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Chern-Simons theory on Seifert 3-manifolds

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Abstract: We study Chern-Simons theory on 3-manifolds M that are circle-bundles over 2-dimensional orbifolds Σ by the method of Abelianisation. This method, which completely sidesteps the issue of having to integrate over the moduli space of non-Abelian flat connections, reduces the complete partition function of the non-Abelian theory on M to a 2-dimensional Abelian theory on the orbifold Σ, which is easily evaluated.

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Contents

1 Introduction

Chern-Simons theory [24] has been with us now for about 25 years. The Chern-Simons path integral, at every level $k \in \mathbb{Z}$ and for Lie group G,

$$Z_{CS}[M, G] = \int_{\mathcal{A}} \exp\left(i\frac{k}{4\pi}\int_M \mathrm{Tr}\, AdA + \frac{2}{3}A^3\right) \tag{1.1}$$

gives us a (framed) invariant of the 3-manifold M. Witten [24] and Reshetikhin and Turaev [21] gave surgery prescriptions for these invariants (the first based on conformal field theory, the second on quantum groups).

Very early on Freed and Gompf [10] expressed the invariant for Seifert 3-manifolds and the group $G = \mathrm{SU}(2)$ in terms of the S and T matrices of conformal field theory. Jeffrey [12] obtained rather more explicit formulae for Lens spaces. Lawrence and Rozansky [15] obtained just as explicit results for Seifert rational homology spheres ($\mathbb{Q}$HS's). Mariño [16] extended the results of [15] to compact simply-laced G. Interestingly enough Lawrence and Rozansky and Mariño were predominantly interested in obtaining asymptotic formulae around the (isolated) trivial connection from the exact result.

Unlike the surgery prescription, strategies for the exact evaluation of the path integral formulation of Chern-Simons theory are few and far between. There have been many studies of the perturbative aspects of the theory from the path integral view point, unfortunately far too many to review here. But, as already mentioned, there is a dearth of exact evaluations based directly on the path integral. There are some exceptions to this however. One such exception is due to Jeffrey [12] who evaluated the partition function of Chern-Simons in the semi-classical approximation for mapping tori (at least formally). Another is our evaluation of the path integral on 3-manifolds of the form $\Sigma \times S^1$ for Σ a genus g Riemann surface [5].

Somewhat more recently Beasley and Witten [3] have developed a localisation procedure for the path integral that extracts the contribution around isolated connections. This method, based on non-Abelian localisation [23], requires a contact structure to be chosen on the 3-manifold and, for calculations, a U(1) action is also required. The two requirements essentially fix one to Seifert $\mathbb{Q}$HS's. Beasley [2] has extended the approach to include the expectation value of Wilson loops along the U(1) fibre.

In [7] we were able to extend the diagonalisation techniques introduced in [5] to manifolds which are circle bundles over smooth Riemann surfaces, in particular to the Lens spaces $L(p, 1)$. This work has some similarity to [2, 3] but perhaps the biggest difference is that while these authors obtain contributions about particular connections our technique evaluates the complete path integral. Furthermore, the formulae obtained, unlike those that come from a semi-classical approximation, do not involve complicated integrals over moduli spaces of flat connections but rather integrals over the Cartan subalgebra of the gauge group.

The present paper is a continuation of [7] to 3-manifolds which are U(1) bundles over orbifolds. One motivation for the present study is, then, to apply the procedure of Abelianisation in the case that the smooth 3-manifold is a circle V-bundle over an orbifold. Understanding the correct condition when diagonalising in this context is the main technical difficulty.

Apart from the intrinsic importance of being able to evaluate the path integral for the Chern-Simons partition function, there is also the benefit from the possibility to use the techniques in other situations. In particular if there is no obvious, or perhaps obviously practical, surgery prescription, then other means are needed to glean non-perturbative information. There are a number of such situations which we will address elsewhere. These include:

- Three dimensional BF theory which is an example of a theory for which there is no known surgery prescription and to which our methods apply. Such theories are of interest because of their relation to gravity [25].

- The $N_T = 2$ topological supersymmetric extension of BF theory on Seifert $\mathbb{Q}$HS's. These theories are presentations of the Casson invariant and its generalisations [8].

- Yang-Mills theories on 2-dimensional orbifolds. Even though these are related to Yang-Mills theories on smooth surfaces with 'parabolic points' [9] they may be of independent interest.

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Another motivation comes from the fact that the Chern-Simons partition function on particular Seifert manifolds is the same as certain intersection pairings on spaces of connections on a Riemann surface. This relationship then gives a geometric meaning to the Chern-Simons invariants of these manifolds (and of knots in them). Certainly one application is a slightly different geometric understanding of the partition function as intersection pairings on the infinite dimensional space of connections modulo the gauge group. This can be established by making use of the topological supersymmetric extension of Chern-Simons introduced in [22].

There is another use for the toplogical supersymmetric extension of [22]. Namely, Källén [14] views the Chern-Simons action as an observable within the topologically twisted Yang-Mills theory and then uses cohomological localisation to reproduce [3] around the trivial connection. Ohta and Yoshida [20] combine the topologically twisted Yang-Mills theory of [14] and diagonalisation [5, 7] to evaluate the path integral for various supersymmetric Yang-Mills Chern-Simons theories.

The contents of this paper are as follows:

We begin with the formula, due to Lawrence and Rozansky [15], for the partition function of SU(2) Chern-Simons theory on a Seifert $\mathbb{Q}$HS and its generalisation due to Mariño [16]. We show how these formulae can be written in various suggestive ways as integrals over the Cartan subalgebra of the group in question. Such a formulation is a preparation for formulae that arise on evaluating the path integral by diagonalisation.

Then we turn to a brief description of Seifert 3-manifolds as S^1 V-bundles over 2-dimensional orbifolds. In particular we introduce the conditions that a Seifert manifold be a $\mathbb{Q}[g]$HS which has, apart from an extra $\mathbb{Z}^{2g}$ summand in $\mathrm{H}_1(M)$, the homology of a $\mathbb{Q}$HS. We are able to evalute the Chern-Simons path integral on this class of Seifert manifolds.

Next we come to the crux of the matter, namely diagonalising a component of the gauge connection so that it lies in the Cartan subalgebra of the Lie algebra of the gauge group. On doing this we are left with a sequence of Gaussian integrals to perform and a price to be paid. That price is that the abelianised field is correctly thought of as a section of certain line bundles over the orbifold and part of our task is to determine which line bundles.

Once the bundles that arise on Abelianisation are clear, apart from some arithmetic the evaluation of the Chern-Simons path integral is almost identical to that presented in [7] and so we will be rather brief about the details. As noted above, we are basically evaluating some Gaussian path integrals which give rise to determinants. The evaluation proceeds in a sequence of steps. Firstly we split the functional determinants into their absolute value and the phase. Then we give a zeta function and eta function type regularisation of these. An application of the Riemann-Roch-Kawasaki index theorem [13] (the extension of the Riemann-Roch theorem to V-manifolds or orbifolds) allows us to push the calculation down to the orbifold. Finally, we can make use of an orbifold version of Hodge theory [1] to evaluate the resulting Abelian theory.

The last section deals with the evaluation of the expectation value of particular Wilson loops. These are the lines that wrap around the S^1 fibration of M. This is quite

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straightforward to do as these Wilson loops do not interfere with the previous method of evaluation as the Gaussian nature of the model is maintained.

There are two appendices. The first gives an example of diagonalisation of a smooth section of a (smooth) SU(2) bundle over M and what that means for summing bundles over the orbifold base of M. This is intended to motivate the choices made in the body of the paper. In the second appendix we give the generators and relations for $\pi_1(M)$ and explicit forms for their abelianisation. Along the way we also give an example of an irreducible non-Abelian connection of the type that we do not have to take into account in our evaluation of the path integral.

2 The formulae of Lawrence-Rozansky and Mariño

The formula found by Lawrence and Rozansky [15] for the partition function of Chern-Simons theory of a Seifert $\mathbb{Q}$HS M, up to an overall constant, is

$$Z_{CS}(M, \mathrm{SU}(2)) = \sum_{r=-Pk_{\mathfrak{g}}}^{Pk_{\mathfrak{g}}} \mathrm{e}^{-\frac{i\pi d}{2Pk_{\mathfrak{g}}}r^2} \left(\mathrm{e}^{\frac{i\pi r}{k_{\mathfrak{g}}}} - \mathrm{e}^{-\frac{i\pi r}{k_{\mathfrak{g}}}} \right)^{2-N} \prod_{i=1}^{N} \left(\mathrm{e}^{\frac{i\pi r}{a_i k_{\mathfrak{g}}}} - \mathrm{e}^{-\frac{i\pi r}{a_i k_{\mathfrak{g}}}} \right) \tag{2.1}$$

where the a_i for $i = 1, \ldots, N$ are part of the data of a Seifert manifold M (see section 3 for more details) and $P = \prod_{i=1}^{N} a_i$, while $|d|$ is the order of $\mathrm{H}_1(M, \mathbb{Q})$ (with $d = \pm|d|$ corresponding to the two choices of orientation). We generally denote $k_{\mathfrak{g}} = k + c_{\mathfrak{g}}$ where $c_{\mathfrak{g}}$ is the dual Coxetor number for the group G. In this formula, and its generalisation to other simply-laced groups G (see [16] and the discussion just after (4.9) there), there are restrictions on the summation. Here we see that for $N > 2$

$$\left(\mathrm{e}^{\frac{i\pi r}{k_{\mathfrak{g}}}} - \mathrm{e}^{-\frac{i\pi r}{k_{\mathfrak{g}}}} \right)^{2-N}$$

diverges whenever $k_{\mathfrak{g}}|r$ and it is these points which are discarded in the sum. This is the analogue of a similar issue that arose in the path integral derivation of the Verlinde formula for the dimension of the space of conformal blocks [5] and we deal with it in the same way here, as we describe below.

As in [7] we introduce a gauge invariant partition function

$$Z_{q,P,d}(f) = \frac{q^{\mathbf{rk}}}{|W|\,V} \sum_{s \in \mathbb{Z}^{\mathbf{rk}}} \int_{\mathfrak{t}} f(\phi) \exp\left(i\frac{q}{4\pi}\frac{d}{P} \mathrm{Tr}\, \phi^2 + iq\, \mathrm{Tr}\, s\phi \right) \tag{2.2}$$

Here W is the Weyl group which acts by permutation on the Cartan elements, $\mathbf{rk}$ is the rank of the group G, q is an integer (later to be identified with $k_{\mathfrak{g}}$), $f(\phi)$ is any function which is invariant under both the shift $\phi \to \phi + 2\pi P$ and the action of W and $V = \mathbf{Vol}(d\mathbb{Z}^{\mathbf{rk}})$. We have set $\phi = \sum_i \phi^i \alpha_i$ and $s = \sum_i s^i \alpha_i$ where α_i are simple roots of a group G (for simplicity in the following we consider G to be simply-laced). The gauge symmetry that is enjoyed by this partition function is, with $n = \sum_i n^i \alpha_i$ and the $n^i \in \mathbb{Z}$,

$$\phi \to \phi + 2\pi n P, \quad s \to s - dn \tag{2.3}$$

Note that this says that we may shift ϕ/P by elements of the integral lattice I. The discrete group that acts is then the affine Weyl group $\Gamma^W = I \rtimes W$.

Now, using I we can either gauge fix ϕ to lie between $-\pi P \leq \phi \leq \pi P$ or we can gauge fix s so that $s \in \mathbb{Z}_d$, or we can use the whole affine Weyl group to restrict ϕ to $\mathfrak{t}/\Gamma^W$. Therefore we arrive at the equalities

$$\begin{aligned} Z_{q,P,d}(f) &= \frac{1}{|W|} \sum_{s\in\mathbb{Z}^{\mathbf{rk}}} \int_{-\pi Pq}^{\pi Pq} \cdots \int_{-\pi Pq}^{\pi Pq} f(\phi/q)\, \exp\left(\frac{i}{4\pi} \frac{d}{Pq} \operatorname{Tr}\phi^2 + i \operatorname{Tr} s\phi \right) \\ &= \frac{1}{|W|} \sum_{s\in\mathbb{Z}_d} \int_{\mathfrak{t}} f(\phi/q)\, \exp\left(\frac{i}{4\pi} \frac{d}{Pq} \operatorname{Tr}\phi^2 + i \operatorname{Tr} s\phi \right) \\ &= \sum_{s\in\mathbb{Z}^{\mathbf{rk}}} \int_{\mathfrak{t}/\Gamma^W} f(\phi)\, \exp\left(i \frac{q}{4\pi} \frac{d}{P} \operatorname{Tr}\phi^2 + iq \operatorname{Tr} s\phi \right) \end{aligned} \tag{2.4}$$

The sum over s in the first of the equalities of (2.4) fixes $\phi = r\pi$ for $r \in \mathbb{Z}^{\mathbf{rk}}$, while the range of integration restricts each of the possible integers r to lie in $-Pq \leq r \leq Pq$. Consequently we have

$$Z_{q,P,d}(f) = \frac{1}{|W|} \sum_{r\in\mathbb{Z}^{\mathbf{rk}}/2Pq\mathbb{Z}^{\mathbf{rk}}} f(\pi r/q)\, \exp\left(i\frac{\pi}{4} \frac{d}{Pq} \operatorname{Tr} r^2 \right) \tag{2.5}$$

A suitable choice of the function f then reproduces the formula (2.1) on taking $q = k_{\mathfrak{g}} = k + c_{\mathfrak{g}}$. More generally, we let $f = \sqrt{T_M(\phi, a_1, \ldots, a_N)}$ (the positive root) where

$$T_M(\phi;\, a_1, \ldots, a_N) = T_{S^1}(\phi)^{2-2g-N}.\prod_{i=1}^{N} T_{S^1}(\phi/a_i) \tag{2.6}$$

to reproduce the formulae of Mariño [16]. This formula relates the Ray-Singer torsion of M, $T_M(\phi,\, a_1, \ldots, a_N)$ to $T_{S^1}(\phi/a)$ which is the Ray-Singer torsion on the circle (modulo $\mathbb{Z}_a$) evaluated at a flat connection $(\phi/a)\, d\theta$.

Our evaluation of the Chern-Simons path integral will eventually lead us to an expression of the form (2.2), with $f(\phi)$ precisely as above, and at that point we can appeal to the above discussion to establish the connection with the results of [15, 16].

3 Seifert 3-manifolds

We will consider Chern-Simons theory on 3-manifolds M which are themselves principal U(1) V-bundles $U(1) \to M \xrightarrow{\pi} \Sigma$ over 2-dimensional orbifolds Σ of genus g. We suppose that there are N orbifold points on Σ which are modeled on $\mathbb{C}/\mathbb{Z}_{a_i}$. A line V-bundle, away from orbifold points is characterised as an ordinary line bundle but is given, in the neighbourhood of the i-th orbifold point, by the identification on $D \times \mathbb{C}$,

$$(z, w) \simeq (\zeta.z,\, \zeta^{b_i}.w), \quad \zeta = \exp(2\pi i/a_i)$$

where $D \subset \mathbb{C}$ is a disc centred at the orbifold point.

$M = S(\mathcal{L})$ is then a circle V-bundle associated to a line V-bundle $\mathcal{L}$. The data which goes into specifying M is $[\deg(\mathcal{L}),\, g,\, (a_1, b_1), \dots, (a_N, b_N)]$.

The Seifert manifold M is smooth if

$$\gcd(a_i, b_i) = 1, \quad \text{for } i = 1, \dots, N$$

which means, in particular, that the $b_i \neq 0$. The Seifert manifold will be an $\mathbb{Z}$HS (integral homology sphere) iff the line bundle $\mathcal{L}_0$ that defines it satisfies

$$g = 0, \quad c_1(\mathcal{L}_0) = \pm \prod_{i=1}^{N} \frac{1}{a_i} \tag{3.1}$$

The last condition implies that the numbers a_i be pairwise relatively prime[1] so that one has the arithmetic condition

$$\gcd(a_i, a_j) = 1, \quad i \neq j$$

As an example the Poincaré $\mathbb{Z}$HS $M = \Sigma(2, 3, 5)$ has the two possible descriptions $[-1, 0, (2, 1), (3, 1), (5, 1)]$ with $c_1(\mathcal{L}_0) = 1/(2.3.5)$ and $[-2, 0, (2, 1), (3, 2), (5, 4)]$ with $c_1(\mathcal{L}_0) = -1/(2.3.5)$. Quite generally, if $[\deg(\mathcal{L}),\, g,\, (a_1, b_1), \dots, (a_N, b_N)]$ is a manifold with $c_1(\mathcal{L}_0) = \pm 1/(a_1 \dots a_N)$, then $[-\deg(\mathcal{L}) - N,\, g,\, (a_1, a_1 - b_1), \dots, (a_N, a_N - b_N)]$ is one with $c_1(\mathcal{L}_0) = \mp 1/(a_1 \dots a_N)$ (we are taking the inverse line bundle).[2]

If one takes M to be the total space of the circle bundle of $\mathcal{L}_0^{\otimes d}$, rather than that of $\mathcal{L}_0$, then M is a $\mathbb{Q}$HS (rational homology sphere) with

$$|d| = |\mathrm{H}_1(M, \mathbb{Z})|$$

In both of these cases, as the a_i are mutually coprime, all line V-bundles on Σ are some tensor power of $\mathcal{L}_0$.

The Gysin sequence played an important role in our previous evaluation of the path integral on U(1) bundles over smooth curves allowing us to count U(1) bundles over the total space which are pullbacks from the base. Likewise, we would like to know the image of the pullback map

$$\mathrm{Pic}(\Sigma) \xrightarrow{\pi^*} [\text{Line bundles over } M] \xrightarrow{c_1} \mathrm{H}^2(M, \mathbb{Z})$$

Fortunately there is a Gysin sequence for U(1) V-bundles over 2-dimensional orbifolds [11] which gives us the required information. The required result is part of Theorem 2.3 in [11] (see also Remark 2.0.20 in [18]) for M smooth,

$$\mathrm{H}^2(M, \mathbb{Z}) \cong \mathrm{Pic}(\Sigma)/\mathbb{Z}[\mathcal{L}] \oplus \mathbb{Z}^{2g} \tag{3.2}$$

[1] From (3.4) and (3.1) we have that for M to be a $\mathbb{Z}$HS that $(\prod_i a_i).(n + \sum_j b_j/a_j) = \pm 1$ where n is the degree of the bundle that defines M. Now suppose that the greatest common divisor of two of the a_i is not unity and, by re-ordering if required, let those two be a_1 and a_2 and such that $a_2 = t a_1$ ($t \in \mathbb{Z}_{>0}$). The equation to be solved becomes $a_1.m = \pm 1$ with $m = (\prod_{j \geq 3} a_j)[t a_1 (n + \sum_{i \geq 3} b_i/a_i) + t b_1 + b_2]$ and clearly $m \in \mathbb{Z}$ so there is no solution. Consequently, for M to be a $\mathbb{Z}$HS the a_i must be pairwise relatively prime.

[2] That the degree in the examples is always negative is not an accident. The equation to solve is $n(\prod_j a_j) + s = \pm 1$ with $s = (\prod_j a_j).\sum_i b_i/a_i$ a positive integer so that $n \leq 0$ given that the $a_i \geq 2$ and the $b_i \geq 1$.

where $M = S(\mathcal{L})$. It is the subgroup $\mathrm{Pic}(\Sigma)/\mathbb{Z}[\mathcal{L}] \subset \mathrm{H}^2(M,\mathbb{Z})$ which is the image of the pullback map and when $c_1(\mathcal{L}) \neq 0$ this is finite and Abelian. When M is a $\mathbb{Q}$HS, then $\mathcal{L} = \mathcal{L}_0^{\otimes d}$ and $\mathrm{Pic}(\Sigma) = \mathbb{Z}[\mathcal{L}_0]$ so that $\mathrm{Pic}(\Sigma)/\mathbb{Z}[\mathcal{L}] \cong \mathbb{Z}_d$.

Choice of manifold. The technique that we make use of is Abelianisation. In particular, we diagonalise sections $\phi : M \longrightarrow \mathrm{ad}\, G$ of the adjoint bundle which are constant along the fibres of M, in the sense of conjugating them into maps into the Cartan sublagebra. For reasons disucssed below we consider the case that G is simply-connected, so that $\mathrm{ad}\, G \cong M \times \mathfrak{g}$.

However, even in this case there are topological obstructions to Abelianisation and, if one insists on diagonalising anyway, the price to be paid is the 'liberation' of non-trivial line bundles on the base of the fibration $S^1 \longrightarrow M \longrightarrow \Sigma$. An important part of the technique, then, is to be able to determine which line bundles we will need to count. In the case of trivial bundles $M = S^1 \times \Sigma$ over a smooth Riemann surface we find that we must count all possible line bundles on Σ. In the case of nontrivial bundles, the circle bundle of a non-trivial line bundle $\mathcal{L}$, over a smooth Riemann surface one counts the $c_1(\mathcal{L})$ available torsion bundles (these arise as $\pi^*(\mathcal{L}^{\otimes c_1(\mathcal{L})}) = \mathcal{O}$).

Hence, we need to be able to follow the line bundles which are available. This is most easily done if there is only one generator that pulls back to the 3-manifold. This is the case in the examples of the previous paragraph. Other examples include particular smooth Seifert manifolds which are constructed as follows. Let Σ be a genus g Riemann surface with N orbifold points $\{p_i\}$ such that the isotropy data a_i at the points p_i are relatively prime $\gcd(a_i, a_j) = 1$ for $i \neq j$. As the line bundle $\mathcal{L}_0$ with $c_1(\mathcal{L}_0) = \prod(a_i)^{-1}$ generates the Picard group of orbifold line bundles on Σ (3.2) the pullback to $S(\mathcal{L}_0)$ of any orbifold line bundle is trivial. (This is the orbifold analogue of the fact that the pullback of any line bundle on S^2 to S^3 is trivial.) However, there is an important caveat. The G bundle that we started with is a smooth bundle and can be thought of as the pullback of an honest G bundle on Σ, i.e. one with trivial isotropy data at the orbifold points. Consequently the line bundles that appear on diagonalisation must be honest line bundles (there is no special discrete action over the orbifold points). All such line bundles on Σ are powers of $\mathcal{L}_P = \mathcal{L}_0^{\otimes P}$ where

$$P = \prod_{i=1}^{N} a_i \tag{3.3}$$

and it is these line bundles, if we are to sum, that we should sum over (though they all pullback to the trivial line bundle).

Notice, in the above discussion, that for diagonalisation we do not need the extra condition that the genus of the Riemann surface vanish, so we are not only dealing with $\mathbb{Z}$HS's (3.1). For brevity we will denote those $M = S(\mathcal{L}_0)$ by $\mathbb{Z}[g]$HS's when we relax the condition on the genus.

On the other hand the manifold $S(\mathcal{L}_0^{\otimes d})$ is such that there are d torsion bundles available and, on diagonalising, we would have to count these (the $S(\mathcal{L}_0^{\otimes d})$ having the same relationship to $S(\mathcal{L}_0)$ as the Lens spaces have to S^3). As we have seen, in order to keep track of the fact that our G bundle is smooth we should consider the torsion bundles to be of the form $\mathcal{L}_P^{\otimes m}$ for $m \in \mathbb{Z}_d$. Once more we may also consider that $g \neq 0$ and we denote such manifolds as $\mathbb{Q}[g]$HS's.

Degree and first Chern number. There is some disparity in the literature regarding the nomenclature used with regards to Chern classes, degree and so on. We adopt the notation that

$$c_1(\mathcal{L}) = \deg(\mathcal{L}) + \sum_{i=1}^{N} \frac{b_i(\mathcal{L})}{a_i} \tag{3.4}$$

where the degree $\deg(\mathcal{L})$ is an integer and the isotropy weights $b_i(\mathcal{L})$ each satisfy

$$0 \leq b_i(\mathcal{L}) < a_i$$

for every line bundle $\mathcal{L}$. Hence, with our definition, $c_1(\mathcal{L}) \in \mathbb{Q}$. Note that $\deg(\mathcal{L}) = c_1(|\mathcal{L}|) \in \mathbb{Z}$ where $|\mathcal{L}|$ is the associated (smooth) line bundle on the smooth curve $|\Sigma|$ (by smoothing the orbifold points and taking no isotropy there, $b_i(\mathcal{L}) = 0$).

One way to think about this is as follows: A line bundle is equivalent to a divisor, which in this case is a (smooth) point on Σ. Each smooth point comes with weight one. A degree n line bundle is the same as the sum of n divisors (say n times one divisor). The orbifold points $\{p_i\}$ have weight $1/a_i$ and so they correspond to line V-bundles with 'degree' $1/a_i$. If one considers the divisor $a_i.\{p_i\}$ (which is now like a smooth point) then the associated line V-bundle is a line bundle and its degree feeds into deg.

A fact which will be important for us later is that while the first Chern class behaves well under tensor product,

$$c_1(\mathcal{L} \otimes \mathcal{K}) = c_1(\mathcal{L}) + c_1(\mathcal{K})$$

the degree does not. Rather, one has from this formula and the definition,

$$c_1(\mathcal{L} \otimes \mathcal{K}) = \deg(\mathcal{L} \otimes \mathcal{K}) + \sum_{i=1}^{N} \frac{b_i(\mathcal{L} \otimes \mathcal{K})}{a_i}$$

with $0 \leq b_i(\mathcal{L} \otimes \mathcal{K}) < a_i$ that

$$\deg(\mathcal{L} \otimes \mathcal{K}) = \deg(\mathcal{L}) + \deg(\mathcal{K}) + \sum_{i=1}^{N} \left\lfloor \frac{b_i(\mathcal{L}) + b_i(\mathcal{K})}{a_i} \right\rfloor \tag{3.5}$$

where $\lfloor x \rfloor$ is the floor function

$$\lfloor x \rfloor = \max\{n \in \mathbb{Z} \,|\, x \geq n\} \tag{3.6}$$

and is such that

$$\lfloor -x \rfloor = \begin{cases} -1 - \lfloor x \rfloor & x \in \mathbb{R} \backslash \mathbb{Z} \\ -\lfloor x \rfloor & x \in \mathbb{Z} \end{cases} \tag{3.7}$$

and where the isotropy weights satisfy

$$b_i(\mathcal{L}\otimes\mathcal{K}) = (b_i(\mathcal{L}) + b_i(\mathcal{K})) \mod a_i, \;\; 0 \le b_i(\mathcal{L}\otimes\mathcal{K}) < a_i$$

Introduce the symbol $((.))$ defined by

$$((x)) = \begin{cases} x - \lfloor x \rfloor - \frac{1}{2}, & x \in \mathbb{R}\backslash\mathbb{Z} \\ 0, & x \in \mathbb{Z} \end{cases}$$

which has unit period $((x+1)) = ((x))$ and is odd under change of sign $((-x)) = -((x))$.

For any line bundle $\mathcal{L}$, as $0 \le b_i(\mathcal{L}) < a_i$,

$$b_i(\mathcal{L}^{-1}) = \begin{cases} a_i - b_i(\mathcal{L}), & b_i(\mathcal{L}) \neq 0 \\ 0, & b_i(\mathcal{L}) = 0 \end{cases} \tag{3.8}$$

so that

$$\frac{b_i(\mathcal{L}) - b_i(\mathcal{L}^{-1})}{a_i} = \begin{cases} 2b_i(\mathcal{L})/a_i - 1, & b_i(\mathcal{L}) \neq 0 \\ 0, & b_i(\mathcal{L}) = 0 \end{cases} = 2\,((b_i(\mathcal{L})/a_i))$$

since $\lfloor b_i(\mathcal{L})/a_i \rfloor = 0$. Consequently,

$$\deg(\mathcal{L}) - \deg(\mathcal{L}^{-1}) = 2\,c_1(\mathcal{L}) - 2\sum_{i=1}^{N}((b_i(\mathcal{L})/a_i))$$

This trick we took from [19].

Likewise, providing that $\gcd(a_i, b_i(\mathcal{L})) = 1$,

$$\deg(\mathcal{L}^{\otimes n}) + \deg(\mathcal{L}^{-\otimes n}) = -N + \sum_{i=1}^{N}\phi_{a_i}(n)$$

where

$$\phi_{a_i}(n) = \begin{cases} 1 \text{ if } a_i | n \\ 0 \text{ otherwise} \end{cases} \tag{3.9}$$

is a function introduced in [3]. Notice that the function $\phi_{a_i}(n)$ does not depend on the line bundle $\mathcal{L}$ but just on the requirement that $\gcd(a_i, b_i(\mathcal{L})) = 1$.

For 'honest' line bundles $\mathcal{K}$ (i.e. having isotropy data $b_i(\mathcal{K}) = 0\ \forall i$) the degree and first Chern class agree

$$c_1(\mathcal{K}) = \deg(\mathcal{K})$$

Moreover, if $\mathcal{L}$ is a V-line bundle and $\mathcal{K}$ a line bundle, we have

$$\deg(\mathcal{K}\otimes\mathcal{L}) = \deg(\mathcal{K}) + \deg(\mathcal{L})$$

so that

$$\deg(\mathcal{L}^{\otimes n}\otimes\mathcal{K}) + \deg(\mathcal{L}^{-\otimes n}\otimes\mathcal{K}^{-1}) = -N + \sum_{i=1}^{N}\phi_{a_i}(n) \tag{3.10}$$

is independent of $\mathcal{K}$ and

$$\deg(\mathcal{L}\otimes\mathcal{K}) - \deg(\mathcal{L}^{-1}\otimes\mathcal{K}^{-1}) = 2\,\deg(\mathcal{K}) + 2\,c_1(\mathcal{L}) - 2\sum_{i=1}^{N}((b_i(\mathcal{L})/a_i)) \tag{3.11}$$

The principal bundle structure on M. Let κ be a connection on the principal U(1) V-bundle $\mathcal{L}$ that defines our 3-manifold $M = S(\mathcal{L})$. We think of κ as a globally defined real-valued 1-form on the total space of the bundle, and denote by ξ the fundamental (or Reeb) vector field on M, i.e. the generator of the U(1)-action. A U(1) connection κ is characterised by the two conditions

$$\iota_\xi \kappa = 1, \quad L_\xi \, \kappa = 0 \tag{3.12}$$

where $L_\xi = \{d\,,\,\iota_\xi\}$ is the Lie derivative in the ξ direction which imply that $\iota_\xi d\kappa = 0$, so that the curvature 2-form $d\kappa$ of κ is horizontal, as behoves the curvature of a connection.

In local coordinates one has

$$\kappa = d\theta + \beta \ , \tag{3.13}$$

where θ is a fibre coordinate, $0 \leq \theta < 1$, and $\beta = \beta_\mu \, dx^\mu$ is a local representative on Σ of the connection κ on M.

Our orientation conventions [7] are such that $d\kappa$ is minus the Euler class or first Chern class of the bundle over Σ,

$$c_1(\mathcal{L}) = \int_\Sigma -d\kappa$$

and since M has $c_1(\mathcal{L}) = (n + \sum_i b_i(\mathcal{L})/a_i)$, we choose β so that the curvature 2-form satisfies

$$d\kappa = -(n + \sum_{i=1}^{N} \frac{b_i(\mathcal{L})}{a_i})\, \pi^*(\omega) \tag{3.14}$$

for ω a unit normalised symplectic form on Σ.

For $c_1(\mathcal{L}) \neq 0$ a choice of κ equips M with a contact structure, such that $\kappa \wedge d\kappa$ is nowhere vanishing on M. Indeed,

$$\kappa \wedge d\kappa = -(n + \sum_{i=1}^{N} \frac{b_i(\mathcal{L})}{a_i})\, d\theta \wedge \pi^*(\omega) \tag{3.15}$$

is nowhere vanishing as required providing that the U(1) V-bundle is non-trivial. We also note that

$$\int_M \kappa \wedge d\kappa = -(n + \sum_{i=1}^{N} \frac{b_i(\mathcal{L})}{a_i}) \int_\Sigma \omega = -c_1(\mathcal{L}) \ . \tag{3.16}$$

4 Chern-Simons theory on Seifert 3-manifolds

Much of the construction that we use has been explained in great detail in [7] so we will be very brief about it here. We fix the gauge group G to be compact, semi-simple, and simply connected so that the principal G-bundle on the 3-manifold M and all its associated vector bundles are trivial. In principle the extension to trivial bundles for non-simply

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connected groups is reasonably straightforward (and will mainly lead to a few extra signs in the formulae), while the extension to non-trivial bundles of non-simply connected groups requires some more thought in relation with diagonalisation and the argument based on the Gysin sequence.

Given that κ is nowhere vanishing, any one-form $\beta \in \Omega^1(M, \mathbb{R})$ may be decomposed as $\beta = \beta_\kappa + \beta_H$ with

$$\beta_\kappa = \kappa \wedge \iota_\xi \, \beta \in \Omega^1_\kappa(M, \mathbb{R}), \quad \beta_H = (1 - \kappa \wedge \iota_\xi) \, \beta \in \Omega^1_H(M, \mathbb{R}). \tag{4.1}$$

One may also decompose connections, thought of as elements of $\Omega^1(M, \mathfrak{g})$,

$$A = A_\kappa + A_H \equiv \phi \, \kappa + A_H. \tag{4.2}$$

and as $\phi \in \Omega^0(M, \mathfrak{g})$ it is correctly thought of as a section of the adjoint bundle $E = M \times \mathfrak{g}$.

The level k Chern-Simons action is

$$\begin{aligned} kS_{CS}[A] &= \frac{k}{4\pi} \int_M \mathrm{Tr} \left(AdA + \frac{2}{3} A^3 \right) \\ &= \frac{k}{4\pi} \int_M \mathrm{Tr} \left(A_H \wedge \kappa \wedge L_\phi \, A_H + 2\phi \, \kappa \wedge d \, A_H + \phi^2 \, \kappa \wedge d \, \kappa \right) \quad . \end{aligned} \tag{4.3}$$

We have changed notation from [7] for the Lie derivative to $L_\xi = \{\iota_\xi, \, d\}$ and for the covariant Lie derivative to $L_\phi = L_\xi + [\phi,$ from $\mathcal{L}_\xi$ and $\mathcal{L}_\phi$ in order to avoid conflict with our notation for bundles.

Gauge conditions. We impose the gauge condition

$$L_\xi A_\kappa = 0 \Leftrightarrow L_\xi \phi = \iota_\xi \, d \, \phi = 0 \quad . \tag{4.4}$$

This gauge condition, $L_\xi \phi = 0$, tells us that ϕ is a U(1)-invariant section of E. Equivalently, it can therefore be regarded as a section of the (trivial) adjoint V-bundle V over Σ. Having pushed down ϕ to Σ in this manner, we can now proceed to the diagonalisation of ϕ as in [5]. Thus let T be some maximal torus of G and $\mathfrak{t}$ the corresponding Cartan subalgebra, with $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{k}$ and set

$$\phi^{\mathfrak{k}} = 0. \tag{4.5}$$

As shown in [5, 6], and discussed previously, there is a price to pay for diagonalising sections of V (in the sense of conjugating them into maps taking values in the Cartan subalgebra $\mathfrak{t}$).

Up to this point we have not imposed any particular conditions on M. However, in order to determine the obstructions in a simple way we ask that $M = S(\mathcal{L}_0)$ with $\gcd(a_i, a_j) = 1$ (for $i \neq j$) and that $c_1(\mathcal{L}_0) = (\prod_i a_i)^{-1}$ or that $M = S(\mathcal{L}_0^{\otimes d})$. With the condition that $M = S(\mathcal{L}_0^{\otimes d})$ we must sum over all T-bundles on M that one gets by pullback of certain T bundles from Σ. The line bundles on Σ are generated by $\mathcal{L}_0$ (by Theorem 2.3 of [11]), and as we have already explained, the ones of interest to us are powers of $\mathcal{L}_P = \mathcal{L}_0^{\otimes P}$ (3.3). The pull-backs of the $\mathcal{L}_P$ from Σ to M are of finite order and all torsion bundles on M arise in this way, so that it is precisely these bundles that we should sum over in the path integral.

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More conditions on ϕ. Those $A_H^{\mathfrak{t}}$ which are U(1) invariant, $L_\xi\, A_H^{\mathfrak{t}} = 0$, do not appear in the kinetic term $A_H \wedge L_\phi A_H$ and so they can only appear in the mixed kinetic term $2\phi\, \kappa \wedge d\, A_H$. The path integral over such $A_H^{\mathfrak{t}}$ then imposes the condition $\iota_\xi d(\kappa\, \phi) = 0$. This delta function constraint on ϕ together with the gauge condition (4.4) imply that ϕ is actually constant,

$$d\phi = 0. \tag{4.6}$$

Now with ϕ constant we have, with $M = S(\mathcal{L}_0^{\otimes d})$,

$$\int_M \kappa \wedge d\kappa \;\mathrm{Tr}\; \phi^2 = -\frac{d}{P}\;\mathrm{Tr}\;\phi^2 \tag{4.7}$$

5 Reduction to an abelian theory on Σ

Having discussed the effect of integrating out the U(1)-invariant modes of $A_H^{\mathfrak{t}}$, we now keep these and investigate what happens upon integrating out the other modes and fields, with the understanding that ϕ will ultimately turn out to be constant. All these fields appear quadratically in the action, and therefore will give rise to ratios of determinants. The definition and regularisation of these determinants for Σ smooth were specified in detail in appendix B of [7] we will take that for granted but augment that discussion here to take the orbifold points into account.

Given the choice of metric

$$g_M = \pi^* g_\Sigma \oplus \kappa \otimes \kappa \tag{5.1}$$

the operator $*\,\kappa \wedge L_\phi$ acts on the space of horizontal $\mathfrak{k}$-valued 1-forms,

$$*\;\kappa \wedge\, L_\phi : \Omega^1_H(M, \mathfrak{k}) \to \Omega^1_H(M, \mathfrak{k}). \tag{5.2}$$

Hence integrating over the $\mathfrak{k}$-components of the ghosts ghosts $(c^{\mathfrak{k}}, \overline{c}^{\mathfrak{k}})$ and the connection $A_H^{\mathfrak{k}}$, one obtains the following ratio of determinants:

$$\frac{\mathrm{Det}\,(iL_\phi)_{\Omega^0(M,\mathfrak{k})}}{\sqrt{\mathrm{Det}\,(*\,\kappa \wedge\, iL_\phi)_{\Omega^1_H(M,\mathfrak{k})}}}\ . \tag{5.3}$$

Integration over the ghosts $(c^{\mathfrak{t}}, \overline{c}^{\mathfrak{t}})$ and those $A_H^{\mathfrak{t}}$ modes which are not U(1) invariant give the following ratio of determinants:

$$\frac{\mathrm{Det}'\,(iL_\xi)_{\Omega^0(M,\mathfrak{t})}}{\sqrt{\mathrm{Det}'\,(*\,\kappa \wedge\, iL_\xi)_{\Omega^1_H(M,\mathfrak{t})}}} \tag{5.4}$$

The notation Det' indicates that the zero mode of the operator is not included.

To evaluate these ratios of determinants we expand all the fields in their Fourier modes in the ξ direction. In particular for the connection we set $A_H = \sum_{n=-\infty}^{\infty} A_n$ where the eigenmodes satisfy $L_\xi\, A_n = -2\pi i n\, A_n$ and $\iota_K\, A_n = 0$ and likewise for the ghosts c and $\overline{c}$.

These eigenmodes can equivalently be regarded as sections of line bundles $\mathcal{L}^{\otimes n}$ (where $\mathcal{L}$ defines M) over Σ (which pull back to the trivial line bundle on M). Hence we have that

$$\Omega^0(M, \mathbb{C}) = \bigoplus_n \Omega^0(\Sigma, \mathcal{L}^{\otimes n}), \tag{5.5}$$

As we have singled out the Cartan subalgebra, the bundles that we are working with effectively 'split' so we think of the charged Lie algebra valued forms on M as sections of the trivial bundle $M \times \mathfrak{k}$. In order to make a Fourier decomposition of such sections we understand each mode to be a section of a trivial bundle $V_{\mathfrak{k}}$ on Σ which pulls back to $M \times \mathfrak{k}$. Consequently, on tensoring (5.5) with the trivial bundles $V_{\mathfrak{k}}$ below and $\pi^*(V_{\mathfrak{k}}) = M \times \mathfrak{k}$ above we have

$$\Omega^0(M, \mathfrak{k}) = \bigoplus_n \Omega^0(\Sigma, \mathcal{L}^{\otimes n} \otimes V_{\mathfrak{k}}) \ . \tag{5.6}$$

A similar discussion shows that each mode n of a horizontal 1-form on M is one to one with a section on Σ, consequently one has

$$\Omega^1_H(M, \mathfrak{k}) = \bigoplus_n \Omega^1(\Sigma, \mathcal{L}^{\otimes n} \otimes V_{\mathfrak{k}}). \tag{5.7}$$

Now, as explained in [7], the ratio of determinants (5.3) and (5.4) need a definition (and regularisation). We set

$$\sqrt{\mathrm{Det}\, Q} = \sqrt{|\mathrm{Det}\, Q|}\, \exp \frac{+i\pi}{2}\, \eta(Q) \tag{5.8}$$

where $\eta(Q) = \frac{1}{2}\sum_{\lambda \in \mathrm{spec}(Q)} \mathrm{sign}(\lambda)$ and the root is the positive root for either of the operators that appear in (5.3) and (5.4). We regularise the absolute value and the phase (assuming that zero is not an eigenvalue) by setting

$$|\mathrm{Det}\, Q|\,(s) = \exp \sum_{\lambda \in \mathrm{spec}(Q)} e^{s\Delta} \ln|\lambda| \tag{5.9}$$

$$\eta(Q,\, s) = \frac{1}{2} \sum_{\lambda \in \mathrm{spec}(Q)} \frac{\mathrm{sign}(\lambda)}{|\lambda|^s} \exp s\Delta \tag{5.10}$$

for Δ an appropriate negative definite operator. As explained in [5] an appropriate choice of Δ is the Laplacian of the twisted Dolbeault operator on Σ.

In order to state the results that we borrow from [5, 7] we need to introduce some notation. Each charged section contributes to the determinant but its contribution depends on the charge, so we decompose the charge space into roots

$$V_{\mathfrak{k}} = \oplus_\alpha V_\alpha \tag{5.11}$$

The regularisation that we have chosen then leads us to considering the index of the Dolbeault operator (how this comes about can be found around (6.14) of [5]). Now the Riemann-Roch-Kawasaki index theorem for a line V-bundle $\mathcal{L}$ on an orbifold [13] states that

$$\mathrm{Index}(\overline{\partial}_{\mathcal{L}}) \equiv \chi(\Sigma,\, \mathcal{L}) \equiv \dim_{\mathbb{C}} \mathrm{H}^0(\Sigma,\, \mathcal{L}) - \dim_{\mathbb{C}} \mathrm{H}^1(\Sigma,\, \mathcal{L}) = \deg(\mathcal{L}) + 1 - g \tag{5.12}$$

and one should note that it is the degree that enters and not the first Chern class.

Returning to the determinants, we find that as far as the norm is concerned it reduces to

$$\sqrt{\prod_{\alpha}\prod_{n}(2\pi n+i\alpha(\phi))^{\chi(\Sigma,\mathcal{L}^{\otimes n}\otimes V_{\alpha})-\chi(\Sigma,K_{\Sigma}\otimes\mathcal{L}^{\otimes n}\otimes V_{\alpha})}} \tag{5.13}$$

with K_{Σ} the canonical bundle of Σ. By Serre duality we have that $\chi(\Sigma, K_{\Sigma}\otimes\mathcal{L}^{\otimes n}\otimes V_{\alpha}) = -\chi(\Sigma,\mathcal{L}^{\otimes -n}\otimes V_{-\alpha})$ so the exponent in the previous expression is

$$\chi(\Sigma,\mathcal{L}^{\otimes n}\otimes V_{\alpha})+\chi(\Sigma,\mathcal{L}^{\otimes -n}\otimes V_{-\alpha}) = 2-2g+\sum_{n}\left[\deg\left(\mathcal{L}^{\otimes n}\right)+\deg\left(\mathcal{L}^{-\otimes n}\right)\right] \tag{5.14}$$

where we have made use of (3.10). By inspection of (5.14) one sees that the absolute value of the determinants is the same for $S(\mathcal{L})$ and $S(\mathcal{L}^{-1})$.

The eta invariant of the phase of the determinant is by (B.26) of [7]

$$\begin{aligned}\eta(L_{\phi},\,s) &= \eta_{(0,1)}(iL_{\phi})(s)+\eta_{(1,0)}(-iL_{\phi})(s)\\ &= -\frac{1}{2}\sum_{n,\,\alpha}\left(\chi(\mathcal{L}^{\otimes n}\otimes V_{\alpha})+\chi(K\otimes\mathcal{L}^{\otimes n}\otimes V_{\alpha})\right)\frac{\mathrm{sign}(2\pi n+i\alpha(\phi))}{|2\pi n+i\alpha(\phi)|^{s}}\\ &= -\frac{1}{2}\sum_{n,\,\alpha}\left(\chi(\mathcal{L}^{\otimes n}\otimes V_{\alpha})-\chi(\mathcal{L}^{\otimes -n}\otimes V_{-\alpha})\right)\frac{\mathrm{sign}(2\pi n+i\alpha(\phi))}{|2\pi n+i\alpha(\phi)|^{s}}\end{aligned} \tag{5.15}$$

the last line following by Serre duality. The subscripts on the η's in (B.26) of [7] are there to indicate whether we are using the index of ∂ or of $\overline{\partial}$.

Without loss of generality we choose ϕ such that $0<i\alpha(\phi)<2\pi$ for the positive roots, so that

$$\begin{aligned}\eta(L_{\phi},s) &= -\sum_{\alpha>0}[\deg(V_{\alpha})-\deg(V_{-\alpha})]\,|i\alpha(\phi)|^{-s}\\ &\quad-\sum_{n\geq 1}\sum_{\alpha>0}[\deg(\mathcal{L}^{\otimes n}\otimes V_{\alpha})-\deg(\mathcal{L}^{\otimes -n}\otimes V_{-\alpha})]\,(2\pi n+i\alpha(\phi))^{-s}\\ &\quad-\sum_{n\geq 1}\sum_{\alpha>0}[\deg(\mathcal{L}^{\otimes n}\otimes V_{-\alpha})-\deg(\mathcal{L}^{\otimes -n}\otimes V_{\alpha})]\,(2\pi n-i\alpha(\phi))^{-s}\end{aligned}$$

By (3.11) we can split the phase as

$$\eta(L_{\phi},s)=\sigma(L_{\phi},V_{\mathfrak{k}},s)+\gamma(L_{\phi},\mathcal{L},s)$$

where

$$\begin{aligned}\sigma(L_{\phi},V_{\mathfrak{k}},s) &= -2\sum_{\alpha>0}\deg(V_{\alpha})|i\alpha(\phi)|^{-s}-2\sum_{\alpha>0}\deg(V_{\alpha})\sum_{n\geq 1}(2\pi n+i\alpha(\phi))^{-s}\\ &\quad+2\sum_{\alpha>0}\deg(V_{\alpha})\sum_{n\geq 1}(2\pi n-i\alpha(\phi))^{-s}\end{aligned} \tag{5.16}$$

and

$$\gamma(L_{\phi},\mathcal{L},s) = -\sum_{n\geq 1}\sum_{\alpha>0}[\deg(\mathcal{L}^{\otimes n})-\deg(\mathcal{L}^{\otimes -n})]\left[(2\pi n+i\alpha(\phi))^{-s}+(2\pi n-i\alpha(\phi))^{-s}\right] \tag{5.17}$$

Now $\sigma(L_\phi, V_{\mathfrak{k}}, s)$ does not depend explicitly on $\mathcal{L}$ so, in particular, we would find the same result had we used any other line V-bundle. However, $\gamma(L_\phi, \mathcal{L}, s)$ is quite a different object depending explicitly on the line V-bundle defining M and in fact we have

$$\gamma(L_\phi, \mathcal{L}^{-1}, s) = -\gamma(L_\phi, \mathcal{L}, s) \tag{5.18}$$

This is as far as we can go in this generality.

5.1 Absolute value of the determinant

In order to determine the absolute value of the determinants we use

$$\begin{aligned}\chi(\Sigma, \mathcal{L}^{\otimes n}) + \chi(\Sigma, \mathcal{L}^{\otimes -n}) &= \chi(\Sigma, \mathcal{L}_0^{\otimes dn}) + \chi(\Sigma, \mathcal{L}_0^{-\otimes dn}) \\ &= 2 - 2g - N + \sum_{i=1}^{N} \phi_{a_i}(dn)\end{aligned}$$

The last line follows from the index theorem as all our assumptions about the bundles for which $\phi_{a_i}(dn)$ is defined hold as we will now see.

We demand that the line bundle $\mathcal{L} = \mathcal{L}_0^{\otimes d}$ which defines our 3-manifold $M = S(\mathcal{L})$ has isotropy invariants such that $\gcd(a_i, b_i(\mathcal{L})) = 1$. However, we also have that $b_i(\mathcal{L}_0^{\otimes d}) = db_i(\mathcal{L}_0) \mod a_i$ so that neither d nor $b_i(\mathcal{L}_0)$ are divisible by a_i otherwise $\gcd(a_i, b_i(\mathcal{L})) \neq 1$. We can do a bit better. Let $b_i(\mathcal{L}_0^{\otimes d}) = db_i(\mathcal{L}_0) + ma_i$ for some m. Suppose that $y \neq \pm 1$ divides a_i then y cannot divide $db_i(\mathcal{L}_0)$ as that would conflict with our assumption that $\gcd(a_i, b_i(\mathcal{L})) = 1$. In particular y cannot divide either d or $b_i(\mathcal{L}_0)$, so we have that $\gcd(a_i, b_i(\mathcal{L}_0)) = 1$ and $\gcd(a_i, d) = 1$. Consequently, we do indeed have that

$$\begin{aligned}\deg(\mathcal{L}^{\otimes n}) + \deg(\mathcal{L}^{\otimes -n}) &= \deg(\mathcal{L}_0^{\otimes dn}) + \deg(\mathcal{L}_0^{-\otimes dn}) \\ &= -N + \sum_{i=1}^{N} \phi_{a_i}(dn)\end{aligned} \tag{5.19}$$

Are there non zero values of $\phi_{a_i}(dn)$ and if so what form do they take? Since $\phi_{a_i}(dn)$ is non-zero only if $a_i | dn$ and we know that a_i does not divide d it must divide n. Whence only those $n = ma_i$ for $m \in \mathbb{Z}$ yield non-zero $\phi_{a_i}(dn)$.

Now, up to normalisation,

$$\prod_\alpha \prod_n (2\pi n + i\alpha(\phi)) \simeq T_{S^1}(\phi) \tag{5.20}$$

where

$$\begin{aligned}T_{S^1}(\phi) = \det{}_{\mathfrak{k}}(1 - \mathrm{Ad}\, \mathrm{e}^{\phi}) &= \prod_{\alpha>0} (1 - \mathrm{e}^{\alpha(\phi)})(1 - \mathrm{e}^{-\alpha(\phi)}) \\ &= \prod_{\alpha>0} 4 \sin^2(i\alpha(\phi)/2)\end{aligned} \tag{5.21}$$

is the Ray-Singer torsion of S^1 (with respect to the flat connection $i\phi d\theta$).

We still need to determine

$$\prod_\alpha \prod_n (2\pi n + i\alpha(\phi))^{\phi_{a_i}(dn)} \tag{5.22}$$

As argued above, the function $\phi_{a_i}(dn)$ vanishes except when $n = ma_i$ for $m \in \mathbb{Z}$ so that

$$\begin{aligned}\prod_\alpha \prod_n (2\pi n + i\alpha(\phi))^{\phi_{a_i}(dn)} &= \prod_\alpha (a_i)^{\sum_n 1} \prod_m (2\pi m + i\alpha(\phi))/a_i) \\ &= \prod_\alpha \prod_m (2\pi m + i\alpha(\phi))/a_i) \\ &\simeq T_{S^1}((\phi/a_i)\end{aligned} \tag{5.23}$$

where we have used the fact that the zeta function regularisation of $\sum_{n=-\infty}^{\infty} 1$ is zero.

Putting the pieces together for the absolute value of the determinant we find it is just what we called $\sqrt{T_M}$ at the end of section 2, namely

$$\sqrt{T_M(\phi,\, a_1, \dots, a_N)} = T_{S^1}(\phi)^{1-g-N/2}.\prod_{i=1}^N T_{S^1}(\phi/a_i)^{1/2} \tag{5.24}$$

Notice that this does not depend on the weights $b_i(\mathcal{L}_0^{\otimes d})$ (and seemingly nor on d, but as we will see ϕ depends on d). So, in particular, (5.24) does not depend on the orientation of M which we explained, in a slightly different way, just after (5.14).

5.2 Phase of the determinant

Nicolaescu [19] has done some of the work for us. In particular the trick of passing from $\lfloor x \rfloor$ to $((x))$ we took from him and this allows us to write certain terms as Dedekind sums later on. We will need to make use of the Hurwitz zeta function

$$\zeta(s, x) = \sum_{m \geq 0} \frac{1}{(m+x)^s},$$

which for negative integral s is related to the Bernoulli functions. In particular,

$$\zeta(0, x) = \frac{1}{2} - x, \quad \text{and} \quad \zeta(-1, x) = -\frac{x^2}{2} + \frac{x}{2} - \frac{1}{12}$$

whence

$$\begin{aligned}\sum_{n\geq 1} \frac{1}{(2\pi n + i\alpha(\phi))^s} &= -\frac{1}{2} - \frac{i}{2\pi}\alpha(\phi) + \mathcal{O}(s) \\ \sum_{n\geq 1} \frac{n}{(2\pi n + i\alpha(\phi))^s} &= -\frac{1}{12} - \frac{1}{8\pi^2}\alpha(\phi)^2 + \mathcal{O}(s)\end{aligned}$$

Clearly for (5.16) as $s \longrightarrow 0$,

$$\sigma(L_\phi, V_{\mathfrak{k}}, s) = -2\sum_{\alpha>0} \deg\left(V_\alpha\right)\left(1 + \frac{1}{\pi} i\alpha(\phi)\right) + \mathcal{O}(s) \tag{5.25}$$

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In order to determine the phase of (5.17) we make use of the difference of degrees formula

$$\deg\left(\mathcal{L}^{\otimes n}\right) - \deg\left(\mathcal{L}^{-\otimes n}\right) = 2n.c_1(\mathcal{L}) - 2\sum_{i=1}^{N}\left(\left(\frac{nb_i(\mathcal{L})}{a_i}\right)\right)$$

We have two types of terms to compute, those proportional to $c_1(\mathcal{L})$ and those proportional to the symbol $((x))$. We start with the ones proportional to $c_1(\mathcal{L})$, namely, from (5.17),

$$\begin{aligned}-2c_1(\mathcal{L})\sum_{\alpha>0}&\left(\sum_{n\geq 1}\left[\frac{n}{(2\pi n + i\alpha(\phi))^s} + \frac{n}{(2\pi n - i\alpha(\phi))^s}\right]\right)\\ &= c_1(\mathcal{L})\sum_{\alpha>0}\left(\frac{1}{3} + \frac{1}{2\pi^2}\alpha(\phi)^2\right) + \mathcal{O}(s)\end{aligned} \tag{5.26}$$

Now the terms in (5.17) proportional to $((.))$ are, with $b_i = b_i(\mathcal{L})$,

$$2\sum_{\alpha>0}\sum_{i=1}^{N}\sum_{n\geq 1}\sum_{\pm}\left[\left(\left(\frac{nb_i}{a_i}\right)\right)\cdot\frac{1}{(2\pi n \pm i\alpha(\phi))^s}\right]$$

One can allow the sum over n to include $n = 0$ since the two contributions cancel (as s goes to zero). Furthermore, the periodicity $((x+1)) = ((x))$ allows us to write

$$\begin{aligned}\sum_{n\geq 0}\left(\left(\frac{nb_i}{a_i}\right)\right)\cdot\frac{1}{(2\pi n \pm i\alpha(\phi))^s} &= \sum_{k=0}^{a_i-1}\left(\left(\frac{kb_i}{a_i}\right)\right)\frac{1}{(2\pi a_i)^s}\zeta\left(s, \frac{k \pm i\alpha(\phi)/2\pi}{a_i}\right)\\ &= \sum_{k=0}^{a_i-1}\left(\left(\frac{kb_i}{a_i}\right)\right)\left(\frac{1}{2} - \frac{k}{a_i} \mp \frac{i\alpha(\phi)/2\pi}{a_i}\right) + \mathcal{O}(s)\\ &= \sum_{k=1}^{a_i-1}\left(\left(\frac{kb_i}{a_i}\right)\right)\left[-\left(\left(\frac{k}{a_i}\right)\right) \mp \frac{i\alpha(\phi)/2\pi}{a_i}\right] + \mathcal{O}(s)\end{aligned}$$

as $1/2 - k/a_i = -((k/a_i))$ for $0 < k < a_i$. We have that

$$\sum_{n\geq 0}\sum_{\pm}\left(\left(\frac{nb_i}{a_i}\right)\right)\cdot\frac{1}{(2\pi n \pm i\alpha(\phi))^s} = -2\sum_{k=1}^{a_i-1}\left(\left(\frac{kb_i}{a_i}\right)\right)\cdot\left(\left(\frac{k}{a_i}\right)\right) + \mathcal{O}(s)$$

The Dedekind sum $s(b, a)$ is defined by

$$s(b,a) = \sum_{k=1}^{a-1}\left(\left(\frac{kb}{a}\right)\right)\left(\left(\frac{k}{a}\right)\right)$$

whence

$$\sum_{n\geq 0}\sum_{\pm}\left(\left(\frac{nb_i}{a_i}\right)\right)\cdot\frac{1}{(2\pi n \pm i\alpha(\phi))^s} = -2s(b_i, a_i) + \mathcal{O}(s)$$

Putting the pieces together, thus far we have

$$\eta(L_\phi,0) = \sum_{\alpha>0}\left(-2c_1(V_\alpha)+\frac{c_1(\mathcal{L})}{3}+\frac{c_1(\mathcal{L})}{2\pi^2}\alpha(\phi)^2+\frac{2c_1(V_\alpha)}{\pi}i\alpha(\phi)\right)$$
$$-4\sum_{i=1}^{N}\sum_{\alpha>0}s(b_i,a_i) \tag{5.27}$$

We also have to consider the contribution from the fields lying in the Cartan subalgebra and these couple neither to the bundles V_α nor to the ϕ. They contribute,

$$\eta(L_\xi,0) = \dim T\left(\frac{c_1(\mathcal{L})}{6}-2s(b_i,a_i)\right) \tag{5.28}$$

Collecting all the contributions we find that the total $\eta(s)=\eta(L_\phi,s)+\eta(L_\xi,s)$, as $s\longrightarrow 0$ is

$$\eta(0) = \sum_{\alpha>0}\left(-2c_1(V_\alpha)+\frac{c_1(\mathcal{L})}{2\pi^2}\alpha(\phi)^2+\frac{2c_1(V_\alpha)}{\pi}i\alpha(\phi)\right)$$
$$+\dim G\left(\frac{c_1(\mathcal{L})}{6}-\sum_{i=1}^{N}s(b_i,a_i)\right) \tag{5.29}$$

so that in this case we have

$$-\frac{i\pi}{2}\eta(0) = 4\pi i\Phi(\mathcal{L})-\frac{ic_{\mathfrak{g}}}{4\pi}\int_\Sigma\left(\frac{d}{P}\,\mathrm{Tr}\,\phi^2\,\omega\right)+\frac{ic_{\mathfrak{g}}}{2\pi}\int_\Sigma\mathrm{Tr}\,\phi F_A \tag{5.30}$$

where

$$\Phi(\mathcal{L}) = \frac{1}{48}\dim G\left(12\sum_{i=1}^{N}s(b_i,a_i)-\frac{d}{P}\right) \tag{5.31}$$

As stated at the beginning of section 4, we consider the case that G is simply-connected. For such groups one has $\sum_{\alpha>0}c_1(V_\alpha)\in 2\mathbb{Z}$ (the Weyl vector is integral), so that this term does not contribute to the phase.

It is important to notice that had we used the line V-bundle $\mathcal{L}^{-1}$ rather than $\mathcal{L}$ to construct M then the Chern class of M would change sign $c_1(\mathcal{L})\to c_1(\mathcal{L}^{-1})=-c_1(\mathcal{L})$ and so too the Dedekind sum $s(b_i,a_i)\to s(a_i-b_i,a_i)=-s(b_i,a_i)$ as required by (5.18).

The expression (5.30) agrees with that obtained in [7] for the Lens spaces $L(p,1)$ on setting $dc_1(\mathcal{L}_0)=-p$, which can be achieved by taking the $a_i=1$ and $b_i=0$ for all i (so that $P=1$), $\deg\mathcal{L}_0=1$ and then $d=-p$.

6 Evaluating the path integral on Σ

Now that we have integrated over all the $\mathfrak{k}$-valued fields as well as all the $\mathfrak{t}$-valued modes which are not constant along the S^1 fibres in M, the Chern-Simons partition function, up to the phase (5.31), reduces to a path integral of an Abelian 2-dimensional gauge theory on Σ with action

$$S_M\to S_\Sigma[A_H,\phi] = \frac{k+c_{\mathfrak{g}}}{4\pi}\int_\Sigma\mathrm{Tr}\left(2\phi\,F_H-\frac{d}{P}\phi^2\omega\right)\ , \tag{6.1}$$

where $A_H=A_H^{\mathfrak{t}}$ and $\phi=\phi^{\mathfrak{t}}$.

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The curvature 2-form F_H includes the contribution of non-trivial line bundles (but not line V-bundles) on Σ. To incorporate those we let

$$F_H \to F_H(A) + 2\pi\, r\, \omega$$

with $r \in \mathbb{Z}^{\mathbf{rk}}$. A_H is now understood to be a $\mathfrak{t}$ valued 1-form on Σ. It might appear to be more natural to have chosen that $r \in \mathbb{Z}_d$ as we did in [7]. However, as we have seen previously [7] and as is also evident from the various equalities in (2.4), the net effect is the same, so for the sake of variety here we choose $r \in \mathbb{Z}^{\mathbf{rk}}$.

Baily [1] tells us that on an orbifold Σ the Hodge decomposition is still available where all sections are understood in the appropriate sense. Locally around an orbifold point (the conic point of $D/\mathbb{Z}_a$) any p-form α is understood to be a $\mathbb{Z}_a$ invariant p-form on D. Then, for two such 1-forms

$$\int_{D^2/\mathbb{Z}_a} \alpha \wedge \beta = \frac{1}{|a|} \int_{D^2} \alpha \wedge \beta$$

and so on.

The Hodge decomposition tells us that the harmonic modes of $A_H^{\mathfrak{t}}$ only contribute to the normalisation of the path integral. The exact components are in the gauge directions of the residual $\mathrm{U}(1)^{\mathbf{rk}}$ gauge symmetry and so may be set to zero by a gauge choice. The co-exact parts of $A_H^{\mathfrak{t}}$ are the only pieces that appear in the action and integrating over those imposes the condition

$$d\phi^i = 0 \ .$$

so that the ϕ are constant. With ϕ constant, the partition function reduces to the finite-dimensional integral over the Cartan subalgebra

$$Z_k[M,G] \sim \mathrm{e}^{4\pi i \Phi(\mathcal{L})} \sum_{r \in \mathbb{Z}^{\mathbf{rk}}} \int_{\mathfrak{t}} \sqrt{T_M(\phi)}\, \exp\left(i \frac{k + c_{\mathfrak{g}}}{4\pi}\, \mathrm{Tr}\left(-\frac{d}{P}\,\phi^2 + 4\pi r\, \phi \right)\right) \tag{6.2}$$

However, this is not the final form of the partition function as there is still a discrete symmetry that we should mod out by. The partition function (6.2) has the form of the gauge invariant partition function (2.2) and so is invariant under the action of the affine Weyl group Γ^W. How does this symmetry arise in the present situation?

- Invariance under the integral lattice I is there since, as we have already noted, the pullback of any multiple of $\mathcal{L}_0^P$ to M is trivial so all such multiples are equivalent. Explicitly we made the substitution

  $$A = A_H + \phi\,\kappa + 2\pi r \frac{P}{d}\,\kappa$$

  which obviously has the symmetry

  $$\phi \to \phi - 2\pi P s, \quad r \to r + ds$$

- The Weyl group makes an appearance since it was part of the original G symmetry. It acts, therefore, by conjugation and on $\mathfrak{t}$ this becomes permutation of the (diagonal) matrix entries. Permutation of both the ϕ and r entries in the same way leaves $\mathrm{Tr}\,(\phi^2)$ and $\mathrm{Tr}\,(r.\phi)$ invariant. The Ray-Singer torsion of the circle is also invariant under W so we have that theory posses the symmetry that we claimed.

The partition function is, therefore,

$$Z_k[M,G] = \Lambda\, \mathrm{e}^{4\pi i\Phi(\mathcal{L})} \sum_{r\in\mathbb{Z}^{\mathbf{rk}}} \int_{\mathfrak{t}/\Gamma^W} \sqrt{T_M(\phi)}\, \exp\left(i\frac{k+c_{\mathfrak{g}}}{4\pi}\,\mathrm{Tr}\left(-\frac{d}{P}\phi^2 + 4\pi r\,\phi\right)\right) \tag{6.3}$$

where $\Gamma^W = I \rtimes W$ is the affine Weyl group and Λ is a real normalisation constant that remains to be determined.

As the Ray-Singer torsion has zeros at the boundary of the Weyl chamber the integrals (6.2), (6.3) diverge when $g + N/2 > 1$. As shown in [5] for the smooth case one ought to regularise by giving a small mass term to the connection, while preserving the residual $\mathrm{U}(1)^{\mathbf{rk}}$ invariance. The same regularisation is applicable when Σ is an orbifold and guarantees the vanishing of the ghost determinant at the boundary while the inverse of the determinant coming from the connection remains finite. The net effect of this procedure is to exclude the boundaries of the Weyl chamber. As the contributions to the path integral are at discrete points this regularisation prescription renders the integrals finite.

Witten [24] shows how at one loop level the Chern-Simons partition function becomes an integral over the moduli space of flat connections with measure the square root of the Ray-Singer Torsion. There is also a phase factor coming from the Chern-Simons function and a framing correction. We note that (6.3) has precisely the form just described with $T_M(\phi,;\, a_1\ldots,a_N)$ being the Ray-Singer Torsion of a Seifert $\mathbb{Q}$[g]HS but with a crucial difference. Rather than the moduli space of flat connections on M we have instead the integral over $\mathfrak{t}/\Gamma^W$ coming from the vertical part of the connection. This is a much simpler integral to perform.

7 The inclusion of Wilson loops along the fibre

We can also easily evaluate Wilson lines which are in the fibre direction of M thought of as a principal bundle. Such Wilson lines only depend on the representation and on the $\kappa\phi$ part of the connection. Since they do not depend on A_H the inclusion of Wilson loops does not change any of the arguments in the evaluation of the path integral. In particular one may just as well take ϕ to be constant and to take values in $\mathfrak{t}$.

The expectation value (normalised or not) of such a Wilson loop $\mathrm{Tr}_{R_j}\left(\mathrm{P}\exp\left(\oint A\right)\right)$ then is the same as evaluating (6.3) with

$$\mathrm{Tr}_{R_j}\left(\exp\left(\oint \kappa\phi\right)\right) \tag{7.1}$$

inserted in the integral (or products of these). Now providing the fibre is not exceptional (i.e. it is based at a regular point on the orbifold) the Wilson loop (7.1) is $\mathrm{Tr}_{R_j}\left(\exp\left(\phi\right)\right)$.

The path integral including such Wilson lines becomes,

$$\Lambda\,\mathrm{e}^{4\pi i\Phi(\mathcal{L})}\sum_{r\in\mathbb{Z}^{\mathbf{rk}}}\int_{\mathfrak{t}/\Gamma^W}\sqrt{T_M(\phi)}\,\exp\left(i\frac{k+c_{\mathfrak{g}}}{4\pi}\,\mathrm{Tr}\left(-\frac{d}{P}\phi^2+4\pi r\,\phi\right)\right)\prod_i\mathrm{Tr}_{R_j}\left(\exp\left(\phi\right)\right)$$

Had any of the Wilson loops been along an exceptional fibre (one which is based at an orbifold point of weight a_i on Σ), in the Wilson loops ϕ would have to be replaced by $\phi\to\phi/a_i$.

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A An example of diagonalisation

In this appendix we wish to explain in some more detail the appearance of non-trivial T-bundles on diagonalisation. This is both a summary and an extension to the orbifold case of one of the arguments given in [6] for this phenomenon.

Let $M=S(\mathcal{L})$ be the circle bundle of a V-line bundle over an orbifold Σ and consider the (trivialised) vector bundle $\mathrm{ad}\,G=M\times\mathfrak{g}$ over M. We equip M with an induced contact structure and let ξ be the Reeb vector field on M. Now with ϕ a section of $\mathrm{ad}\,G$ we mean $s: M\longrightarrow M\times\mathfrak{g}$ such that on M it is the identity map. In this case a section is the combination $s\cong(\mathrm{Id}_M,\,\phi)$ where $\phi: M\longrightarrow\mathfrak{g}$. Now restrict attention to those sections which satisfy $\iota_\xi d\phi=0$. Such ϕ are still just maps to $\mathfrak{g}$. Or put another way, s is still a section of the trivial bundle $\mathrm{ad}\,G$ on M.

However, such a ϕ also defines a section $\hat{s}:\Sigma\longrightarrow\Sigma\times\mathfrak{g}$ with the map automorphism on Σ being the identity map Id_Σ. We are still dealing with a trivial bundle albeit over an orbifold. In this context ϕ is a map from Σ to $\mathfrak{g}$. Notice, that at this point, the information about the V-line bundle $\mathcal{L}$ (i.e. its isotropy weights b_i) no longer appears and so too then information about M is lost. (This is just as it is in the case of Lens spaces $L(p,1)$ as for all of them the base is S^2.)

In case that $\Sigma=\widehat{\Sigma}/\Gamma$ the map ϕ is equivalent to a Γ invariant map from $\widehat{\Sigma}$ to $\mathfrak{g}$. A good example of this situation is the orbifold $S^2/\mathbb{Z}_a$ which has two marked points both with isotropy a. Now a section of the trivial $\mathfrak{g}$ bundle over $S^2/\mathbb{Z}_a$ is the same as an $\mathbb{Z}_a$ invariant section of the trivial $\mathfrak{g}$ bundle over S^2. If, for $\zeta\in\mathbb{Z}_a$, the section were simply equivariant $\phi(\zeta.z)=\zeta.\phi(z)$ where the action on the Lie algebra is non-trivial, then one would have a non-trivial $\mathfrak{g}$ bundle over $S^2/\mathbb{Z}_a$.

Let ϕ then be a map from $S^2\to\mathfrak{g}$. For simplicity we take $\mathfrak{g}$ to be $su(2)$. Given a connection on the bundle we can define an invariant

$$n(\phi,\,A)=\frac{1}{2\pi i}\int_{S^2}\mathrm{Tr}\left(\phi F_A-\frac{1}{4}\phi\,d_A\phi\wedge d_A\phi\right)\tag{A.1}$$

for those ϕ such that $\phi^2=-\mathrm{Id}_{2\times 2}$ (these are maps to S^2). This invariant exhibits the non-trivial line bundles that arise on diagonalisation. On the one hand the maps of interest are maps from S^2 to S^2 and so they fall into homotopy π_2 classes. In that case $n(\phi, 0)$ just measures the winding number of the map. Upon diagonalisation, with group map g, the map $g^{-1}\phi g$ is just to a single point and $n(g^{-1}\phi g, 0 + g^{-1}dg)$ is the first Chern class of the connection $A = 0 + g^{-1}dg$. As $n(\phi, A)$ is gauge invariant the winding number of ϕ and the first Chern class of the liberated line bundles must agree.

We are interested in maps to S^2 which are $\mathbb{Z}_a$ invariant. Embed S^2 in $\mathbb{R}^3$ so that it is the solution to $x^2 + y^2 + z^2 = 1$. The action of $\mathbb{Z}_a$ is $\zeta.(w, z) = (\zeta.w, z)$ where $w = x + iy = \exp(i\theta). \sin\varphi$, $z = \cos\varphi$, where $0 \le \theta < 2\pi$, $-\pi \le \varphi \le 0$ and $\zeta = \exp(2\pi i/a)$. As part of our example let

$$\phi(\theta, \varphi) = \sin\theta. \sin\varphi.i\sigma_1 + \cos\theta. \sin\varphi.i\sigma_2 + \cos\varphi.i\sigma_3 \tag{A.2}$$

be the identity map. The identity map is not $\mathbb{Z}_a$ invariant since the action of $\mathbb{Z}_a$ is

$$\zeta : (\theta, \varphi) \longrightarrow \left(\theta + \frac{2\pi}{a}, \varphi\right) \tag{A.3}$$

However, it is quite straightforward to create maps which are $\mathbb{Z}_a$ invariant, these are

$$\widetilde{\phi}(\theta, \varphi) = \sin(an\theta). \sin\varphi.i\sigma_1 + \cos(an\theta). \sin\varphi.i\sigma_2 + \cos\varphi.i\sigma_3 \tag{A.4}$$

It will not come as a surprise that their winding numbers are elements of $a\mathbb{Z}$, indeed

$$\frac{-1}{8\pi i}\int_{S^2} \mathrm{Tr}\left(\widetilde{\phi}\, d\widetilde{\phi} \wedge d\widetilde{\phi}\right) = an \tag{A.5}$$

On $S^2/\mathbb{Z}_a$ the maps $\widetilde{\phi}$ become 'winding' number n maps to the 2-sphere. On diagonalising (A.1) tells us that the first Chern class of the liberated line bundle is integral and it is over such line bundles that we must sum.

B The fundamental group and representations in SU(2)

Our evaluation of the path integral does not involve the moduli space of non-Abelian flat connections as diagonalisation forces us to consider Abelian connections. By way of example we exhibit a non-trivial irreducible SU(2) flat connection on the Poincaré $\mathbb{Z}$HS in order to convince the reader that such connections are there even though we manage to sidestep having to face their existence in the course of our evaulation of the path integral.

To do this we start with the presentation of the generators and relations of the fundamental group $\pi_1(M)$ for M a $\mathbb{Q}$HS. After the example we move on to determine the first cohomology group of these manifolds as this is what really enters in the body of the paper. We use the fact that the first homology group is the abelianisation of the fundamental group $\mathrm{H}_1 = \pi_1/[\pi_1, \pi_1]$.

As we are in the situation where $g = 0$ the generators of $\pi_1(M)$ are c_j, $j = 1, \ldots, N$ and h subject to the relations $[c_j, h] = 1$ and

$$c_j^{a_j} h^{b_j} = 1, \quad \prod_{j=1}^{N} c_j = h^n \tag{B.1}$$

where, n is related to the degree of the line V-bundle that defines M and, h is central. If $h=1$ the relations are just those for the fundamental group of the orbifold Σ so that h is the generator along the fibre.

We give an example of a representation of π_1 for the Poincaré 3-sphere $\Sigma(2,3,5)$ in SU(2). The presentation of π_1 is given by

$$X^2 = H^{-1}, \quad Y^3 = H^{-1}, \quad Z^5 = H^{-1}, \quad XYZ = H^{-1} \tag{B.2}$$

and H commutes with X, Y and Z. We take H to be central (indeed one can show that for an irreducible representation it must be) and for concreteness let $H = -\mathbb{I}_2$. We diagonalise Z

$$Z = \exp\left(im\pi/5\,.\,\sigma_3\right), \quad m = 1,3,5 \tag{B.3}$$

(one is quickly led to a contradiction if one takes Z central). If we write

$$X = a\mathbb{I}_2 + i\underline{b}.\sigma, \quad a^2 + |\underline{b}|^2 = 1 \tag{B.4}$$

then the condition on X in (B.2) implies $a=0$, so $X \in S^2$. We may still act by conjugation by elements in the torus defined by σ_3 without changing Z and so we may rotate X into an S^1 of our choice, i.e. we simply set $b_1 = 0$, $b_2 > 0$ and we have that

$$X = ib_2\sigma_2 + ib_3\sigma_3 \tag{B.5}$$

We write Y in the same way as we did X

$$Y = c\mathbb{I}_2 + i\underline{d}.\sigma, \quad c^2 + |\underline{d}|^2 = 1 \tag{B.6}$$

then the fact that $Y^3 = -\mathbb{I}_2$ implies that $c = 1/2$. The only relation still to satisfy is $XYZ = -\mathbb{I}_2$. This is straightforward and we find, with $\lambda_0 = \cos\left(\pi m/5\right)$ and $\lambda_1 = -\sin\left(\pi m/5\right)$,

$$b_3 = \frac{1}{2\lambda_1} \quad d_3 = -\frac{\lambda_0}{2\lambda_1}, \quad d_1 = b_2\lambda_1, \quad d_2 = -b_2\lambda_0 \tag{B.7}$$

We now have a point in the space of flat SU(2) connections on $\Sigma(2,3,5)$. The matrices X, Y and Z are essentially the holonomies of the flat connection in question around the non-trivial cycles of $\Sigma(2,3,5)$.

As in [3] let $c_{N+1} = h$ so that the relations in $\mathrm{H}_1(M,\mathbb{Z})$ may be written as

$$\prod_{j=1}^{N+1} c_j^{A_{jk}} = 1 \tag{B.8}$$

where

$$A = \begin{pmatrix} a_1 & 0 & \cdots & b_1 \\ 0 & \ddots & 0 & \vdots \\ 0 & \cdots & a_N & b_N \\ 1 & \cdots & 1 & -n \end{pmatrix}$$

If $v \in \mathrm{H}_1(M, \mathbb{Z})$ then v^d must be trivial (in the multiplicative sense so that $v^d = 1$). An element $w = \prod_j c_j^{m_j}$ is trivial iff $m_j = A_{jk} l_k$ and will be of the form v^d providing $\mathrm{Det}\, A \propto d$.

Calculating, we find that

$$\mathrm{Det}\, A = -P \left(n + \sum_{i=1}^{N} \frac{b_i}{a_i} \right) \tag{B.9}$$

We have then that $|\,\mathrm{Det}\, A| = |d|$ is the order of $\mathrm{H}_1(M)$. One can be rather more explicit about this. Going back to (B.1) we have

$$c_j^P h^{P b_j / a_j} = 1, \quad \prod_{j=1}^{N} c_j^P = h^{nP} \tag{B.10}$$

plugging the first into the second gives $h^d = 1$. As we have abelianised $\pi_1(M)$ to pass to $\mathrm{H}_1(M) = \pi_1(M)/[\pi_1(M), \pi_1(M)]$ we may as well represent the generators as

$$c_j = \exp\left(2\pi i \, \frac{b_j}{a_j} . \frac{P}{d}\right), \quad h = \exp\left(-2\pi i \, \frac{P}{d}\right)$$

References

[1] W.L. Baily, *The decomposition theorem for V-manifolds*, *Amer. J. Math.* **78** (1956) 862.

[2] C. Beasley, *Localization for Wilson loops in Chern-Simons theory*, arXiv:0911.2687 [INSPIRE].

[3] C. Beasley and E. Witten, *Non-abelian localization for Chern-Simons theory*, *J. Diff. Geom.* **70** (2005) 183 [hep-th/0503126] [INSPIRE].

[4] D. Birmingham, M. Blau, M. Rakowski and G. Thompson, *Topological field theory*, *Phys. Rept.* **209** (1991) 129 [INSPIRE].

[5] M. Blau and G. Thompson, *Derivation of the Verlinde formula from Chern-Simons theory and the G/G model*, *Nucl. Phys.* **B 408** (1993) 345 [hep-th/9305010] [INSPIRE].

[6] M. Blau and G. Thompson, *On diagonalization in* Map(M, G), *Commun. Math. Phys.* **171** (1995) 639 [hep-th/9402097] [INSPIRE].

[7] M. Blau and G. Thompson, *Chern-Simons theory on S1-bundles: abelianisation and q-deformed Yang-Mills theory*, *JHEP* **05** (2006) 003 [hep-th/0601068] [INSPIRE].

[8] M. Blau and G. Thompson, *$N = 2$ topological gauge theory, the Euler characteristic of moduli spaces and the Casson invariant*, *Commun. Math. Phys.* **152** (1993) 41 [hep-th/9112012] [INSPIRE].

[9] M. Blau and G. Thompson, *Lectures on 2D gauge theories: topological aspects and path integral techniques*, in the proceedings of the *1993 Trieste Summer School on High Energy Physics and Cosmology*, E. Gava et al. eds., World Scientific, Singapore (1994), hep-th/9310144 [INSPIRE].

[10] D. Freed and R. Gompf, *Computer calculation of Witten's three manifold invariant*, *Commun. Math. Phys.* **141** (1991) 79 [INSPIRE].

[11] M. Furuta and B. Steer, *Seifert fibred homolog 3-spheres and the Yang-Mills equations on Riemann surfaces with marked points*, *Adv. Math.* **96** (1992) 38.

[12] L. Jeffrey, *Chern-Simons-Witten invariants of lens spaces and torus bundles, and the semiclassical approximation*, *Commun. Math. Phys.* **147** (1992) 563.

[13] T. Kawasaki, *The Riemann-Roch theorem for complex V-manifolds*, *Osaka J. Math.* **16** (1979) 151.

[14] J. Kallen, *Cohomological localization of Chern-Simons theory*, *JHEP* **08** (2011) 008 [arXiv:1104.5353] [INSPIRE].

[15] R. Lawrence and L. Rozansky, *Witten-Reshetikhin-Turaev invariants of Seifert manifolds*, *Commun. Math. Phys.* **205** (1999) 287.

[16] M. Mariño, *Chern-Simons theory, matrix integrals and perturbative three manifold invariants*, *Commun. Math. Phys.* **253** (2004) 25 [hep-th/0207096] [INSPIRE].

[17] J. Martinet, *Formes de Contact sur les varietés de dimension 3*, *Springer Lect. Notes Math.* **209** (1971) 142.

[18] T.S. Mrowka, P.S. Ozsváth and B. Yu, *Seiberg–Witten Monopoles on Seifert Fibered Spaces*, *Comm. Anal. Geom.* **5** (1997) 685 [math/9612221].

[19] L. Nicolaescu, *Finite energy Seiberg-Witten moduli spaces on 4-manifolds bounding seifert fibrations*, *Comm. Anal. Geom.* **8** (2000) 1027 [dg-ga/9711006].

[20] K. Ohta and Y. Yoshida, *Non-abelian localization for supersymmetric Yang-Mills-Chern-Simons theories on Seifert manifold*, *Phys. Rev.* **D 86** (2012) 105018 [arXiv:1205.0046] [INSPIRE].

[21] N. Reshetikhin and V. Turaev, *Invariants of 3-manifolds via link polynomials and quantum groups*, *Invent. Math.* **103** (1991) 357.

[22] G. Thompson, *Intersection pairings on spaces of connections and Chern-Simons theory on Seifert manifolds*, arXiv:1001.2885 [INSPIRE].

[23] E. Witten, *Two-dimensional gauge theories revisited*, *J. Geom. Phys.* **9** (1992) 303 [hep-th/9204083] [INSPIRE].

[24] E. Witten, *Quantum field theory and the Jones polynomial*, *Commun. Math. Phys.* **121** (1989) 351 [INSPIRE].

[25] E. Witten, *Topology changing amplitudes in* $(2+1)$*-dimensional gravity*, *Nucl. Phys.* **B 323** (1989) 113 [INSPIRE].

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