, f_k(\xi^*))$ are zonoid equivalent as long as one of these vectors is integrable.*

Proof. It suffices to use the fact that $f(x) = |u_1 f_1(x) + \cdots + u_k f_k(x)|$ belongs to $\mathcal{H}_e$ and $f(\xi)$ is integrable. □

The following easy fact is also worth noticing.

Proposition 5. *Two random vectors are zonoid equivalent if and only if all their linear transformations are zonoid equivalent.*

Proof. For each matrix A, we have $\mathbf{E}|\langle A\xi, u\rangle| = \mathbf{E}|\langle \xi, A^\top u\rangle|$ and so $A\xi$ and $A\xi^*$ are zonoid equivalent if ξ and ξ^* are. □

In the following, we often consider random vectors with positive coordinates (shortly called positive vectors), which are usually denoted by the letter η.

Proposition 6. *Two positive integrable random vectors η and η^* are zonoid equivalent if and only if $\mathbf{E} f(\eta) = \mathbf{E} f(\eta^*)$ for each $f \in \mathcal{H}$. In particular, the zonoid equivalence implies $\mathbf{E}\eta = \mathbf{E}\eta^*$.*

Proof. While the sufficiency is evident, the necessity can be proved similarly to Theorem 2 with $\mathbf{Q}$ having density $\eta_1/\mathbf{E}\eta_1$. The equality of expectations is obtained by setting $f(x) = (x_i)_+$ for any $i = 1, \dots, d$. □

For positive random vectors, the concept of a *max-zonoid* is also useful. The max-zonoid M_η of a positive random vector $\eta = (\eta_1, \dots, \eta_d)$ is defined as the expectation of the crosspolytope in $\mathbb{R}^d$, which is the convex hull of the origin and the standard basis vectors scaled by $\eta_1, \dots, \eta_d$, see [24]. The support function of M_η is given by

$$h_{M_\eta}(u) = \mathbf{E} \max(0, u_1\eta_1, \dots, u_d\eta_d), \qquad u = (u_1, \dots, u_d) \in \mathbb{R}^d. \tag{2.1}$$

This support function is most interesting for positive $u_1, \dots, u_d$, where it is possible to omit zero in the right-hand side of (2.1). The following result has been proved analytically in [39], Theorem 1.1. An alternative proof (using a geometric argument combined with the change of measure technique) has recently been given in [28], Proposition 1.

Proposition 7. *Two positive integrable random vectors η and η^* have identical max-zonoids if and only if η and η^* are zonoid equivalent.*

3. Isometries, representations of stable processes, and zonoid stationarity

A result of Hardin ([13], Theorem 1.1) reformulated for random vectors implies that, for any given positive $p \notin 2\mathbb{Z}$, the values $\mathbf{E}|1 + \langle u, \xi\rangle|^p$ for all $u \in \mathbb{R}^d$ determine uniquely the distribution of random vector $\xi \in \mathbb{R}^d$. If $p = 1$, this result means that the centred zonoid of $(1, \xi)$ uniquely identifies the distribution of ξ, cf. [21,29]. This also means that if two zonoid equivalent random vectors contain the same coordinate being exactly one, then these random vectors are identically distributed. Below we provide a generalisation of this result for $p = 1$ and symmetric random vectors showing that it is possible to replace the constant 1 with any random variable taking values ± 1.

Theorem 8. *Let ξ be a symmetric random vector in $\mathbb{R}^d$. If ε is any random variable with values ± 1, then the centred zonoid of (ε, ξ), that is, the values of*

$$\mathbf{E}\bigl|u_0\varepsilon + \langle u, \xi\rangle\bigr|, \qquad u_0 \in \mathbb{R}, u \in \mathbb{R}^d,$$

determines uniquely the distribution of ξ.

Proof. For each function $f(\varepsilon, \xi)$ we have $f(\varepsilon, \xi) + f(-\varepsilon, \xi) = f(1, \xi) + f(-1, \xi)$, so that

$$\mathbf{E}\bigl|u_0\varepsilon + \langle u, \xi\rangle\bigr| + \mathbf{E}\bigl|-u_0\varepsilon + \langle u, \xi\rangle\bigr| = \mathbf{E}\bigl|u_0 + \langle u, \xi\rangle\bigr| + \mathbf{E}\bigl|-u_0 + \langle u, \xi\rangle\bigr|.$$

Since ξ is symmetric,

$$\mathbf{E}\bigl|-u_0 + \langle u, \xi\rangle\bigr| = \mathbf{E}\bigl|u_0 + \langle u, -\xi\rangle\bigr| = \mathbf{E}\bigl|u_0 + \langle u, \xi\rangle\bigr|.$$

Thus,

$$\mathbf{E}\bigl|u_0 + \langle u, \xi\rangle\bigr| = \tfrac{1}{2}\bigl(\mathbf{E}\bigl|u_0\varepsilon + \langle u, \xi\rangle\bigr| + \mathbf{E}\bigl|-u_0\varepsilon + \langle u, \xi\rangle\bigr|\bigr)$$

for all $u_0 \neq 0$ and $u \in \mathbb{R}^d$. Therefore, the right-hand side is determined by the zonoid of (ε, ξ), and it remains to note that the left-hand side uniquely identifies the distribution of ξ by [13], Theorem 1.1. $\square$

An integrable random vector ξ in $\mathbb{R}^d$, which is not a.s. zero, generates a norm on $\mathbb{R}^d$ by

$$\|u\|_\xi = \mathbf{E}\big|\langle u, \xi\rangle\big|.$$

With this definition, zonoid equivalence of ξ and ξ^* means that $\|\cdot\|_\xi$ and $\|\cdot\|_{\xi^*}$ are two identical norms on $\mathbb{R}^d$. The uniqueness result in [13] is used to characterise isometries of subspaces of L^1 that contain the function identically equal one. Theorem 8 makes it possible to obtain similar results for subspaces of L^1 that consist of symmetric random variables and contain random variables taking values $\pm c$ for any fixed $c > 0$. The characterisation of linear isometries defined on families of random variables are important for the studies of symmetric stable laws, see [13,14,32].

A collection of integrable random elements $\{\xi_t, t \in T\}$ is a subset of the space $L^1 = L^1(\Omega, \mathfrak{F}, \mathbf{P})$. Denote by F_ξ the L^1-closure of the linear space generated by this collection. Assume that Ω is a Borel space with $\mathfrak{F}$ being the Borel σ-algebra.

Assume that $\{\xi_t\}$ is rigid, that is, any linear isometry $U_0 : F_\xi \mapsto L^1$ is uniquely extendable to the isometry $U : L^1 \mapsto L^1$. It is well known [13,32] that the rigidity is guaranteed by imposing that the random elements $\{\xi_t\}$ have full support, the union of its supports is Ω up to a null set (see [14] for details), and that $\xi_t/\bar{\xi}$, $t \in T$, generate the σ-algebra $\mathfrak{F}$, where $\bar{\xi} \in F_\xi$ is a random variable with full support (its existence is guaranteed by [13], Lemma 3.2). Note that the family $\{\xi_t\}$ is often called *minimal* instead of *rigid*, as it gives rise to a *minimal spectral representation* of a $S\alpha S$ process via (1.1), see also [14].

Consider another rigid collection $\{\xi_t^*, t \in T\}$, which is zonoid equivalent to $\{\xi_t, t \in T\}$. Then the isometry between F_ξ and F_{ξ^*} can be characterised as follows, see Theorem 3.2 in [32]. For every $t \in T$,

$$\xi_t^*(\omega) = h(\omega)\xi_t\big(\phi(\omega)\big) \qquad \mathbf{P}\text{-a.s.}, \tag{3.1}$$

where $\phi : \Omega \to \Omega$, $h : \Omega \to \mathbb{R} \setminus \{0\}$ are measurable and $|\bar{\xi}| \mathrm{d}\mathbf{P} = |\bar{\xi}|(|h| \mathrm{d}\mathbf{P}) \circ \phi^{-1}$, for a random variable $\bar{\xi} \in F_\xi$ with full support.

A similar construction of isometries can be carried over for max-zonoids and non-negative integrable functions, see [10], where such isometries are called pistons. Since for positive random vectors the zonoid equivalence and the max-zonoid equivalence are identical (see Proposition 7), the isometries corresponding to max-zonoids are also characterised by (3.1).

Recall that each *symmetric* 1-*stable* (i.e., $S\alpha S$ with $\alpha = 1$) random vector X in $\mathbb{R}^d$ can be represented as the LePage series

$$X = \sum_{k=1}^{\infty} \Gamma_k^{-1} \xi^{(k)}, \tag{3.2}$$

where $\Gamma_k = \zeta_1 + \cdots + \zeta_k$ are successive sums of i.i.d. standard exponential random variables and $\xi, \xi^{(1)}, \xi^{(2)}, \ldots$ are i.i.d. integrable symmetric random vectors, see [22]. Note that the ξ's are often assumed to be distributed on the unit sphere with an extra normalisation constant in front of the sum, see [33], Corollary 1.4.3. A similar series representation with the sum replaced by the maximum, and positive ξ yields simple (i.e., having unit Fréchet marginals) max-stable random vectors, see [9] and [12]. If ξ is a stochastic process, similar series representations yield symmetric 1-stable processes and simple max-stable processes. For instance, a result of [9] says that each stochastically continuous simple max-stable process Y can be represented as

$$Y_t = \max_{k\geq 1} \Gamma_k^{-1} \xi_t^{(k)}, \qquad t \in \mathbb{R}^d, \tag{3.3}$$

where $\{\xi_t^{(k)}, t \in \mathbb{R}^d\}$ are i.i.d. copies of an integrable positive process $\{\xi_t, t \in \mathbb{R}^d\}$. In the following, we refer to (3.2), its variant for stochastic processes or their max-analogues as the LePage series.

Theorem 9. *Two LePage series X and X^* given by* (3.2) (*resp*., *their max-analogues*) *with integrable symmetric* (*resp*., *positive*) *summands distributed as ξ and ξ^* coincide in distribution if and only if ξ and ξ^* are zonoid equivalent.*

Proof. It suffices to consider the case of ξ being a random vector in $\mathbb{R}^d$. The points $\{(\Gamma_k^{-1}, \xi^{(k)}), k \geq 1\}$ build the Poisson point process on $(0, \infty)$ with intensity t^{-2}, $t > 0$, and independent marks $\xi^{(k)}$, $k \geq 1$. The formula for the probability generating functional of the marked Poisson process (see [8]) yields the characteristic function of X

$$\begin{aligned}\mathbf{E}e^{\imath\langle u,X\rangle} &= \exp\left\{-\int_0^\infty \mathbf{E}\big(1 - e^{\imath t\langle u,\xi\rangle}\big)t^{-2}\,dt\right\} \\ &= \exp\left\{-\int_0^\infty \mathbf{E}\big(1-\cos\big(t\langle u,\xi\rangle\big)\big)t^{-2}\,dt\right\} = \exp\left\{-\frac{\pi}{2}\mathbf{E}\big|\langle u,\xi\rangle\big|\right\},\end{aligned}$$

since $\int_0^\infty (1-\cos(s))s^{-2}\,ds = \pi/2$, where $\imath$ denotes the imaginary unit. Thus, the distribution of X is determined by $\mathbf{E}|\langle u,\xi\rangle|$, $u \in \mathbb{R}^d$.

The result for max-stable random vectors follows from the association argument from [15] or [39] or a direct calculation of the cumulative distribution functions combined with Proposition 7. □

Let $\{\xi_t, t \in T\}$ be a stochastic process such that ξ_t is integrable for all $t \in T$, where T is either integer grid $\mathbb{Z}^d$ or $\mathbb{R}^d$.

Definition 10. *The process $\{\xi_t, t \in T\}$ is called* zonoid stationary *if $\{\xi_t, t \in T\}$ and $\{\xi_{t+s}, t \in T\}$ are zonoid equivalent for all $s \in T$.*

Obviously all integrable stationary processes are zonoid stationary. If both ξ and ξ^* are centred Gaussian processes, then by Corollary 35 their zonoid equivalence implies the equality of all finite-dimensional distributions, so their zonoid stationarity is equivalent to the conventional stationarity. The same holds for symmetric α-stable processes with given $\alpha > 1$. The fact that zonoid does not uniquely determine the general distribution suggests that there exist non-stationary but zonoid stationary processes. The next result follows from Theorem 9.

Corollary 11. *A symmetric* 1*-stable process* (*resp.*, *max-stable process with unit Fréchet marginals*) *obtained as the LePage series* (3.2) (*resp.*, (3.3)) *is stationary if and only if* ξ *is zonoid stationary.*

If the max-stable process Y given by (3.3) is stationary, the process $\log \xi$ is called *Brown–Resnick stationary*, see [17]. Corollary 11 shows that the Brown–Resnick stationarity of $\log \xi$ is equivalent to the zonoid stationarity of a positive stochastic process ξ.

Example 12. The geometric Brownian motion $e^{W_t - |t|/2}$, where W_t, $t \in \mathbb{R}$, is a double-sided Brownian motion, is zonoid stationary. The corresponding stationary process given by (3.3) was introduced by Brown and Resnick [3]. Kabluchko *et al.* [17] replaced W_t by a Gaussian process ξ_t with mean μ_t and variance σ_t^2. Their result implies that e^{ξ_t} is zonoid stationary if and only if $\xi_t - \mu_t$ has stationary increments and $\mu_t + \frac{1}{2}\sigma_t^2$ is constant for all t.

For a zonoid stationary process ξ the spaces generated by $\{\xi_t, t \in T\}$ and $\{\xi_{t+h}, t \in T\}$ are isometric for all $h \in T$. This gives rise to a representation of ξ in term of isometries. Following [30], a measurable function $\phi : \Omega \times T \to \Omega$ is said to be a measurable flow if $\phi_{t_1+t_2}(\omega) = \phi_{t_1}(\phi_{t_2}(\omega))$ and $\phi_0(\omega) = \omega$ for all $t_1, t_2 \in T$ and $\omega \in \Omega$. The flow ϕ is said to be non-singular if $\mathbf{P} \circ \phi_t^{-1}$ is equivalent to $\mathbf{P}$ for all $t \in T$. A measurable function $r : \Omega \times T \to \mathbb{R}$ is said to be a cocycle for a measurable flow ϕ if $r_{t_1+t_2}(\omega) = r_{t_1}(\omega) r_{t_2}(\phi_{t_1}(\omega))$ for all $t_1, t_2 \in T$ and for $\mathbf{P}$-almost all $\omega \in \Omega$. By replicating the proofs of [30], Theorem 3.1, and [31], Theorem 2.2, it is easy to show that a zonoid stationary process ξ with rigid (minimal) family F_ξ satisfies

$$\xi_t(\omega) = r_t(\omega) \left(\frac{\mathrm{d}\mathbf{P} \circ \phi_t}{\mathrm{d}\mathbf{P}} \right)(\omega)(\xi_0 \circ \phi_t)(\omega) \qquad \mathbf{P}\text{-a.s.},$$

where $\{\phi_t, t \in T\}$ is a measurable non-singular flow and $\{r_t, t \in T\}$ is a cocycle for ϕ taking values in $\{-1, 1\}$.

4. Swap invariant sequences

A finite or infinite random sequence $\xi = (\xi_1, \xi_2, \dots)$ of random elements is said to be *exchangeable* if its distribution is invariant under finite permutations, that is, the distribution of any finite subsequence is invariant under any permutation of its elements, see, for example, [20], Section 1.1.

Definition 13. *An integrable random vector is called* swap-invariant *if all random vectors obtained by permutations of its coordinates are zonoid equivalent. A sequence of integrable random variables is called swap-invariant if all its finite subsequences are swap-invariant.*

An integrable random vector ξ with positive components exhibiting the swap-invariance property restricted to permutation of its two components ξ_i and ξ_j is called ij-swap-invariant. This weaker variant of the swap-invariance property has been already introduced and applied in a financial context in [27] and [36]. The swap-invariance property of the vector of asset prices ensures that different financial derivatives share the same price and can be freely exchanged, which is an essential tool for semi-static hedging of barrier options, see [4].

The swap-invariance property of ξ immediately implies that $\mathbf{E}|\xi_1| = \cdots = \mathbf{E}|\xi_d|$. It is obvious that the exchangeable sequence is swap-invariant. The following examples show that the swap-invariance is weaker than the exchangeability property.

Example 14 (See [7]). On the probability space $\Omega = [0, 1]$ with the Lebesgue measure define

$$\xi_n = n(n+1)\mathbb{1}_{\omega \in ((n+1)^{-1}, n^{-1}]}, \qquad n \geq 1. \tag{4.1}$$

By a direct computation it is easy to see that

$$\mathbf{E}|u_1\xi_1 + \cdots + u_n\xi_n| = \sum_{i=1}^{n} |u_i|,$$

so that the sequence is indeed swap-invariant, but not exchangeable. Further examples of this type can be obtained for general sequences of non-negative random variables with equal expectations and disjoint supports.

Example 15. Let $Z_1, Z_2, \dots$ be a sequence of i.i.d. standard normal random variables and let $\{b_k, k \geq 1\}$ be a sequence of real numbers such that $\sum b_k^2 < \infty$. Define $\eta_i = \mathrm{e}^{\xi_i}$, $i \geq 1$, where

$$\xi_i = Z_i + \sum_{k=1}^{\infty} b_k Z_k + \mu_i$$

and

$$\mu_i = -\frac{1}{2}\operatorname{Var}(\xi_i) = -\frac{1}{2}\left(1 + \sum_{k=1}^{\infty} b_k^2 + 2b_i\right).$$

By Corollary 38, η is swap-invariant. Note that no two components η_i and η_j are identically distributed unless $b_i = b_j$.

If the extended sequence $(1, \xi)$ (or (ε, ξ) with $\varepsilon \in \{-1, 1\}$ and symmetric ξ) is swap-invariant, then ξ is exchangeable. Actually, the swap invariance of such extended sequence is stronger than the exchangeability of ξ, see Section 5.

It is well known that each exchangeable sequence of integrable random variables satisfies several ergodic theorems. Given an infinite random sequence $\{\xi_n, n \geq 1\}$, denote the corresponding tail σ-algebra by $\mathcal{T}_\xi$, the shift-invariant σ-algebra by $\mathcal{I}_\xi$, and the permutation-invariant σ-field by $\mathcal{E}_\xi$. These σ-algebras are identical modulo null sets for exchangeable sequences, see [20], Corollary 1.6. Since an infinite exchangeable sequence is stationary, the following result is a direct consequence of [19], Theorem 10.6, and [20], Corollary 1.6.

Theorem 16. *Let $\xi_1, \xi_2, \dots$ be an exchangeable sequence of integrable random variables. Then*

$$n^{-1}\sum_{i=1}^{n}\xi_i \to \mathbf{E}(\xi_1 \mid \mathcal{E}_\xi) \qquad \text{a.s. and in } L^1 \text{ as } n\to\infty.$$

In the following, we extend this fact to swap-invariant sequences. Recall that these sequences by definition consist of integrable random variables.

Theorem 17. *Let $\xi_1, \xi_2, \dots$ be a swap-invariant sequence of random variables. Then $n^{-1}(\xi_1 + \cdots + \xi_n)$ converges almost surely to an integrable random variable X as $n \to \infty$.*

Proof. Assume first that all random variables $\xi_1, \xi_2, \dots$ are symmetric and that at least one random variable (say ξ_1) is non-zero with probability one. Recall that $\mathbf{E}|\xi_i|$ is the same for all i. Define an equivalent to $\mathbf{P}$ probability measure $\mathbf{P}^1$ by

$$\frac{\mathrm{d}\mathbf{P}^1}{\mathrm{d}\mathbf{P}} = \frac{|\xi_1|}{\mathbf{E}|\xi_1|}. \tag{4.2}$$

For any finite subsequence $\xi = (\xi_1, \xi_{k_1}, \dots, \xi_{k_d})$,

$$\frac{\mathbf{E}|\langle u, \xi\rangle|}{\mathbf{E}|\xi_1|} = \mathbf{E}_{\mathbf{P}^1}\left|u_1\varepsilon + u_2\frac{\xi_{k_1}}{|\xi_1|} + \cdots + u_d\frac{\xi_{k_d}}{|\xi_1|}\right|, \tag{4.3}$$

where $\varepsilon = \xi_1/|\xi_1|$ is the sign of ξ_1 and $\mathbf{E}_{\mathbf{P}^1}$ denotes the expectation with respect to $\mathbf{P}^1$. By Theorem 8, the right-hand side of (4.3) determines the distribution of $(\xi_{k_1}, \dots, \xi_{k_d})/|\xi_1|$ under $\mathbf{P}^1$. By writing (4.3) for a permutation $\xi_{k_{i_1}}, \dots, \xi_{k_{i_d}}$ we arrive at the conclusion that the sequence $\frac{\xi_2}{|\xi_1|}, \frac{\xi_3}{|\xi_1|}, \dots$ is exchangeable under $\mathbf{P}^1$. Theorem 16 yields that

$$\frac{1}{n}\left(\frac{\xi_2}{|\xi_1|} + \cdots + \frac{\xi_n}{|\xi_1|}\right) \to Z \qquad \mathbf{P}^1\text{-a.s. as } n\to\infty$$

for some random variable Z. Since $\mathbf{P}^1$ and $\mathbf{P}$ are equivalent, the same holds $\mathbf{P}$-a.s. Thus,

$$\frac{\xi_2 + \cdots + \xi_n}{n} \to X = |\xi_1|Z \qquad \text{a.s. as } n\to\infty.$$

It is obviously possible to add ξ_1 in the numerator without altering the limit.

If the sequence $\{\xi_n\}$ is no longer symmetric, consider an independent symmetric random variable ε with values ± 1. Then the sequence $\{\varepsilon\xi_n, n \geq 1\}$ is symmetric and swap-invariant, which is seen by the total probability formula. As shown above, $\{\varepsilon\xi_n\}$ satisfies the ergodic theorem with limit X_ε. Then the original sequence $\{\xi_n\}$ satisfies the ergodic theorem with the limit $\varepsilon X_\varepsilon$ (note that ε and X_ε may be dependent).

It remains to consider the case when all ξ_i have an atom at zero. Fix any $k \geq 1$ and define a new measure $\mathbf{P}^k$ by

$$\frac{\mathrm{d}\mathbf{P}^k}{\mathrm{d}\mathbf{P}} = \frac{|\xi_k|}{\mathbf{E}|\xi_k|}.$$

The function $(x_1,\dots,x_d)\mapsto |u_1x_1+\dots+u_dx_d|\mathbb{1}_{x_k\neq 0}$ is in $\mathcal{H}_e$, hence

$$\begin{aligned}&\mathbf{E}|u_1\xi_1+u_2\xi_2+\dots+u_k\xi_k+\dots+u_d\xi_d|\mathbb{1}_{\xi_k\neq 0}\\&\quad=\mathbf{E}|u_1\xi_1+u_2\xi_{i_2}+\dots+u_k\xi_k+\dots+u_n\xi_{i_d}|\mathbb{1}_{\xi_k\neq 0}\end{aligned}$$

for all $u_1,\dots,u_d\in\mathbb{R}$ and all permutations $i_1,\dots,i_d$ with $i_k=k$. Thus, the sequence $(\xi_1,\dots,\xi_{k-1},\xi_{k+1},\dots)/|\xi_k|$ is exchangeable under $\mathbf{P}^k$. Since $\mathbf{P}^k$ is equivalent to $\mathbf{P}$ restricted on $\{\xi_k\neq 0\}$, $n^{-1}(\xi_1+\dots+\xi_n)$ converges to some random variable X for almost all $\omega\in\{\xi_k\neq 0\}$. Note that the same limit appears under $\mathbf{P}^m$ for $m\neq k$ for almost all ω such that $\xi_k(\omega)\neq 0$ and $\xi_m(\omega)\neq 0$. Finally, set $X(\omega)=0$ for all $\omega\in\Omega$ such that $\xi_n(\omega)=0$ for all $n\geq 1$.

Since $\xi_1,\xi_2,\dots$ have the same first absolute moment, the integrability of X follows trivially by Fatou's lemma and the triangle inequality. □

Remark 18. A proof of Theorem 17 for almost surely positive swap-invariant sequences can be alternatively carried over by using ξ_1 to change the measure and then referring to [13], Theorem 1.1.

Theorem 19. *Assume that a swap-invariant sequence $\xi_1,\xi_2,\dots$ satisfies one of the following conditions*:

(a) $\xi_k\neq 0$ *a.s. for some* $k\geq 1$,
(b) $\xi_1,\xi_2,\dots$ *is uniformly integrable*.

Then the convergence of $n^{-1}(\xi_1+\dots+\xi_n)\to X$ also holds in L^1.

Proof. (a) The proofs of Theorems 17 and 16 yield that

$$\mathbf{E}\big|n^{-1}(\xi_1+\dots+\xi_n)-X\big|\mathbb{1}_{\xi_k\neq 0}\to 0\qquad\text{as } n\to\infty,$$

while $\mathbf{P}(\xi_k\neq 0)=1$.

(b) It is well known that the uniform integrability of $\{\xi_n,n\geq 1\}$ implies the uniform integrability of $\{(\xi_1+\dots+\xi_n)/n,n\geq 1\}$. The a.s. convergence implies the L^1-convergence in view of the uniform integrability property, see [19], Proposition 4.12. □

Example 20 (Example 14 continuation). For the sequence (4.1), $n^{-1}(\xi_1+\dots+\xi_n)\to 0$ a.s., but $\mathbf{E}n^{-1}(\xi_1+\dots+\xi_n)=1$, so the ergodic theorem holds almost surely but not in L^1.

The following theorem characterises the limits in Theorem 17 for the case when at least one random variable in the sequence does not have an atom at zero.

Theorem 21. *Let $\xi=(\xi_1,\xi_2,\dots)$ be a symmetric swap-invariant sequence such that $\xi_1\neq 0$ a.s. Then*

$$\frac{1}{n}\sum_{i=1}^n\xi_i\to\frac{|\xi_1|}{\mathbf{E}(|\xi_1|\mid\mathcal{E}_{\tilde{\xi}})}\mathbf{E}(\xi_2\mid\mathcal{E}_{\tilde{\xi}})\qquad\text{a.s. and in } L^1 \text{ as } n\to\infty,\tag{4.4}$$

where $\tilde{\xi}=(\xi_2/|\xi_1|,\xi_3/|\xi_1|,\dots)$.

Proof. The sequence $\tilde{\xi}$ is exchangeable under $\mathbf{P}^1$ defined by (4.2) and Theorem 16 implies

$$\frac{1}{n}\sum_{i=1}^{n}\frac{\xi_i}{|\xi_1|}\to \mathbf{E}_{\mathbf{P}^1}\left[\frac{\xi_2}{|\xi_1|}\,\Big|\,\mathcal{E}_{\tilde{\xi}}\right] \qquad \text{a.s. and in } L^1 \text{ as } n\to\infty. \tag{4.5}$$

Let Z be a $\mathcal{E}_{\tilde{\xi}}$ measurable and $\mathbf{P}^1$-integrable random variable. Then

$$\mathbf{E}_{\mathbf{P}^1} Z = \mathbf{E}\frac{|\xi_1|Z}{\mathbf{E}|\xi_1|} = \mathbf{E}\left[\mathbf{E}\left(\frac{|\xi_1|Z}{\mathbf{E}|\xi_1|}\,\Big|\,\mathcal{E}_{\tilde{\xi}}\right)\right] = \mathbf{E}\left[Z\frac{\mathbf{E}(|\xi_1|\mid\mathcal{E}_{\tilde{\xi}})}{\mathbf{E}|\xi_1|}\right]. \tag{4.6}$$

Let $A\in\mathcal{E}_{\tilde{\xi}}$. By the definition of the conditional expectation

$$\begin{aligned}\mathbf{E}_{\mathbf{P}^1}\left(\mathbb{1}_A\mathbf{E}_{\mathbf{P}^1}\left(\frac{\xi_2}{|\xi_1|}\,\Big|\,\mathcal{E}_{\tilde{\xi}}\right)\right) &= \mathbf{E}_{\mathbf{P}^1}\big(\mathbb{1}_A\xi_2/\mathbf{E}|\xi_1|\big) = \mathbf{E}_{\mathbf{P}^1}\big(\mathbb{1}_A\mathbf{E}\big(\xi_2/\mathbf{E}|\xi_1|\mid\mathcal{E}_{\tilde{\xi}}\big)\big)\\ &= \mathbf{E}_{\mathbf{P}^1}\left[\mathbb{1}_A\frac{\mathbf{E}(\xi_2\mid\mathcal{E}_{\tilde{\xi}})}{\mathbf{E}(|\xi_1|\mid\mathcal{E}_{\tilde{\xi}})}\frac{\mathbf{E}(|\xi_1|\mid\mathcal{E}_{\tilde{\xi}})}{\mathbf{E}|\xi_1|}\right] = \mathbf{E}_{\mathbf{P}^1}\left[\mathbb{1}_A\frac{\mathbf{E}(\xi_2\mid\mathcal{E}_{\tilde{\xi}})}{\mathbf{E}(|\xi_1|\mid\mathcal{E}_{\tilde{\xi}})}\right],\end{aligned}$$

where the last equality follows from (4.6). The uniqueness of the conditional expectation yields

$$\mathbf{E}_{\mathbf{P}^1}\left[\frac{\xi_2}{|\xi_1|}\,\Big|\,\mathcal{E}_{\tilde{\xi}}\right] = \frac{\mathbf{E}(\xi_2\mid\mathcal{E}_{\tilde{\xi}})}{\mathbf{E}(|\xi_1|\mid\mathcal{E}_{\tilde{\xi}})} \qquad \text{a.s.}$$

This equation together with (4.5) yield the claim. □

With a similar proof, we arrive at the following result for positive sequences.

Proposition 22. *Let $\xi=(\xi_1,\xi_2,\dots)$ be a positive swap-invariant sequence. Then*

$$\frac{1}{n}\sum_{i=1}^{n}\xi_i\to\frac{\xi_1}{\mathbf{E}(\xi_1\mid\mathcal{E}_{\tilde{\xi}})}\mathbf{E}(\xi_2\mid\mathcal{E}_{\tilde{\xi}}) \qquad \textit{a.s. and in } L^1 \textit{ as } n\to\infty, \tag{4.7}$$

where $\tilde{\xi}=(\xi_2/\xi_1,\xi_3/\xi_1,\dots)$.

For non-symmetric swap-invariant sequences, we obtain the following result by applying the total probability formula and Theorem 21.

Corollary 23. *Let $\xi=(\xi_1,\xi_2,\dots)$ be a swap-invariant sequence such that $\xi_1\neq 0$ a.s. Then*

$$\frac{1}{n}\sum_{i=1}^{n}\xi_i\to\frac{|\xi_1|}{\mathbf{E}(|\xi_1|\mid\mathcal{E}_{\varepsilon\tilde{\xi}})\varepsilon}\mathbf{E}(\varepsilon\xi_2\mid\mathcal{E}_{\varepsilon\tilde{\xi}}) \qquad \textit{a.s. and in } L^1 \textit{ as } n\to\infty, \tag{4.8}$$

where ε is the Rademacher random variable independent of ξ under $\mathbf{P}$.

Corollary 24. *Let $\xi = (\xi_1, \xi_2, \dots)$ be a swap-invariant sequence. If $n^{-1}(\xi_1 + \cdots + \xi_n)$ converges in L^1 to a deterministic non-zero limit c, then (c, ξ) is swap-invariant and so ξ is exchangeable.*

Proof. For $m, n \geq 1$, the swap-invariance property implies

$$\mathbf{E}\left|u_1\xi_1 + \cdots + u_n\xi_n + u_0\frac{1}{m}\sum_{k=1}^{m}\xi_{n+k}\right| = \mathbf{E}\left|u_{i_1}\xi_1 + \cdots + u_{i_n}\xi_n + u_{i_0}\frac{1}{m}\sum_{k=1}^{m}\xi_{n+k}\right|$$

for all permutations $(i_0, i_1, \dots, i_n)$ of $(0, 1, \dots, n)$. The L^1-convergence then yields as $m \to \infty$

$$\mathbf{E}|u_0c + u_1\xi_1 + \cdots + u_n\xi_n| = \mathbf{E}|u_{i_0}c + u_{i_1}\xi_1 + \cdots + u_{i_n}\xi_n|,$$

so that (c, ξ) is swap-invariant. Its exchangeability follows from [13], Theorem 1.1. $\square$

Example 25 (Example 15 continuation). We show that $n^{-1}(\eta_1 + \cdots + \eta_n)$ converges a.s. to

$$X = \exp\left(\sum_{i=1}^{\infty} b_i Z_i - \frac{1}{2}\sum_{i=1}^{\infty} b_i^2\right).$$

By [20], Corollary 1.6, we can consider the tail σ-field $\mathcal{T}_{\tilde{\eta}}$, where

$$\tilde{\eta} = \left(\frac{\eta_2}{\eta_1}, \frac{\eta_3}{\eta_1}, \dots\right) = \left(\mathrm{e}^{Z_2 - Z_1 - (b_2 - b_1)}, \mathrm{e}^{Z_3 - Z_1 - (b_3 - b_1)}, \dots\right).$$

Since the functions $x \mapsto \mathrm{e}^{x-(b_i-b_1)}$, $i \geq 2$, are bijective, $\mathcal{T}_{\tilde{\eta}}$ can be written as $\mathcal{T}_{\tilde{\eta}} = \bigcap_{n\geq 2}\mathcal{F}_n$, where $\mathcal{F}_n = \sigma(Z_n - Z_1, Z_{n+1} - Z_1, \dots)$. For each $n \geq 2$, the random variable

$$\tilde{Z}_n = \lim_{k\to\infty} k^{-1}\sum_{i=0}^{k-1}(Z_1 - Z_{n+i})$$

is clearly $\mathcal{F}_n$-measurable and by the strong law of large numbers $\tilde{Z}_n = Z_1$ a.s. Thus, Z_1 is measurable with respect to the completion $\bar{\mathcal{F}}_n$ of $\mathcal{F}_n$ for all $n \geq 2$, and hence $\bar{\mathcal{T}}_{\tilde{\eta}}$ measurable. On the other hand, for all $n \geq 2$, the vector $(Z_2, \dots, Z_n)$ is independent of $\mathcal{F}_{n+1}$ and therefore independent of $\mathcal{T}_{\tilde{\eta}}$. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and bounded. Then for all $A \in \mathcal{T}_{\tilde{\eta}}$, the dominated convergence theorem yields

$$\begin{aligned}
\mathbf{E}\mathbb{1}_A f\left(\sum_{i=2}^{\infty} b_i Z_i\right) &= \lim_{k\to\infty}\mathbf{E}\mathbb{1}_A f\left(\sum_{i=2}^{k} b_i Z_i\right) \\
&= \lim_{k\to\infty}\mathbf{P}(A)\mathbf{E}f\left(\sum_{i=2}^{k} b_i Z_i\right) = \mathbf{P}(A)\mathbf{E}f\left(\sum_{i=2}^{\infty} b_i Z_i\right),
\end{aligned}$$

which shows the independence of $\sum_{i=2}^{\infty} b_i Z_i$ and $\mathcal{T}_{\tilde{\eta}}$. Since $\mathbf{E}(Z \mid \mathcal{T}_{\tilde{\eta}}) = \mathbf{E}(Z \mid \bar{\mathcal{T}}_{\tilde{\eta}})$ a.s. for all integrable Z,

$$\mathbf{E}(\eta_1 \mid \mathcal{T}_{\tilde{\eta}}) = \mathrm{e}^{(1+b_1)Z_1}\mathrm{e}^{-(1+b_1^2+2b_1)/2},$$

$$\mathbf{E}(\eta_2 \mid \mathcal{T}_{\tilde{\eta}}) = \mathrm{e}^{b_1 Z_1}\mathrm{e}^{-b_1^2/2}.$$

By Proposition 22,

$$\frac{1}{n}\sum_{i=1}^{n}\eta_i \to \frac{X\mathrm{e}^{Z_1}\mathrm{e}^{-(1+2b_1)/2}}{\mathrm{e}^{(1+b_1)Z_1}\mathrm{e}^{-(1+b_1^2+2b_1)/2}}\mathrm{e}^{b_1 Z_1}\mathrm{e}^{-b_1^2/2} = X \qquad \text{a.s. and in } L^1 \text{ as } n \to \infty.$$

5. Non-centred zonoids and lift swap invariance

It is possible to relate the centred and non-centred zonoids as $Z_\xi^o = Z_\xi + Z_{-\xi}$, that is, the centred zonoid is the Minkowski (elementwise) sum of the zonoid of ξ and the zonoid of $-\xi$ being the central symmetric version $Z_{-\xi} = \{-x\colon x \in Z_\xi\}$ of Z_ξ. If ξ has a symmetric distribution, then $Z_\xi^o = 2Z_\xi$ is a scaled zonoid of ξ. For a general integrable ξ, its centred zonoid equals $2Z_{\varepsilon\xi}$, where ε is the Rademacher random variable taking values ± 1 with equal probability and independent of ξ. Note that the conventional symmetrisation $\xi - \xi'$ for i.i.d. ξ and ξ' is not helpful in this context.

Proposition 26. *If ξ and ξ^* are two integrable random vectors, then $Z_\xi = Z_{\xi^*}$ if and only if $\mathbf{E}\xi = \mathbf{E}\xi^*$ and $Z_\xi^o = Z_{\xi^*}^o$.*

Proof. Since $2a_+ = |a| + a$ for any real a,

$$h_{Z_\xi}(u) = \tfrac{1}{2}\big(\mathbf{E}\big|\langle \xi, u\rangle\big| + \langle \mathbf{E}\xi, u\rangle\big).$$

It remains to note that the equality $Z_\xi = Z_{\xi^*}$ implies the equality of expectations by [29], Proposition 2.11. □

In view of the above fact, Proposition 6 implies that for positive random vectors the equivalences of centred and non-centred zonoids are identical concepts.

The centred zonoid of $(1, \xi)$ (also called the centred lift zonoid of ξ) determines uniquely the distribution of ξ by [13], Theorem 1.1. In particular, the invariance of $\mathbf{E}|1 + u_1\xi_1 + \cdots + u_d\xi_d|$ with respect to permutations of any $u_1, \ldots, u_d$ is equivalent to the exchangeability of ξ. If the lifted random vector $(1, \xi)$ is swap-invariant, that is, $\mathbf{E}|u_0 + u_1\xi_1 + \cdots + u_d\xi_d|$ is invariant for all permutations of $u_0, u_1, \ldots, u_d$, then ξ is called *lift swap-invariant*.

The lift swap-invariance property is slightly weaker than the *joint self-duality* of ξ meaning the permutation invariance of $\mathbf{E}(u_0 + u_1\xi_1 + \cdots + u_d\xi_d)_+$ for all $u_0, u_1, \ldots, u_d$. The relation between these two properties is exactly the same as the relation between the equality of centred and non-centred zonoids. For instance, the lift swap-invariance implies that $\mathbf{E}|\xi_1| = \cdots = \mathbf{E}|\xi_d| = 1$, while the joint self-duality yields that $\mathbf{E}\xi_1 = \cdots = \mathbf{E}\xi_d = 1$. The both properties are identical for random vectors with positive components.

By construction, the lift swap-invariance property implies the exchangeability of ξ and is actually much stronger. For instance a vector of i.i.d. positive random variables is exchangeable, but is neither jointly self-dual nor is lift swap-invariant unless all random variables equal 1 almost surely, see [26].

A weaker version of the self-duality property corresponding to the permutation of the lifting (constant) coordinate and *one* fixed other coordinate was studied in [26]. In particular, its univariate version is often called *put-call symmetry* and is intensively discussed and applied in the financial literature, see, for example, [5,38] and further references cited in [26].

Proposition 27. *If a non-trivial random vector ξ is either jointly self-dual or is lift swap-invariant with $\mathbf{E}\xi_i = 1$ for any i, then all its components are almost surely positive random variables with expectation being one.*

Proof. It suffices to prove this for random variable ξ. If $(1,\xi)$ is swap-invariant and $\mathbf{E}\xi = 1$, then $(1,\xi)$ is jointly self-dual by Proposition 26, so it suffices to consider only the case of a self-dual ξ. The self-duality property of ξ implies that

$$\mathbf{E}\big(0+(-1)\xi\big)_+ = \mathbf{E}(-1+0\xi)_+,$$

so that $\mathbf{E}\xi_- = 0$ and so ξ is almost surely non-negative. Since

$$\mathbf{E}(0+1\xi)_+ = \mathbf{E}(1+0\xi)_+,$$

it follows that $1 = \mathbf{E}\xi_+ = \mathbf{E}\xi$.

If ξ has an atom at zero, then $\mathbf{E}(1-a\xi)_+$, $a \in \mathbb{R}$, is bounded from below by a positive number. The self-duality implies that $\mathbf{E}(-a+\xi)_+$ is also bounded from below by the same number, which is not possible for large a in view of the integrability of ξ. $\square$

For integrable random vectors with positive components the symmetry properties can be related to each other. Following the notation of [27], define functions

$$\tilde{\kappa}_j(x) = \left(\frac{x_1}{x_j},\ldots,\frac{x_{j-1}}{x_j},\frac{x_{j+1}}{x_j},\ldots,\frac{x_d}{x_j}\right), \qquad j=1,\ldots,d,$$

on $x \in (0,\infty)^d$. For any $j=1,\ldots,d$ define a new probability measure by

$$\frac{\mathrm{d}\mathbf{P}^j}{\mathrm{d}\mathbf{P}} = \frac{\eta_j}{\mathbf{E}\eta_j}. \tag{5.1}$$

This measure change was used in [11] in order to reduce the dimensionality when calculating option prices. Consider an integrable random vector η with positive components. If $\mathbf{E}\eta_j = 1$, then the zonoid of η coincides with the lift zonoid of $\tilde{\kappa}_j(\eta)$ under $\mathbf{P}^j$, see [28], Proposition 3.

Theorem 28. *Assume that η is an integrable random vector of dimension $d \geq 2$ with positive components. The following conditions are equivalent*:

(a) *η is swap-invariant under* $\mathbf{P}$.
(b) *$\tilde{\kappa}_j(\eta)$ is lift swap-invariant* (*equivalently jointly self-dual*) *under $\mathbf{P}^j$ for any* (*and then all*) *$j \in \{1, \dots, d\}$*.
(c) *In case $d \geq 3$, for at least two $j \in \{1, \dots, d\}$* (*and then automatically for all j*), *$\tilde{\kappa}_j(\eta)$ is exchangeable under $\mathbf{P}^j$*.

Proof. The equivalence of (a) and (b) is obtained (for $j = 1$) by

$$\mathbf{E}|u_1\eta_1 + \cdots + u_d\eta_d| = \mathbf{E}\eta_1\mathbf{E}_{\mathbf{P}^1}\left|u_1 + u_2\frac{\eta_2}{\eta_1} + \cdots + u_d\frac{\eta_d}{\eta_1}\right|,$$

so that permutations of coordinates in the left-hand side corresponds to permutations in the right-hand side. The invariance with respect to the latter is equivalent to the lift swap invariance of $\tilde{\kappa}_1(\eta)$ under $\mathbf{P}^1$, since the right-hand side identifies the distribution of $\tilde{\kappa}_1(\eta)$.

It is easy to see that (a) implies (c) for all j, since the exchangeability is a weaker property than (b). Assuming (c) for $j = 1, 2$ without loss of generality, we see that $(\eta_2/\eta_1, \dots, \eta_d/\eta_1)$ is $\mathbf{P}^1$-exchangeable and $(\eta_1/\eta_2, \eta_3/\eta_2, \dots, \eta_d/\eta_2)$ is $\mathbf{P}^2$-exchangeable. The first fact implies that $\mathbf{E}|\langle u, \eta\rangle|$ is invariant with respect to permutation all but first coordinates of u, while the second fact implies the invariance with respect to permutations of all coordinates excluding the second one, so η is swap-invariant. $\square$

6. Equality of zonoids

6.1. Location-scale families

Consider family of random variables $\xi = \mu + \sigma X$ for an integrable random variable X and $\mu \in \mathbb{R}$, $\sigma > 0$. These random variables are said to form a location-scale family.

Theorem 29. *Assume that the distribution of X has infinite essential infimum and essential supremum. Then the zonoid Z_ξ of a random variable ξ from the location-scale family generated by X uniquely determines the location and scale parameters of the distribution.*

Proof. Without loss of generality, set $\mathbf{E}X = 0$. Assume that the random variables $\mu + \sigma X$ and $\mu^* + \sigma^* X$ share the same zonoid. By Proposition 26, $\mu = \mu^*$.

In order to finish the proof, we show that $\mathbf{E}(\mu + \sigma X)_+$ is strictly increasing in σ for each fixed $\mu \in \mathbb{R}$. This is obvious if $\mu = 0$, since $\mathbf{E}(\sigma X)_+ = \sigma\mathbf{E}X_+$, which is strictly increasing in σ since $\mathbf{E}X_+ > 0$.

Assume that $\mu < 0$ and $\sigma_1 > \sigma_2$. Then

$$\begin{aligned}
&\mathbf{E}\big((\mu + \sigma_1 X)_+ - (\mu + \sigma_2 X)_+\big) \\
&\quad = \mathbf{E}\big((\mu + \sigma_1 X)\mathbb{1}_{\{-\mu/\sigma_1 < X \leq -\mu/\sigma_2\}}\big) + (\sigma_1 - \sigma_2)\mathbf{E}(X\mathbb{1}_{\{-\mu/\sigma_2 < X\}}) > 0,
\end{aligned}$$

where the last expectation is positive because X has unbounded support and $\mathbf{E}X = 0$.

If $\mu > 0$, the same argument applied to $\mathbf{E}(\mu + \sigma_1 X)_-$ yields that the expectation of the negative part is strictly decreasing in σ and the equality $\mathbf{E}(\mu + \sigma_1 X)_+ = \mu - \mathbf{E}(\mu + \sigma_1 X)_-$ concludes the proof. □

Note that Theorem 29 does not hold for the centred zonoid Z^o_ξ unless it is assumed that the expectation of ξ is known and so Z_ξ is also identified.

Corollary 30. *Assume that random variable ξ has infinite essential infimum and essential supremum. If $Z_\xi = Z_{\sigma\xi+\mu}$, then $\mu = 0$ and $\sigma = 1$.*

Corollary 31. *Two normally distributed d-dimensional random vectors ξ and ξ^* coincide in distribution if and only if $Z_\xi = Z_{\xi^*}$.*

Proof. For $u \in \mathbb{R}^d$ the random variables $\langle \xi, u\rangle$ and $\langle \xi^*, u\rangle$ belong to the same location-scale family. The proof is finished by referring to Theorem 29 and noticing that all one-dimensional projection of a random vector uniquely determine its distribution. □

The uniqueness holds also for the location scale family obtained as $\mu + \sigma X$ for a symmetric stable random variable X.

Example 32 (Distribution with bounded support). Assume that $\mathbf{E}X = 0$ and that X has finite essential infinum, that is, there exists a constant c such that $X \geq c$ a.s. Choose $\mu > 0$. Then for all $\sigma < -\mu/c$ the random variable $\xi = \mu + \sigma X$ is a.s. positive and so the expectation of its negative part is zero and the expectation of its positive part is μ. Thus, the zonoid Z_ξ does not uniquely determine the scale parameter σ.

Note that all above results are formulated for non-centred zonoids. In the rest of this section, we consider centred zonoids, and the corresponding zonoid equivalence concept. The following result concerns random vectors that can be represented as product of a scaling random variable and an independent random vector.

Proposition 33. *Two random vectors $\xi = R\zeta$ and $\xi^* = R^*\zeta^*$, where R and R^* are positive random variables independent of ζ and ζ^*, respectively, are zonoid equivalent if and only if $(\mathbf{E}R)\zeta$ and $(\mathbf{E}R^*)\zeta^*$ are zonoid equivalent.*

Proof. It suffices to note that

$$\mathbf{E}\big|\langle u, \xi\rangle\big| = \mathbf{E}R\mathbf{E}\big|\langle u, \zeta\rangle\big| = \mathbf{E}\big|\big\langle u, (\mathbf{E}R)\zeta\big\rangle\big|. \qquad \square$$

Consider random vectors with centred *elliptical* distributions, that is, assume that $\xi = R(AU)$, where U is uniformly distributed on the unit sphere, A is a (deterministic) matrix and R is a positive random variable independent of U.

Proposition 34. *Two centred elliptically distributed random vectors* $\xi = R(AU)$ *and* $\xi^* = R^*(A^*U)$ *are zonoid equivalent if and only if* $(\mathbf{E}R)^2 AA^\top = (\mathbf{E}R^*)^2 A^*(A^*)^\top$.

Proof. Using rescaling, it is possible to assume that $\mathbf{E}R = \mathbf{E}R^*$. By Proposition 33, it suffices to consider zonoid equivalence of AU and A^*U. By Proposition 5, this is the case if and only if random variables $\langle A^\top u, U\rangle$ and $\langle (A^*)^\top u, U\rangle$ are zonoid equivalent. Since U is uniformly distributed on the unit sphere, $\langle v, U\rangle$ is distributed as a certain random variable with a fixed distribution scaled by $\|v\|$ for all v. Thus, $\|A^\top u\| = \|(A^*)^\top u\|$ for all u, which implies the statement. □

Corollary 35. *Two symmetric normally distributed random vectors* ξ *and* ξ^* *coincide in distribution if and only if they are zonoid equivalent.*

Zonoid of $S\alpha S$ random ξ with $\alpha \in (1, 2]$ is computed in [25], Section 6.4, as

$$Z_\xi = \frac{1}{\pi}\Gamma\left(1 - \frac{1}{\alpha}\right)K,$$

where Γ is the gamma-function and K is a convex body that, together with α, characterises the distribution of ξ. Thus, if α is fixed, then the zonoid determines uniquely the corresponding symmetric α-stable distribution. However, two symmetric stable vectors with the same zonoid are not necessarily identically distributed if their stability indices are different.

6.2. Log-infinitely divisible distributions with equal zonoids

A random vector with positive components can be written as the coordinate-wise exponential $\eta = \mathrm{e}^\xi$. In the following, φ_ξ stands for the characteristic function of ξ. The following result immediately follows from [17], Proposition 6, see also [27], Theorem 3.2.

Theorem 36. *Two integrable random vectors* e^ξ *and* e^{ξ^*} *are zonoid equivalent if and only if*

$$\varphi_\xi(u - \imath w) = \varphi_{\xi^*}(u - \imath w) \tag{6.1}$$

for all $u \in \mathbb{R}^d$ *with* $\sum u_i = 0$ *and for at least one (and then necessarily for all)* w, *such that* $\sum w_k = 1$ *and both sides in* (6.1) *are finite.*

Assume that e^ξ and e^{ξ^*} are two random vectors, where ξ and ξ^* are infinitely divisible random variables. Then

$$\varphi_\xi(u) = \mathbf{E}\mathrm{e}^{\imath\langle u,\xi\rangle} = \exp\left\{\imath\langle b, u\rangle - \frac{1}{2}\langle u, Au\rangle + \int_{\mathbb{R}^d}\left(\mathrm{e}^{\imath\langle u,x\rangle} - 1 - \imath\langle u, x\rangle \mathbb{1}_{\|x\|\le 1}\right)\mathrm{d}\nu(x)\right\}$$

for $u \in \mathbb{R}^d$, where $A = (a_{ij})$ is a symmetric non-negative definite $d \times d$ matrix, $b \in \mathbb{R}^d$ is a constant vector and ν is a measure on $\mathbb{R}^d$ (called the Lévy measure) satisfying $\nu(\{0\}) = 0$ and

$$\int_{\mathbb{R}^d} \min\left(\|x\|^2, 1\right)\mathrm{d}\nu(x) < \infty.$$

Then ξ is said to have the Lévy triplet (A, ν, b). In this section, we translate the equality of the zonoids of two log-infinitely divisible random vectors into conditions on their Lévy triplets. Note that the conditions on the Lévy triplet of infinitely divisible random vectors apply also for Lévy processes with time one values ξ and ξ^*.

In order to formulate the condition on the Gaussian terms in a compact form it is helpful to use the *variogram*

$$\gamma_{ij} = a_{ii} + a_{jj} - 2a_{ij}.$$

If ξ is normally distributed, then γ_{ij} is the variance of $\xi_i - \xi_j$. In order to state the condition on the Lévy measure define $(d-1) \times d$-dimensional matrix, $d \geq 2$

$$U = \begin{pmatrix} 1 & 0 & \cdots & 0 & -1 \\ 0 & 1 & \cdots & 0 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -1 \end{pmatrix}. \tag{6.2}$$

Theorem 37. *Let* e^{ξ} *and* e^{ξ^*} *be integrable random vectors such that* ξ *and* ξ^* *are infinitely divisible with characteristic triplets* (A, ν, γ) *and* (A^*, ν^*, γ^*). *Then for* $d \geq 2$ e^{ξ} *and* e^{ξ^*} *are zonoid equivalent if and only if the following three conditions hold.*

(a) $\gamma_{ij} = \gamma^*_{ij}$ *for all* $i, j \in \{1, \dots, d\}$.
(b) *The images* $\hat{\nu}U^{-1}$ *and* $\hat{\nu}^*U^{-1}$ *under* U *of measures* $\mathrm{d}\hat{\nu}(x) = \mathrm{e}^{x_d}\,\mathrm{d}\nu(x)$ *and* $\mathrm{d}\hat{\nu}^*(x) = \mathrm{e}^{x_d}\,\mathrm{d}\nu^*(x)$, $x \in \mathbb{R}^d$, *restricted to* $\mathbb{R}^{d-1} \setminus \{0\}$ *coincide.*
(c) $\mathbf{E}\mathrm{e}^{\xi_i} = \mathbf{E}\mathrm{e}^{\xi^*_i}$ *for all* $i = 1, \dots, d$, *that is,*

$$\begin{aligned} b_i + \frac{1}{2}a_{ii} + \int_{\mathbb{R}^d} \bigl(\mathrm{e}^{x_i} - 1 - x_i \mathbb{1}_{\|x\| \leq 1}\bigr)\,\mathrm{d}\nu(x) \\ = b^*_i + \frac{1}{2}a^*_{ii} + \int_{\mathbb{R}^d} \bigl(\mathrm{e}^{x_i} - 1 - x_i \mathbb{1}_{\|x\| \leq 1}\bigr)\,\mathrm{d}\nu^*(x). \end{aligned} \tag{6.3}$$

For $d = 1$, e^{ξ} *and* e^{ξ^*} *are zonoid equivalent if and only if* (c) *holds.*

The following result is closely related to and can be alternatively derived following the proof of [17], Theorem 10, see also [16], Theorem 1.1.

Corollary 38. *Two lognormal random vectors* e^{ξ} *and* e^{ξ^*} *are zonoid equivalent if and only if* $\mu_i + \frac{1}{2}a_{ii} = \mu^*_i + \frac{1}{2}a^*_{ii}$ *for all* i *and* $\gamma_{ij} = \gamma^*_{ij}$ *for all* i, j, *that is,* ξ *and* ξ^* *have identical variogram.*

In particular, in the lognormal case the zonoid equivalence does not even imply the equality of the marginal distributions, quite differently to the case of normal distributions where the zonoid uniquely determines the joint distribution, see Corollary 35.

Furthermore, note that the kernel of U given by (6.2) is the family of vectors with all equal components. Hence, if the support of ν is a subset of the kernel of U, then the corresponding log-infinitely divisible distribution shares the same zonoid with a lognormal distribution, meaning that two rather different distributions are zonoid equivalent.

Proof of Theorem 37. For $d \ge 2$, the zonoid equivalence of e^{ξ} and e^{ξ^*} implies $\mathbf{E}\mathrm{e}^{\xi} = \mathbf{E}\mathrm{e}^{\xi^*}$, see Proposition 6, and in particular $c = \mathbf{E}\mathrm{e}^{\xi_d} = \mathbf{E}\mathrm{e}^{\xi_d^*}$. Note that this is also implied by (c). Since also $Z_{\mathrm{e}^{\xi}} = Z_{\mathrm{e}^{\xi^*}}$ by Proposition 6,

$$\mathbf{E}\big(u_1\mathrm{e}^{\xi_1} + \cdots + u_d\mathrm{e}^{\xi_d}\big)_+ = \mathbf{E}\mathrm{e}^{\xi_d}\big(u_1\mathrm{e}^{\xi_1-\xi_d} + \cdots + u_{d-1}\mathrm{e}^{\xi_{d-1}-\xi_1} + u_d\big)_+,$$

the zonoid of e^{ξ} uniquely determines and is uniquely determined by the probability distribution of $U\xi = (\xi_1 - \xi_d, \dots, \xi_{d-1} - \xi_d)$ under the probability measure $\mathbf{P}^d$ with density e^{ξ_d}/c.

In order to identify the distribution of $U\xi$ under $\mathbf{P}^d$ first note that the distribution of ξ under $\mathbf{P}^d$ has the characteristic triplet $(A, \hat{\nu}, \hat{b})$, where $\mathrm{d}\hat{\nu}(x) = \mathrm{e}^{x_d}\,\mathrm{d}\nu(x)$ and

$$\hat{b} = b + \int_{\|x\|\le 1} x\big(\mathrm{e}^{x_d} - 1\big)\nu(\mathrm{d}x) + Ae_d,$$

where e_d is the dth standard basis vector, see [35], Example 7.3. By [34], Proposition 11.10, the Lévy triplet of $U\xi$ under $\mathbf{P}^d$ is given by $A_U = UAU^\top$, $\hat{\nu}U^{-1}$ restricted onto $\mathbb{R}^{d-1} \setminus \{0\}$ and

$$b_U = U\hat{b} + \int_{\mathbb{R}^d} Ux(\mathbb{1}_{\|Ux\|\le 1} - \mathbb{1}_{\|x\|\le 1})\hat{\nu}(\mathrm{d}x).$$

The corresponding formula holds for ξ^*.

Equating the centred Gaussian terms, the Lévy measures, and simplifying $b_U = b_U^*$ yields that $U\xi$ under $\mathbf{P}^d$ coincides in distribution with $U\xi^*$ under $\mathbf{P}^{d*}$ if and only if

$$a_{ij} + a_{dd} - a_{di} - a_{jd} = a_{ij}^* + a_{dd}^* - a_{di}^* - a_{jd}^*, \qquad i, j = 1, \dots, d-1, \tag{6.4}$$

condition (b) holds and, for all $i = 1, \dots, d-1$,

$$\begin{aligned} &b_i - b_d + a_{id} - a_{dd} + \int_{\mathbb{R}^d} (x_i - x_d)\big(\mathbb{1}_{\|Ux\|\le 1}\mathrm{e}^{x_d} - \mathbb{1}_{\|x\|\le 1}\big)\,\mathrm{d}\nu(x) \\ &\quad = b_i^* - b_d^* + a_{id}^* - a_{dd}^* + \int_{\mathbb{R}^d} (x_i - x_d)\big(\mathbb{1}_{\|Ux\|\le 1}\mathrm{e}^{x_d} - \mathbb{1}_{\|x\|\le 1}\big)\,\mathrm{d}\nu^*(x). \end{aligned} \tag{6.5}$$

Adding equations (6.4) with $k, l = i, i$; $k, l = j, j$ (for given i and j), and subtracting (6.4) multiplied by two, we arrive at the equality of the variograms. Furthermore, noticing that

$$(a_{ij} + a_{dd} - a_{di} - a_{jd})_{ij=1}^{d-1} = \tfrac{1}{2}(\gamma_{id} + \gamma_{jd} - \gamma_{ij})_{ij=1}^{d-1}$$

we obtain that the equality of variograms implies (6.4). The equality of zonoids implies the equality of expectations, which exactly corresponds to (6.3). It remains to show that (6.3) together with other two conditions (a) and (b) imply (6.5).

By (6.3), we have for all $i = 1, \ldots, d-1$

$$\begin{aligned} &b_i + \frac{1}{2}a_{ii} + \int_{\mathbb{R}^d} \big(\mathrm{e}^{x_i} - 1 - x_i \mathbb{1}_{\|x\|\le 1}\big)\,\mathrm{d}\nu(x) \\ &\quad = b_i^* + \frac{1}{2}a_{ii}^* + \int_{\mathbb{R}^d} \big(\mathrm{e}^{x_i} - 1 - x_i \mathbb{1}_{\|x\|\le 1}\big)\,\mathrm{d}\nu^*(x), \end{aligned} \tag{6.6}$$

$$\begin{aligned} &b_d + \frac{1}{2}a_{dd} + \int_{\mathbb{R}^d} \big(\mathrm{e}^{x_d} - 1 - x_d \mathbb{1}_{\|x\|\le 1}\big)\,\mathrm{d}\nu(x) \\ &\quad = b_d^* + \frac{1}{2}a_{dd}^* + \int_{\mathbb{R}^d} \big(\mathrm{e}^{x_d} - 1 - x_d \mathbb{1}_{\|x\|\le 1}\big)\,\mathrm{d}\nu^*(x), \end{aligned} \tag{6.7}$$

while condition (a) implies

$$a_{ii} + a_{dd} - 2a_{id} = a_{ii}^* + a_{dd}^* - 2a_{id}^* \tag{6.8}$$

for all $i = 1, \ldots, d-1$. Furthermore, condition (b) implies

$$\begin{aligned} &\int_{\mathbb{R}^d} \big(\mathrm{e}^{x_i - x_d} - 1 - (x_i - x_d)\mathbb{1}_{\|Ux\|\le 1}\big)\,\mathrm{d}\hat{\nu}(x) \\ &\quad = \int_{\mathbb{R}^d} \big(\mathrm{e}^{x_i - x_d} - 1 - (x_i - x_d)\mathbb{1}_{\|Ux\|\le 1}\big)\,\mathrm{d}\hat{\nu}^*(x), \end{aligned} \tag{6.9}$$

where $d\hat{\nu}(x) = \mathrm{e}^{x_d}\,\mathrm{d}\nu(x)$, since by changing variables

$$\int_{\mathbb{R}^{d-1}} \big(\mathrm{e}^{y} - 1 - y\mathbb{1}_{\|y\|\le 1}\big)\,\mathrm{d}\big(\hat{\nu}U^{-1}\big)(y) = \int_{\mathbb{R}^{d-1}} \big(\mathrm{e}^{y} - 1 - y\mathbb{1}_{\|y\|\le 1}\big)\,\mathrm{d}\big(\hat{\nu}^*U^{-1}\big)(y).$$

Now (6.5) is obtained by subtracting from (6.6) the sum of (6.9), (6.7) and a half of (6.8).

Recall that equality of the zonoids is equivalent to equality of their support functions for all u on the unite sphere. Hence, for positive random variables e^{ξ} and e^{ξ^*} ($d = 1$) equality of their zonoids is equivalent to equality of their expectations, which in turn, is equivalent to condition (c). □

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References

[1] Borell, C. (2009). Zonoids induced by Gauss measure with an application to risk aversion. *ALEA Lat. Am. J. Probab. Math. Stat.* **6** 133–147. MR2506861

[2] Breeden, D.T. and Litzenberger, R.H. (1978). Prices of state-contingent claims implicit in options prices. *J. Business* **51** 621–651.

[3] Brown, B.M. and Resnick, S.I. (1977). Extreme values of independent stochastic processes. *J. Appl. Probab.* **14** 732–739. MR0517438

[4] Carr, P., Ellis, K. and Gupta, V. (1998). Static hedging of exotic options. *J. Finance* **53** 1165–1190.

[5] Carr, P. and Lee, R. (2009). Put-call symmetry: Extensions and applications. *Math. Finance* **19** 523–560. MR2583519

[6] Cascos, I. (2010). Data depth: Multivariate statistics and geometry. In *New Perspectives in Stochastic Geometry* (W.S. Kendall and I. Molchanov, eds.) 398–423. Oxford: Oxford Univ. Press. MR2654685

[7] Dacunha-Castelle, D. (1982). A survey on exchangeable random variables in normed spaces. In *Exchangeability in Probability and Statistics* (*Rome*, 1981) (G. Koch and F. Spizzichino, eds.) 47–60. Amsterdam: North-Holland. MR0675964

[8] Daley, D.J. and Vere-Jones, D. (1988). *An Introduction to the Theory of Point Processes*. *Springer Series in Statistics*. New York: Springer. MR0950166

[9] de Haan, L. (1984). A spectral representation for max-stable processes. *Ann. Probab.* **12** 1194–1204. MR0757776

[10] de Haan, L. and Pickands, J. III (1986). Stationary min-stable stochastic processes. *Probab. Theory Related Fields* **72** 477–492. MR0847381

[11] Eberlein, E., Papapantoleon, A. and Shiryaev, A.N. (2009). Esscher transform and the duality principle for multidimensional semimartingales. *Ann. Appl. Probab.* **19** 1944–1971. MR2569813

[12] Falk, M., Hüsler, J. and Reiss, R.D. (2004). *Laws of Small Numbers*: *Extremes and Rare Events*, extended ed. Basel: Birkhäuser. MR2104478

[13] Hardin, C.D. Jr. (1981). Isometries on subspaces of L^p. *Indiana Univ. Math. J.* **30** 449–465. MR0611233

[14] Hardin, C.D. Jr. (1982). On the spectral representation of symmetric stable processes. *J. Multivariate Anal.* **12** 385–401. MR0666013

[15] Kabluchko, Z. (2009). Spectral representations of sum- and max-stable processes. *Extremes* **12** 401–424. MR2562988

[16] Kabluchko, Z. (2010). Stationary systems of Gaussian processes. *Ann. Appl. Probab.* **20** 2295–2317. MR2759735

[17] Kabluchko, Z., Schlather, M. and de Haan, L. (2009). Stationary max-stable fields associated to negative definite functions. *Ann. Probab.* **37** 2042–2065. MR2561440

[18] Kabluchko, Z. and Stoev, S. (2012). Minimal spectral representations of infinitely divisible and max-infinitely divisible processes. Technical report. Available at arXiv:1207.4983.

[19] Kallenberg, O. (2002). *Foundations of Modern Probability*, 2nd ed. *Probability and Its Applications* (*New York*). New York: Springer. MR1876169

[20] Kallenberg, O. (2005). *Probabilistic Symmetries and Invariance Principles*. *Probability and Its Applications* (*New York*). New York: Springer. MR2161313

[21] Koshevoy, G. and Mosler, K. (1998). Lift zonoids, random convex hulls and the variability of random vectors. *Bernoulli* **4** 377–399. MR1653276

[22] LePage, R., Woodroofe, M. and Zinn, J. (1981). Convergence to a stable distribution via order statistics. *Ann. Probab.* **9** 624–632. MR0624688

[23] Molchanov, I. (2005). *Theory of Random Sets*. *Probability and Its Applications* (*New York*). London: Springer. MR2132405

[24] Molchanov, I. (2008). Convex geometry of max-stable distributions. *Extremes* **11** 235–259. MR2429906

[25] Molchanov, I. (2009). Convex and star-shaped sets associated with multivariate stable distributions. I. Moments and densities. *J. Multivariate Anal.* **100** 2195–2213. MR2560363

[26] Molchanov, I. and Schmutz, M. (2010). Multivariate extension of put-call symmetry. *SIAM J. Financial Math.* **1** 396–426. MR2652071

[27] Molchanov, I. and Schmutz, M. (2011). Exchangeability-type properties of asset prices. *Adv. in Appl. Probab.* **43** 666–687. MR2858216

[28] Molchanov, I. and Schmutz, M. (2013). Multiasset derivatives and joint distributions of asset prices. In *Musiela Festschrift* (Y. Kabanov, ed.). Berlin: Springer. To appear.

[29] Mosler, K. (2002). *Multivariate Dispersion, Central Regions and Depth: The Lift Zonoid Approach. Lecture Notes in Statistics* **165**. Berlin: Springer. MR1913862

[30] Rosiński, J. (1995). On the structure of stationary stable processes. *Ann. Probab.* **23** 1163–1187. MR1349166

[31] Rosiński, J. (2000). Decomposition of stationary α-stable random fields. *Ann. Probab.* **28** 1797–1813. MR1813849

[32] Rosiński, J. (2006). Minimal integral representations of stable processes. *Probab. Math. Statist.* **26** 121–142. MR2301892

[33] Samorodnitsky, G. and Taqqu, M.S. (1994). *Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance. Stochastic Modeling*. New York: Chapman & Hall. MR1280932

[34] Sato, K.i. (1999). *Lévy Processes and Infinitely Divisible Distributions. Cambridge Studies in Advanced Mathematics* **68**. Cambridge: Cambridge Univ. Press. MR1739520

[35] Sato, K.i. (2000). Density transformation in Lévy processes. MaPhySto, Lecture Notes, 7.

[36] Schmutz, M. (2011). Semi-static hedging for certain Margrabe-type options with barriers. *Quant. Finance* **11** 979–986. MR2813831

[37] Schneider, R. (1993). *Convex Bodies: The Brunn–Minkowski Theory. Encyclopedia of Mathematics and Its Applications* **44**. Cambridge: Cambridge Univ. Press. MR1216521

[38] Tehranchi, M.R. (2009). Symmetric martingales and symmetric smiles. *Stochastic Process. Appl.* **119** 3785–3797. MR2568296

[39] Wang, Y. and Stoev, S.A. (2010). On the association of sum- and max-stable processes. *Statist. Probab. Lett.* **80** 480–488. MR2593589

[40] Wang, Y. and Stoev, S.A. (2010). On the structure and representations of max-stable processes. *Adv. in Appl. Probab.* **42** 855–877. MR2779562

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