Cut-Free Sequent Systems for Temporal Logic

Kai Brünnler [a], Martin Lange [b]

[a] *Institut für angewandte Mathematik und Informatik*
Universität Bern
Neubrückstr. 10, CH – 3012 Bern, Switzerland

[b] *Institut für Informatik*
Ludwig-Maximilians-Universität München
Oettingenstr. 67, 80538 München, Germany

Abstract

Currently known sequent systems for temporal logics such as linear time temporal logic and computation tree logic either rely on a *cut* rule, an *invariant* rule, or an *infinitary* rule. The first and second violate the subformula property and the third has infinitely many premises. We present finitary cut-free invariant-free weakening-free and contraction-free sequent systems for both logics mentioned. In the case of linear time all rules are invertible. The systems are based on annotating fixpoint formulas with a history, an approach which has also been used in game-theoretic characterisations of these logics.

Key words: sequent calculus, temporal logic, cut-free

1 Introduction

Temporal logics are used for specification and verification of reactive systems. Two prominent representatives are linear time temporal logic, LTL for short, and computation tree logic, CTL for short. Both are well-studied and several axiomatisations and decision procedures are available. For LTL we just refer to Lichtenstein and Pnueli [13], which is a recent overview of the development of this subject and contains further references. For CTL we refer to Emerson's handbook article [1].

In this paper, we give cut-free sequent systems for propositional temporal logics. A cut-free sequent system is a valuable tool that helps us understand a

URLs: `www.iam.unibe.ch/~kai/` (Kai Brünnler),
`http://www.tcs.ifi.lmu.de/~mlange` (Martin Lange).

logic. It is typically much easier to carry out proof search in a cut-free sequent system than in a Hilbert-style axiom system. Proof search in the sequent calculus is typically easy to understand because of the clear logical reading of the inference rules. We feel that the same cannot be said, for example, for automata theoretic constructions or procedures that compute strongly connected components in a graph, which typically happens in tableau procedures. So, even when Hilbert-style axiom systems as well as decision procedures are available, it is worthwhile to look for a cut-free sequent system.

Several sequent systems have been known for a long time for temporal logics, but they all have their problems. Some are *infinitary*, i.e. they contain a rule with an infinite number of premises such as

$$\omega \frac{\vdash \Gamma, \varphi \qquad \vdash \Gamma, \bigcirc\varphi \qquad \vdash \Gamma, \bigcirc\bigcirc\varphi \quad \ldots}{\vdash \Gamma, \Box\varphi} \quad ,$$

where $\bigcirc\varphi$ means that φ holds at the next moment and $\Box\varphi$ means that φ holds always from now on.

Other systems are not truly cut-free, that is they contain a rule such as

$$\Box \frac{\vdash \Gamma, \psi \qquad \vdash \overline{\psi}, \bigcirc\psi \qquad \vdash \overline{\psi}, \varphi}{\vdash \Gamma, \Box\varphi} \quad ,$$

which clearly violates the subformula property since ψ is an arbitrary formula. Systems of the first kind can be found for example in Kawai [9]. Gudzhinskas [6] and Paech [14] give systems of the second kind. An exception is Pliuškevičius [15], who gives a finitary and truly cut-free system for a fragment of linear temporal logic with first-order quantifiers. However, this fragment does not include full propositional linear temporal logic.

Jäger et al. propose using the small model property to finitise the ω-rule for the logic of common knowledge [8] and for the full modal μ-calculus in [7]. This leads to finitary cut-free systems, which have a form of the subformula property. However, the finitary version of the ω-rule has a number of premises which is exponential in the size of the conclusion. Also, this approach relies on the finite model property, rather than allowing us to prove it.

Typically a tableau system closely corresponds to a cut-free sequent system. Unfortunately this is not true in the case of temporal logics, cf. Goré [5]. Here the tableau procedure usually consists of two passes: in the first it constructs a certain graph, from which it deletes certain strongly connected components in the second pass. This two-pass nature gets in the way of a correspondence to a sequent system. Schwendimann gives a one-pass tableau procedure for LTL in [16], but it works on a more intricate data structure than sets of formulas and thus here as well it is hard to see a correspondence to a sequent system.

Here we use the idea behind *focus games* put forward by Lange and Stirling in [12] in order to obtain finitary cut-free sequent systems for LTL and CTL.

Essentially, we just reformulate focus games as a sequent system, or, in game-theoretic terms, reduce a two-player game with a winning condition that is not history-free to a history-free one-player game – and thus not much of a game anymore.

Focus games are based on two observations. Consider unary LTL in negation normal form and the "naive" sequent calculus, where we essentially just add the rules which unfold the fixpoint formulas to a sequent calculus for propositional logic:

$$\bigcirc\ \frac{\Gamma}{\bigcirc\Gamma,\Sigma} \qquad\qquad \Diamond\ \frac{\Gamma,\bigcirc\Diamond\varphi,\varphi}{\Gamma,\Diamond\varphi} \qquad\qquad \Box\ \frac{\Gamma,\varphi \qquad \Gamma,\bigcirc\Box\varphi}{\Gamma,\Box\varphi}$$

The first observation is that this system almost works: the only thing that goes wrong in a standard completeness argument is that we cannot extract a countermodel from a failed branch if we always choose the right premisse in the $\Box$-rule. Thus there is no proof for the induction axiom

$$\varphi \wedge \Box(\varphi \supset \bigcirc\varphi) \ \supset\ \Box\varphi \quad ,$$

which is of course valid. Proof search fails as follows, notice how the endsequent reappears on the upper right:

$$\Box\ \dfrac{\Diamond(\varphi\wedge\bigcirc\overline{\varphi}),\overline{\varphi},\varphi \qquad \Diamond\ \dfrac{\wedge\ \dfrac{\bigcirc\Diamond(\varphi\wedge\bigcirc\overline{\varphi}),\varphi,\overline{\varphi},\bigcirc\Box\varphi \qquad \bigcirc\ \dfrac{\Diamond(\varphi\wedge\bigcirc\overline{\varphi}),\overline{\varphi},\Box\varphi}{\bigcirc\Diamond(\varphi\wedge\bigcirc\overline{\varphi}),\bigcirc\overline{\varphi},\overline{\varphi},\bigcirc\Box\varphi}}{\bigcirc\Diamond(\varphi\wedge\bigcirc\overline{\varphi}),\varphi\wedge\bigcirc\overline{\varphi},\overline{\varphi},\bigcirc\Box\varphi}}{\Diamond(\varphi\wedge\bigcirc\overline{\varphi}),\overline{\varphi},\bigcirc\Box\varphi}}{\Diamond(\varphi\wedge\bigcirc\overline{\varphi}),\overline{\varphi},\Box\varphi} \quad .$$

However, the obvious idea of just closing a cyclic branch as axiomatic will lead to an unsound system. Consider for example the following derivation, where the endsequent is not valid:

$$\Box\ \dfrac{\bigcirc\ \dfrac{\Diamond\ \dfrac{\Box\varphi,\bigcirc\Diamond\Box\varphi}{\Diamond\Box\varphi}}{\varphi,\bigcirc\Diamond\Box\varphi} \qquad \bigcirc\ \dfrac{\Diamond\ \dfrac{\Box\varphi,\bigcirc\Diamond\Box\varphi}{\Box\varphi,\Diamond\Box\varphi}}{\bigcirc\Box\varphi,\bigcirc\Diamond\Box\varphi}}{\Box\varphi,\bigcirc\Diamond\Box\varphi} \quad .$$

The idea behind focus games is to close a cyclic branch if there is an $\Box$-formula such that whenever the $\Box$-rule is applied to it between the two occurrences of the cyclic sequent, the branch is along the right premise. Thus, the rightmost branch of the derivation of the induction axiom would be closed, as would be the right branch in the second derivation, but not the left branch in the second derivation.

The second observation which is crucial to focus games is that the following rule is sound:

$$\frac{\Gamma, \psi(\nu x.\overline{\Gamma} \vee \psi(x))}{\Gamma, \nu x.\psi(x)} \quad ,$$

and thus whenever we unfold an $\Box$-formula (a greatest fixpoint), we can weaken it with the context. This observation goes back to Kozen [10] and is behind the soundness proofs for both focus games and the sequent systems that we consider here.

2 Linear-Time Temporal Logic, LTL

Formulas. The set of *(LTL) formulas* is given by the grammar

$$\varphi \quad ::= \quad p \mid \overline{p} \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \mathtt{X}\varphi \mid \varphi \,\mathtt{U}\, \varphi \mid \varphi \,\mathtt{R}\, \varphi \quad .$$

Propositions p and their negations $\overline{p}$ are also called *atoms*. Atoms are denoted by a and b. A formula with $\mathtt{X}$ (next) as its main connective is called a *next formula*. Formulas with $\mathtt{U}$ (until) or $\mathtt{R}$ (release) as their main connective are respectively called *release formulas* and *until formulas* and collectively they are called *fixpoint formulas*. The *negation* $\overline{\varphi}$ of a formula φ is defined as usual:

$$\begin{array}{lll} \overline{\overline{p}} = p & \overline{\varphi \vee \psi} = \overline{\varphi} \wedge \overline{\psi} & \overline{\varphi \,\mathtt{U}\, \psi} = \overline{\varphi} \,\mathtt{R}\, \overline{\psi} \\ \overline{\mathtt{X}\varphi} = \mathtt{X}\overline{\varphi} & \overline{\varphi \wedge \psi} = \overline{\varphi} \vee \overline{\psi} & \overline{\varphi \,\mathtt{R}\, \psi} = \overline{\varphi} \,\mathtt{U}\, \overline{\psi} \quad . \end{array}$$

Models. A *model*, denoted by σ, is an ω-sequence of sets of propositions. The element of the sequence at position i is denoted by $\sigma(i)$ and $\sigma(0)$ denotes the first element. We now define the relation $\models$. We have $\sigma, i \models p$ iff $p \in \sigma(i)$, $\sigma, i \models \overline{p}$ iff $p \notin \sigma(i)$, $\sigma, i \models \mathtt{X}\varphi$ iff $\sigma, i+1 \models \varphi$, $\sigma, i \models \varphi \vee \psi$ iff $\sigma, i \models \varphi$ or $\sigma, i \models \psi$, $\sigma, i \models \varphi \wedge \psi$ iff $\sigma, i \models \varphi$ and $\sigma, i \models \psi$. Further we have

$$\begin{array}{lll} \sigma, i \models \varphi \,\mathtt{U}\, \psi & \text{iff} & \exists j \geq i \;\; (\sigma, j \models \psi \;\text{ and }\; \forall i \leq k < j \;\; \sigma, k \models \varphi) \quad , \\ \sigma, i \models \varphi \,\mathtt{R}\, \psi & \text{iff} & \forall j \geq i \;\; (\sigma, j \models \psi \;\text{ or }\; \exists i \leq k < j \;\; \sigma, k \models \varphi) \quad , \end{array}$$

and $\sigma \models \varphi$ iff $\forall i \; \sigma, i \models \varphi$. A formula φ is *valid*, denoted by $\models \varphi$, if $\forall \sigma \forall i \; \sigma, i \models \varphi$ and it is *satisfiable* if $\exists \sigma \exists i \; \sigma, i \models \varphi$.

Sequents and annotated formulas. Our sequents will provide a way to store the history of a release formula, which is the set of the contexts in which a rule has been applied to it. To that end we define *annotated formulas*, which are given by the grammar

$$\varphi_H \quad ::= \quad \varphi \,\mathtt{R}_H\, \varphi \mid \mathtt{X}(\varphi \,\mathtt{R}_H\, \varphi) \quad ,$$

$$\mathsf{aid}\,\frac{}{\Gamma, a, \overline{a}} \qquad \vee\,\frac{\Gamma, \varphi, \psi}{\Gamma, \varphi \vee \psi} \qquad \wedge\,\frac{\Gamma, \varphi \qquad \Gamma, \psi}{\Gamma, \varphi \wedge \psi}$$

$$\mathtt{U}\,\frac{\Gamma, \varphi, \psi \qquad \Gamma, \mathtt{X}(\varphi\,\mathtt{U}\,\psi), \psi}{\Gamma, \varphi\,\mathtt{U}\,\psi} \qquad \mathtt{R}\,\frac{\Gamma, \psi \qquad \Gamma, \mathtt{X}(\varphi\,\mathtt{R}\,\psi), \varphi}{\Gamma, \varphi\,\mathtt{R}\,\psi}$$

$$\mathsf{rep}\,\frac{}{\Gamma, \varphi\,\mathtt{R}_{H,\Gamma}\,\psi} \qquad \mathsf{foc}\,\frac{\Gamma, \varphi\,\mathtt{R}_{\emptyset}\,\psi}{\Gamma, \varphi\,\mathtt{R}\,\psi} \qquad \mathtt{X}\,\frac{\Gamma}{\mathtt{X}\,\Gamma, a_1, \dots, a_n}$$

$$\mathtt{RN}_H\,\frac{\Gamma, \psi \qquad \Gamma, \mathtt{X}(\varphi\,\mathtt{R}_{H,\Gamma}\,\psi), \varphi}{\Gamma, \varphi\,\mathtt{R}_H\,\psi}$$

Fig. 1. System LT1.

where the *annotation* or *history* H is a finite set of finite sets of formulas. Put differently, an annotated formula is a pair, consisting of an annotation and of a release formula which is possibly prefixed with an next.

Let the empty disjunction of formulas denote the formula $p \wedge \overline{p}$ for some p and let the the empty conjunction of formulas be its negation. If $\Gamma = \{\varphi_1, \dots, \varphi_n\}$ is a finite set of formulas then the disjunction of all its formulas $\varphi_1 \vee \dots \vee \varphi_n$ is called its *corresponding formula* and is also denoted by Γ. The *corresponding formula* of an annotation $H = \{\Gamma_1, \dots, \Gamma_n\}$ is the formula $\Gamma_1 \wedge \dots \wedge \Gamma_n$ which is also denoted by H. The *corresponding formula* of an annotated formula is obtained by replacing

$$\varphi\,\mathtt{R}_H\,\psi \quad \text{by} \quad (\varphi \vee \overline{H})\;\mathtt{R}\;(\psi \vee \overline{H}) \quad .$$

The semantics of an annotated formula is given by its corresponding formula.

A *presequent* is a finite set of formulas and annotated formulas. A *sequent*, denoted by Γ, Δ, Σ, is a presequent which contains at most one annotated formula. If a sequent contains an annotated formula then this annotated formula is said to be *in focus*. A sequent is *history-free* if it does not contain an annotated formula. Notation for sequents is as usual, i.e. Γ, φ denotes $\Gamma \cup \{\varphi\}$ and is also used for annotations, i.e. H, Γ denotes $H \cup \{\Gamma\}$. The *corresponding formula* of a sequent is the disjunction of the corresponding formulas of its elements and the semantics of a sequent is given by its corresponding formula. A sequent inside a formula denotes its corresponding formula.

A *system* is a set of inference rules and a *derivation* in a system is a possibly infinite tree built according to the rules in that system where the nodes are labelled with sequents (so they contain at most one annotated formula). The

$\mathsf{LT1} - \{\mathsf{aid}, \mathsf{rep}, \mathtt{RN}_H\} +$

$$\mathsf{id}\,\frac{}{\Gamma, \varphi, \overline{\varphi}} \qquad\qquad \mathtt{R}_H\,\frac{\Gamma, \psi, \overline{H} \qquad \Gamma, \mathtt{X}(\varphi\,\mathtt{R}_{H,\Gamma}\,\psi), \varphi, \overline{H}}{\Gamma, \varphi\,\mathtt{R}_H\,\psi}$$

Fig. 2. System LT2.

sequent at the bottom of a rule is called *conclusion* and the sequents at the top are called *premises*. The sequent at the root of a derivation is called *endsequent* and the sequents at the leaves are called *initial sequents*. The conclusion of a rule without premisses is also called *axiom*. A sequent is called *axiomatic* for a system if it is an instance of an axiom in this system and it is *irreducible* for a system if it is not an instance of the conclusion of any rule in this system. A *proof* for the sequent Γ in some system is a finite derivation in that system with the endsequent Γ and with all initial sequents axiomatic. We write $\mathcal{S} \vdash \Gamma$ to denote that there is a proof for Γ in system $\mathcal{S}$.

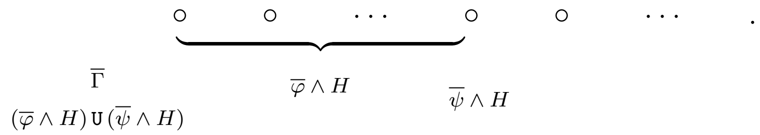
Sequent systems for LTL. System LT1 is shown in Figure 1. It turns out to be complete for history-free sequents but not for arbitrary sequents. Also, the $\mathtt{R}_H$-rule is not invertible. This is remedied in system LT2 shown in Figure 2. The names id, aid, rep and foc respectively stand for *identity, atomic identity, repeat* and *focus*. The N in RN is for *Non-invertible*. Systems in which sequents are sets typically include a hidden contraction. The systems in this paper do not: all rules carry the proviso that the active formula in the conclusion is not an element of the context. For example

$$\frac{\varphi \vee \psi, \varphi, \psi}{\varphi \vee \psi}$$

is not an instance of the $\vee$-rule. In the X-rule, the conclusion is not allowed to be axiomatic. $\mathtt{X}\Gamma$ is obtained from Γ by applying a next connective to every element of Γ. Notice that, since sequents can contain at most one annotated formula, the foc rule is not applicable to a sequent which already contains an annotated formula.

Theorem 1 (Soundness)
(i) If $\mathsf{LT1} \vdash \Gamma$ *then* $\models \Gamma$.
(ii) If $\mathsf{LT2} \vdash \Gamma$ *then* $\models \Gamma$.

PROOF. Soundness of all rules besides rep and $\mathtt{R}_H$ is easy to see: soundness of $\mathtt{RN}_H$ follows from soundness of $\mathtt{R}_H$. For rep note that the negation of the corresponding formula is $\overline{\Gamma} \wedge ((\overline{\varphi} \wedge \Gamma \wedge \dots)\,\mathtt{U}\,(\overline{\psi} \wedge \Gamma \wedge \dots))$, and thus unsatisfiable: because of the first conjunct each model of this formula satisfies $\overline{\Gamma}$ but because of the second conjunct it also satisfies Γ. This is a contradiction, thus there is no such model. For soundness of $\mathtt{R}_H$ we also argue by contradiction. Suppose that the premises are valid and thus 1) $\models \Gamma, \psi, \overline{H}$ and 2) $\models \Gamma, \mathtt{X}(\varphi \, \mathtt{R}_{H,\Gamma} \, \psi), \varphi, \overline{H}$ but that $\not\models \Gamma, \varphi \, \mathtt{R}_H \, \psi$. Then there is a model π with $\pi, 0 \models \overline{\Gamma} \wedge (\overline{\varphi} \wedge H) \, \mathtt{U} \, (\overline{\psi} \wedge H)$. Hence, there is a k such that $\pi, k \models \overline{\psi} \wedge H$ and for all $j < k$: $\pi, j \models \overline{\varphi} \wedge H$, as follows:



Thus, by 1) we have $\pi, 0 \models \psi$. Thus $k > 0$ and $\pi, 0 \models \overline{\varphi} \wedge H$. But then by 2) $\pi, 0 \models \mathtt{X}(\varphi \, \mathtt{R}_{\Gamma,H} \, \psi)$ and thus $\pi, 1 \models \varphi \, \mathtt{R}_{\Gamma,H} \, \psi$, which is equivalent to $(\psi \vee \overline{H} \vee \overline{\Gamma}) \wedge ((\varphi \vee \overline{H} \vee \overline{\Gamma}) \vee \mathtt{X}(\varphi \, \mathtt{R}_{\Gamma,H} \, \psi))$. From the first conjunct together with 1) we get $\pi, 1 \models \psi$ and from the second conjunct together with 2) we get $\pi, 1 \models \mathtt{X}(\varphi \, \mathtt{R}_{\Gamma,H} \, \psi)$.

This argument can now be iterated showing that for all $i \geq 0$ we have $\pi, i \models \psi$. But this contradicts the assumption that $\pi, k \models \overline{\psi}$. □

Completeness. It is easy to see that LT1 is not complete for arbitrary sequents: $a \, \mathtt{R}_{\{\{a\}\}} \, a$ is equivalent to $(a \vee \overline{a}) \, \mathtt{R} \, (a \vee \overline{a})$ and thus valid and clearly provable in LT2 but equally clearly not provable in LT1. We will prove completeness for LT2 via a more restricted system LT2', which we define now. It proves statements of the form $\Gamma : l$, where Γ is a sequent and l is a finite list which contains all release formulas that occur in Γ. The rules of LT2' are just like those of LT2 and they simply pass on the list from the conclusion to all premises. The exception is the foc-rule. We want to focus on the release formula that occurs earliest in the list, so the rule is defined as follows:

$$\mathsf{foc}\,\frac{\Gamma, \varphi \, \mathtt{R}_\emptyset \, \psi : l_1, l_2, \varphi \, \mathtt{R} \, \psi}{\Gamma, \varphi \, \mathtt{R} \, \psi : l_1, \varphi \, \mathtt{R} \, \psi, l_2} \quad \text{no formula in } \Gamma \text{ occurs in } l_1 \quad .$$

This ensures that each release formula which keeps occuring in a branch will be annotated eventually.

Definition 2 (closure) *Let $sf(\varphi)$ be the set of all subformulas of φ and their negations. The closure $cl(\varphi)$ is then defined as*

$$sf(\varphi) \cup \{ \mathtt{X}\psi \mid \psi \textit{ is a fixpoint formula in } sf(\varphi)\} \quad .$$

The closure of a sequent is the union of the closures of its elements and the closure of an annotated formula $\varphi \mathtt{R}_{\Gamma_1 \ldots \Gamma_n} \psi$ is defined as

$$cl(\varphi \mathtt{R} \psi) \cup cl(\Gamma_1) \cup \cdots \cup cl(\Gamma_n) \quad .$$

Clearly, the closure of a sequent is finite. Fix an arbitrary linear ordering of formulas. We denote by $l(\Gamma)$ the list of all release formulas in $cl(\Gamma)$ ordered according to that order.

Theorem 3 (Completeness)
(i) If Γ *is history-free and* $\models \Gamma$ *then* $\mathsf{LT1} \vdash \Gamma$.
(ii) If $\models \Gamma$ *then* $\mathsf{LT2} \vdash \Gamma$.
(iii) If $\models \Gamma$ *then* $\mathsf{LT2}' \vdash \Gamma : l(\Gamma)$.

PROOF. We just show (ii) since (i) is very similar and easier. Statement (iii) implies (ii) since we can just drop all the lists. We show the contrapositive of (iii). Assume $\mathsf{LT2}' \nvdash \Gamma : l(\Gamma)$. Build a possibly infinite derivation with the conclusion $\Gamma : l(\Gamma)$ applying the rules of LT' repeating the following ad infinitum:

(1) apply $\mathsf{id}, \wedge, \vee, \mathtt{X}$ as long as possible,
(2) apply foc if possible,
(3) apply $\mathtt{U}, \mathtt{R}, \mathtt{R}_H$ as long as possible.

Notice that with this strategy only subsets of the closure of the endsequent will enter the histories.

By assumption the above algorithm will not yield a proof, and thus there will be either 1) a finite branch ending in a leaf to which no rule applies or 2) an infinite branch. In both cases define a sequence π of sequents with length $|\pi| \leq \omega$ such that $\pi(i)$ contains exactly those formulas and annotated formulas which occur in some sequent along this branch between the i-th and $(i+1)$-th application of the $\mathtt{X}$-rule. In the second case this sequence is infinite and we define the model $\tilde{\pi}$ as $\tilde{\pi}(i) = \{p \,|\, \overline{p} \in \pi(i)\}$. In the first case this sequence is finite, with $\pi(n)$ its last element, and we define the model $\tilde{\pi}$ as before but with $\tilde{\pi}(i) = \{a \,|\, \overline{a} \in \pi(n)\}$ for $i > n$. It is a countermodel for Γ since $\Gamma \subseteq \pi(0)$ and we prove the following two claims:

Claim 1 For all $i < |\pi|$ and all formulas φ we have $\varphi \in \pi(i) \;\Rightarrow\; \tilde{\pi}, i \not\models \varphi$.

Claim 2 For all $i < |\pi|$ and all annotated formulas φ_H we have $\varphi_H \in \pi(i) \;\Rightarrow\; \tilde{\pi}, i \not\models \varphi_H$.

We prove the first claim by induction on the structure of φ. The claim is true by definition for propositions, it is true for negated proposition since by assumption no element of π can be axiomatic, and it follows easily from the induction hypothesis for conjunctions and disjunctions.

Suppose $\mathtt{X}\varphi \in \pi(i)$. If π is finite and $\pi(n)$ is its last element, then no next-formula can occur in $\pi(n)$, and thus $i < n$. But then $\varphi \in \pi(i+1)$ follows by the $\mathtt{X}$-rule, $\tilde{\pi}, i+1 \not\models \varphi$ follows from the induction hypothesis and thus $\tilde{\pi}, i \not\models \mathtt{X}\varphi$. The same holds for the case where π is infinite.

Suppose that $\varphi \mathtt{U} \psi \in \pi(i)$. Assume that there is a $k \geq i$ s.t. $\varphi \in \pi(k)$. Take the least such k. Then for all $i \leq j \leq k$ we have $\psi \in \pi(j)$. Thus by induction hypothesis for all $i \leq j \leq k$ we have $\pi, j \not\models \psi$ and $\tilde{\pi}, k \not\models \varphi$ and thus $\tilde{\pi}, i \not\models \varphi \, \mathtt{U} \, \psi$. Now assume that there is no such k, i.e. that for all $k \geq i$ we have $\varphi \notin \pi(k)$. Then for all $k \geq i$ we have $\psi \in \pi(k)$ and thus by induction hypothesis $\tilde{\pi}, k \not\models \psi$ and thus $\tilde{\pi}, i \not\models \varphi \, \mathtt{U} \, \psi$.

Suppose $\varphi \, \mathtt{R} \, \psi \in \pi(i)$. First we show that then there is a $k \geq i$ such that $\psi \in \pi(k)$. So assume otherwise, i.e. that for all $k \geq i$ we have $\psi \notin \pi(k)$. Thus, whenever a $\mathtt{R}$ or $\mathtt{R}_H$ rule is applied to the formula $\varphi \, \mathtt{R} \, \psi$ or the annotated formula $\varphi \, \mathtt{R}_H \, \psi$, then our branch is along the premise to the right. Thus for all $k \geq i$ we have either $\varphi \, \mathtt{R} \, \psi \in \pi(k)$ or for some annotation H we have $\varphi \, \mathtt{R}_H \, \psi \in \pi(k)$. Also, this branch cannot end in an irreducible leaf and thus has to be infinite. Notice that a formula cannot be annotated forever: since the $\mathtt{R}_H$-rule always adds the context to the history, and there are only finitely many subsets of the closure of the endsequent, some sequent along this branch would eventually be axiomatic:

$$\mathtt{R}_H \, \frac{\Gamma, \psi, \overline{H} \vee \overline{\Gamma} \qquad \Gamma, \mathtt{X}(\varphi \, \mathtt{R}_{H,\Gamma} \, \psi), \varphi, \overline{H} \vee \overline{\Gamma}}{\Gamma, \varphi \, \mathtt{R}_{H,\Gamma} \, \psi}$$

Thus, by our strategy and the foc-rule, every release formula which occurs in every $\pi(k)$ for $k \geq i$ is eventually annotated and later unannotated. Thus, we in particular have $\varphi \, \mathtt{R}_H \, \psi \in \pi(j)$ for some annotation H and some $j \geq i$. The only way to drop the annotation is by taking the branch along the left premise of the $\mathtt{R}_H$-rule. Thus there is a $k \geq i$ such that $\psi \in \pi(k)$. We take the least such k and thus for all j with $i \leq j < k$ we have $\varphi \in \pi(j)$. By induction hypothesis, $\tilde{\pi}(k) \not\models \psi$ and $\tilde{\pi}(j) \not\models \varphi$ for all such j. But then $\tilde{\pi}, i \not\models \varphi \, \mathtt{R} \, \psi$.

To prove the second claim, suppose $\varphi \, \mathtt{R}_H \, \psi \in \pi(i)$. Similarly to the proof of the first claim we know that there is a $k \geq i$ such that $\psi \vee \overline{H} \in \pi(k)$. We take the least such k and thus for all j with $i \leq j < k$ we have $\varphi \vee \overline{H} \in \pi(j)$. By claim 1, $\tilde{\pi}(k) \not\models \psi \vee \overline{H}$ and $\tilde{\pi}(j) \not\models \varphi \vee \overline{H}$ for all such j. But then $\tilde{\pi}, i \not\models \varphi \, \mathtt{R}_H \, \psi$. The case where we have an annotated release formula prefixed with a next is similar. □

Clearly, completeness of system $\mathsf{LT2}$ gives us admissibility of any sound rule, in particular we get the following corollary.

Corollary 4 (Weakening- and cut-admissibility) *The rules weakening and cut*

$$\mathsf{w} \, \frac{\Gamma}{\Gamma, \varphi} \qquad \textit{and} \qquad \mathsf{cut} \, \frac{\Gamma, \varphi \quad \Delta, \overline{\varphi}}{\Gamma, \Delta}$$

are admissible for system $\mathsf{LT2}$.

Completeness also gives us invertibility of all rules, we just need to check that the validity of the conclusion implies the validity of all premisses:

Corollary 5 (Invertibility) *All rules of system* $\mathsf{LT2}$ *are invertible.*

3 Computational Tree Logic, CTL

Formulas. The set of *(CTL) formulas* is given by the grammar

$$\varphi \ ::= \ p \mid \overline{p} \mid \varphi{\vee}\varphi \mid \varphi{\wedge}\varphi \mid \mathtt{EX}\varphi \mid \mathtt{AX}\varphi \mid \mathtt{E}(\varphi\,\mathtt{U}\,\varphi) \mid \mathtt{E}(\varphi\,\mathtt{R}\,\varphi) \mid \mathtt{A}(\varphi\,\mathtt{U}\,\varphi) \mid \mathtt{A}(\varphi\,\mathtt{R}\,\varphi)\,.$$

In the following, $\mathtt{Q}$ will be an element of $\{\mathtt{E},\mathtt{A}\}$, which are the existential and universal path quantifiers, respectively. Formulas with either $\mathtt{QX}$ as main connective are called *next-formulas.* The negation $\overline{\varphi}$ of a formula φ is defined as usual by the De Morgan laws, in particular $\overline{\mathtt{A}\varphi} = \mathtt{E}\overline{\varphi}$ and $\overline{\mathtt{E}\varphi} = \mathtt{A}\overline{\varphi}$ where φ is a formula in the union of the languages of CTL and LTL.

Models. A *model*, denoted by $\mathcal{M}$, consists of a set of *states* S, a total binary *transition* relation R on S and a *valuation* V which assigns to each state a set of propositions. The relation $\models$ for CTL formulas is defined as usual on the boolean connectives and on the temporal connectives as follows, where s ranges over states in $\mathcal{M}$:

$$\mathcal{M}, s \models \mathtt{EX}\varphi \iff \exists t \text{ with } sRt \text{ and } \mathcal{M}, t \models \varphi$$

$$\mathcal{M}, s \models \mathtt{AX}\varphi \iff \forall t \text{ if } sRt \text{ then } \mathcal{M}, t \models \varphi$$

$$\begin{aligned}
\mathcal{M}, s_0 \models \mathtt{E}(\varphi\,\mathtt{U}\,\psi) &\iff \exists \text{ infinite path } s_0 s_1 s_2 \ldots \\
&\qquad \exists i \geq 0\; s_i \models \psi \text{ and } \forall\, 0 \leq j < i\; s_j \models \varphi \\
\mathcal{M}, s_0 \models \mathtt{A}(\varphi\,\mathtt{U}\,\psi) &\iff \forall \text{ infinite paths } s_0 s_1 s_2 \ldots \\
&\qquad \exists i \geq 0\; s_i \models \psi \text{ and } \forall\, 0 \leq j < i\; s_j \models \varphi \\
\mathcal{M}, s_0 \models \mathtt{E}(\varphi\,\mathtt{R}\,\psi) &\iff \exists \text{ infinite path } s_0 s_1 s_2 \ldots \\
&\qquad \forall i \geq 0\; s_i \models \psi \text{ or } \exists\, 0 \leq j < i\; s_j \models \varphi \\
\mathcal{M}, s_0 \models \mathtt{A}(\varphi\,\mathtt{R}\,\psi) &\iff \forall \text{ infinite paths } s_0 s_1 s_2 \ldots \\
&\qquad \forall i \geq 0\; s_i \models \psi \text{ or } \exists\, 0 \leq j < i\; s_j \models \varphi
\end{aligned}$$

$$\mathcal{M} \models \varphi \iff \forall s\; \mathcal{M}, s \models \varphi \qquad .$$

We write $s \models \varphi$ if $\mathcal{M}$ is clear from the context. A formula φ is *valid*, denoted by $\models \varphi$, if $\forall \mathcal{M} \forall s\; \mathcal{M}, s \models \varphi$ and it is *satisfiable* if $\exists \mathcal{M} \exists s\; \mathcal{M}, s \models \varphi$.

Sequents and annotated formulas. *Annotated formulas* are given by the grammar

$$\varphi_H \quad ::= \quad \mathtt{E}(\varphi\,\mathtt{R}_H\,\varphi) \mid \mathtt{A}(\varphi\,\mathtt{R}_H\,\varphi) \mid \mathtt{EX}(\mathtt{E}(\varphi\,\mathtt{R}_H\,\varphi)) \mid \mathtt{AX}(\mathtt{A}(\varphi\,\mathtt{R}_H\,\varphi)) \quad ,$$

where the *annotation* H is a finite set of finite sets of formulas. The *corresponding formula* of an annotated formula is obtained as in the case for LTL by replacing

$$\varphi\,\mathtt{R}_H\,\psi \quad \text{by} \quad (\varphi \vee \overline{H})\,\mathtt{R}\,(\psi \vee \overline{H}) \quad .$$

A sequent system for CTL. The sequent system for CTL is shown in Figure 3. In the $\mathtt{X}$-rule an α denotes a possibly annotated formula and the case where $m = 0$ is allowed. Also, $\mathtt{EX}\,\Gamma$ is obtained from Γ by applying a $\mathtt{EX}$ to every

$$\textsf{aid}\,\frac{}{\Gamma, a, \overline{a}} \qquad \vee\,\frac{\Gamma, \varphi, \psi}{\Gamma, \varphi \vee \psi} \qquad \wedge\,\frac{\Gamma, \varphi \qquad \Gamma, \psi}{\Gamma, \varphi \wedge \psi}$$

$$\mathtt{QU}\,\frac{\Gamma, \varphi, \psi \qquad \Gamma, \mathtt{QX}(\mathtt{Q}(\varphi\,\mathtt{U}\,\psi)), \psi}{\Gamma, \mathtt{Q}(\varphi\,\mathtt{U}\,\psi)} \qquad \mathtt{QR}\,\frac{\Gamma, \psi \qquad \Gamma, \mathtt{QX}(\mathtt{Q}(\varphi\,\mathtt{R}\,\psi)), \varphi}{\Gamma, \mathtt{Q}(\varphi\,\mathtt{R}\,\psi)}$$

$$\mathtt{X}\,\frac{\alpha_i, \Gamma}{\mathtt{AX}\alpha_1, \ldots, \mathtt{AX}\alpha_m, \mathtt{EX}\,\Gamma, a_1, \ldots, a_n}$$

$$\textsf{rep}\,\frac{}{\Gamma, \mathtt{Q}(\varphi\,\mathtt{R}_{H,\Gamma}\,\psi)} \qquad \textsf{foc}\,\frac{\Gamma, \mathtt{Q}(\varphi\,\mathtt{R}_{\emptyset}\,\psi)}{\Gamma, \mathtt{Q}(\varphi\,\mathtt{R}\,\psi)}$$

$$\mathtt{QR}_H\,\frac{\Gamma, \psi \qquad \Gamma, \mathtt{QX}(\mathtt{Q}(\varphi\,\mathtt{R}_{H,\Gamma}\,\psi)), \varphi}{\Gamma, \mathtt{Q}(\varphi\,\mathtt{R}_H\,\psi)}$$

Fig. 3. System CT.

element of Γ.

Theorem 6 (Soundness)
(i) The rep*-rule is sound.*
(ii) The $\mathtt{QR}_H$*-rule is sound.*
(iii) If $\mathsf{CT} \vdash \Gamma$ *then* $\models \Gamma$*.*

PROOF. The proof is similar to the corresponding one for LTL. Let us see (ii) in detail. Let $\mathtt{Q} = \mathtt{E}$, the proof for the dual case is similar. We argue by contradiction. Suppose that the premises of the $\mathtt{R}_H$-rule are valid and thus 1) $\models \Gamma, \psi$ and 2) $\models \Gamma, \mathtt{EX}(\mathtt{E}(\varphi\,\mathtt{R}_{H,\Gamma}\,\psi)), \varphi$ but that $\not\models \Gamma, \mathtt{E}(\varphi\,\mathtt{R}_H\,\psi)$. Then there is a model $\mathcal{M}$ with a state s with $s \models \overline{\Gamma}$ and $s \models \mathtt{A}((\overline{\varphi} \wedge H)\,\mathtt{U}\,(\overline{\psi} \wedge H))$. Hence for each path π starting from s there is a k such that $\pi(k) \models \overline{\psi} \wedge H$ and for all $j < k$: $\pi(j) \models \overline{\varphi} \wedge H$. By 1) we thus have $s \models \psi$.

Thus we have for each such path that $k > 0$ and that $s \models \overline{\varphi}$. But then by 2) $s \models \mathtt{EX}(\mathtt{E}(\varphi\,\mathtt{R}_{\Gamma,H}\,\psi))$ and thus there is a successor t of s such that $t \models \mathtt{E}(\varphi\,\mathtt{R}_{\Gamma,H}\,\psi)$, which is equivalent to

$$(\psi \vee \overline{H} \vee \overline{\Gamma}) \wedge (\varphi \vee \overline{H} \vee \overline{\Gamma} \vee \mathtt{EXE}(\varphi\,\mathtt{R}_{\Gamma,H}\,\psi)) \qquad .$$

From the first conjunct together with 1) we get $t \models \psi$ and from the second conjunct together with 2) we get $t \models \mathtt{EXE}(\varphi\,\mathtt{R}_{\Gamma,H}\,\psi)$.

This argument can now be iterated to build an infinite path π starting from s

such that for all $i \geq 0$ we have $\pi(i) \models \psi$. But this contradicts the assumption that for all paths π from s there is a k such that $\pi(k) \models \overline{\psi}$. $\square$

Completeness. Like for LTL, we prove completeness for a different system CT', which proves statements of the form $\Gamma : l$, where Γ is a sequent and l is a finite list of release formulas. The rules of CT' are just like those of CT and they simply pass on the list from the conclusion to all premises. There are two exceptions. The first exception is the foc-rule, which is defined as in the previous section:

$$\mathsf{foc}\,\frac{\Gamma, \mathtt{Q}(\varphi\,\mathtt{R}_{\emptyset}\,\psi) : l_1, l_2, \mathtt{Q}(\varphi\,\mathtt{R}\,\psi)}{\Gamma, \mathtt{Q}(\varphi\,\mathtt{R}\,\psi) : l_1, \mathtt{Q}(\varphi\,\mathtt{R}\,\psi), l_2}\quad \text{no formula in } \Gamma \text{ occurs in } l_1 \quad .$$

The second exception is the X-rule. Since it is not invertible we have to keep track of all possible premises:

$$\mathtt{X}\,\frac{\alpha_1, \Gamma : l \qquad \dots \qquad \alpha_m, \Gamma : l}{\mathtt{AX}\alpha_1, \dots, \mathtt{AX}\alpha_m, \mathtt{EX}\,\Gamma, a_1, \dots, a_n : l} \quad .$$

A node in a derivation is called *cyclic* if there is another node on the path to the root which carries the same sequent and is the conclusion of an X-rule. The X-rule applies only if its conclusion is not cyclic. Notice that the applicability of the X-rule is only well-defined for bottom-up proof search, and this is the only way we will use it. Notice also that the branching in this X-rule is very different from the branching of the other rules: the conclusion is provable if one of the premisses is provable. To capture that, we now inductively define when a node in a finite derivation in CT' is *successful.* Axiomatic leaves are successful. For the X-rule the conclusion is successful iff there is a successful premise and for all other rules the conclusion is successful iff all premises are successful. For CT' we also redefine the notion of proof: a *proof* in CT' is a finite derivation with a successful endsequent. The *closure* of a sequent is defined in a similar way to the one for LTL-formulas.

Theorem 7 (Completeness)
(i) For history-free sequents Γ *if* $\models \Gamma$ *then* $\mathsf{CT} \vdash \Gamma$*.*
(ii) For history-free sequents Γ *if* $\models \Gamma$ *then* $\mathsf{CT}' \vdash \Gamma : l(\Gamma)$*.*

PROOF. Statement (ii) implies (i) since we can just drop all the lists and for each application of X we can drop all premises except for one successful premise. We prove the contrapositive of (ii). Assume $\mathsf{CT}' \not\vdash \Gamma : l(\Gamma)$. Build a derivation with the conclusion $\Gamma : l(\Gamma)$ applying the rules of CT' repeating the following until no rule is applicable:

(1) apply aid,rep, $\wedge, \vee,$ X as long as possible,

(2) apply foc if possible,
(3) apply $\mathtt{QU}, \mathtt{QR}, \mathtt{QR}_H$ as long as possible.

Since there are only finitely many expressions $\Gamma : l$ which can occur in the derivation and because of the proviso on the $\mathtt{X}$-rule this yields a finite derivation. By assumption this derivation is not a proof and thus its endsequent is not successful. By construction each non-axiomatic leaf is either irreducible or cyclic.

We now construct a countermodel. First, starting from the endsequent, recursively extract a subtree from the given derivation: for each rule instance which is not an instance of $\mathtt{X}$ drop all premises except for one unsuccessful premise. This unsuccessful premise exists because the endsequent is unsuccessful. The resulting subtree only branches for instances of $\mathtt{X}$ and only contains unsuccessful nodes. Second, collapse all nodes between two adjacent instances of $\mathtt{X}$ and mark the resulting node with the union of all the sequents from these nodes. From this resulting tree we now define a Kripke model $\mathcal{M} = (S, R, V)$. Let S be the set of nodes of the tree, let $V(s) = \{p \mid \overline{p} \in s\}$, and let sRt iff 1) t is a child node of s or 2) s is a leaf and t is a child node of a node which caused s to be cyclic or 3) s contains no formula of the form $\mathtt{AX}\varphi$ and $t = s$. It is clearly total and it is a countermodel for Γ since Γ is a subset of the root of $\mathcal{M}$ and we prove the following claim:

Claim For all $s \in S$ and all formulas φ we have $\varphi \in s \;\Rightarrow\; \mathcal{M}, s \not\models \varphi$.

We prove this by induction on the structure of φ. The claim is true by definition for propositions, it is true for negated propositions since by assumption no element of S can be axiomatic, and it follows easily from the induction hypothesis for conjunctions and disjunctions.

Suppose $\mathtt{EX}\varphi \in s$. But then for all t with sRt we have $\varphi \in t$ by the $\mathtt{X}$-rule, $t \not\models \varphi$ follows from the induction hypothesis and thus $s \not\models \mathtt{EX}\varphi$. The argument is similar for $\mathtt{AX}\varphi$.

Suppose that $\mathtt{E}(\varphi \,\mathtt{U}\, \psi) \in s$. Take an arbitrary infinite path π starting from s. Assume that there is a k such that $\varphi \in \pi(k)$. Then for the least such k and for all $j \leq k$ we have $\psi \in \pi(j)$. Thus by induction hypothesis for all $j \leq k$ we have $\pi(j) \not\models \psi$ and we have $\pi(k) \not\models \varphi$. Now assume that there is no such k, namely that for all k we have $\varphi \notin \pi(k)$. Then for all k we have $\psi \in \pi(k)$ and thus by induction hypothesis $\pi(k) \not\models \psi$. For each path one of these two cases is true and thus $s \not\models \mathtt{E}(\varphi \,\mathtt{U}\, \psi)$. The case for $\mathtt{A}(\varphi \,\mathtt{U}\, \psi)$ is similar.

Suppose $\mathtt{E}(\varphi \,\mathtt{R}\, \psi) \in s$. First we show that for each path π starting from s there is a k such that $\psi \in \pi(k)$. So assume otherwise, namely that there is such a path π such that for all k we have $\psi \notin \pi(k)$. Thus, whenever a $\mathtt{ER}$ or $\mathtt{ER}_H$ rule is applied to the formula $\mathtt{E}(\varphi \,\mathtt{R}\, \psi)$ or the annotated formula $\mathtt{E}(\varphi \,\mathtt{R}_H\, \psi)$, then the premise we chose to keep is the premise to the right. Because of that and because the $\mathtt{X}$-rule keeps this possibly annotated formula from conclusion to all premises, for all k we have either $\mathtt{E}(\varphi \,\mathtt{R}\, \psi) \in \pi(k)$ or for some annotation H we have $\mathtt{E}(\varphi \, \mathtt{R}_H \, \psi) \in \pi(k)$. Notice that a formula can not be annotated forever since the $\mathtt{ER}_H$-rule always adds one of the finitely many subsets of the closure of the endsequent to the history and then some element of π would be successful by rep. Thus by our strategy and the foc-rule every release formula which occurs in every element of π will eventually be annotated and later unannotated. Thus we in particular have $\mathtt{E}(\varphi \, \mathtt{R}_H \, \psi) \in \pi(j)$ for some annotation H and some j. The only way to drop the annotation is taking the branch along the left premise of the $\mathtt{ER}_H$-rule. Thus there is a k such that $\psi \in \pi(k)$. We take the least such k and thus for each path π and the corresponding k and for all j with $j < k$ we have $\varphi \in \pi(j)$. By induction hypothesis, $\pi(k) \not\models \psi$ and $\pi(j) \not\models \varphi$ for all such j. But then $s \not\models \mathtt{E}(\varphi \, \mathtt{R} \, \psi)$. The argument for $\mathtt{A}(\varphi \, \mathtt{R} \, \psi)$ is similar. $\square$

4 Discussion

We have seen sound and complete cut-free sequent systems for LTL and CTL. It seems a safe bet that epistemic logics with common knowledge [2] and also PDL [3] can receive a similar treatment. Focus games have already been defined for PDL with converse [11]. Common knowledge is a fairly simple fixpoint and should be similar to CTL. More interesting are CTL* and, of course, the full modal μ-calculus.

Another interesting problem is syntactic cut elimination. This is quite challenging: even weakening admissibility is problematic syntactically. Typically, weakening is easily shown admissible by a trivial induction on proof depth which simply adds a formula to every sequent in a proof and checks that this does not break any rule instance. However, in system LT1 for example, this does not work. Weakening does not permute over the $\mathtt{RN}_H$-rule. Permuting weakening up would require a rule ? which is not even sound:

$$\mathsf{w}\,\frac{\mathtt{RN}_H \dfrac{\Gamma, \psi \quad \Gamma, \mathtt{X}(\varphi \, \mathtt{R}_{H,\Gamma} \, \psi), \varphi}{\Gamma, \varphi \, \mathtt{R}_H \, \psi}}{\Gamma, \varphi \, \mathtt{R}_H \, \psi, \chi} \quad \rightsquigarrow \quad \mathtt{RN}_H \, \frac{\mathsf{w}\,\dfrac{\Gamma, \psi}{\Gamma, \psi, \chi} \qquad ?\,\dfrac{\Gamma, \mathtt{X}(\varphi \, \mathtt{R}_{H,\Gamma} \, \psi), \varphi}{\Gamma, \mathtt{X}(\varphi \, \mathtt{R}_{H,(\Gamma,\chi)} \, \psi), \varphi, \chi}}{\Gamma, \varphi \, \mathtt{R}_H \, \psi, \chi} \quad .$$

This problem is to be expected. The $\mathtt{RN}_H$-rule incorporates an induction principle and the fact that a certain statement is provable by induction does not imply that a weaker statement is also provable by induction. Nevertheless, for LT2 weakening is admissible, and it would be interesting to have a syntactic proof of that.

When this work was completed, we became aware of the work by Gaintzarain et al., who prove the completeness of a cut-free sequent system for LTL in [4] which is called FC. In contrast to our systems, in this system formulas get arbitrarily larger during a root-first, bottom-up proof-search. So the system FC is far from having a reasonable subformula property. The completeness proof for FC goes along the lines of [13] and is arguably not as simple as the proof that we presented.

Acknowledgements. We thank Rajeev Goré and Florian Widmann for finding a mistake in an earlier version of the completeness proof for CTL. We also thank an anonymous referee for useful comments.

References

[1] E. A. Emerson. Temporal and modal logic. In J. van Leeuwen, editor, *Handbook of Theoretical Computer Science: Volume B: Formal Models and Semantics*, pages 995–1072. Elsevier, Amsterdam, 1990.

[2] Ronald Fagin, Joseph Y. Halpern, Yoram Moses, and Moshe Y. Vardi. *Reasoning about Knowledge.* The MIT Press, Boston, 1995.

[3] M. J. Fischer and R. E. Ladner. Propositional dynamic logic of regular programs. *J. Comput. Syst. Sci.*, 18(2)(2):194–211, 1979.

[4] Joxe Gaintzarain, Montserrat Hermo, Paqui Lucio, Marisa Navarro, and Fernando Orejas. A cut-free and invariant-free sequent calculus for pltl. In Jacques Duparc and Thomas A. Henzinger, editors, *CSL*, volume 4646 of *Lecture Notes in Computer Science*, pages 481–495. Springer, 2007.

[5] Rajeev Goré. Tableau methods for modal and temporal logics. In M. D'Agostino, D. Gabbay, R. Haehnle, and Posegga J., editors, *Handbook of Tableau Methods*, pages 297–396. Kluwer Academic Publishers, 1999.

[6] Éigirdas Gudzhinskas. Syntactical proof of the elimination theorem for von Wright's temporal logic. *Math. Logika Primenen*, 2:113–130, 1982.

[7] Gerhard Jäger, Mathis Kretz, and Thomas Studer. Cut-free systems for the propositional modal mu-calculus. Manuscript, 2006.

[8] Gerhard Jäger, Mathis Kretz, and Thomas Studer. Cut-free common knowledge. *J. Applied Logic*, 5(4):681–689, 2007.

[9] Hiroya Kawai. Sequential calculus for a first order infinitary temporal logic. *Zeitschr. für Math. Logik und Grundlagen der Math.*, 33:423–432, 1987.

[10] Dexter Kozen. Results on propositional mu-calculus. *Theoretical Computer Science*, 27:333–354, 1983.

[11] Martin Lange. Satisfiability and completeness of converse-PDL replayed. In A. Günther, R. Kruse, and B. Neumann, editors, *Proc. 26th German Conf. on Artificial Intelligence, KI'03*, volume 2821 of *Lecture Notes in Artificial Intelligence*, pages 79–92, Hamburg, Germany, 2003. Springer.

[12] Martin Lange and Colin Stirling. Focus games for satisfiability and completeness of temporal logic. In *Proc. 16th Symp. on Logic in Computer Science, LICS'01*, Boston, MA, USA, June 2001. IEEE.

[13] Orna Lichtenstein and Amir Pnueli. Propositional temporal logics: decidability and completeness. *Logic Jnl IGPL*, 8(1):55–85, 2000.

[14] Barbara Paech. Gentzen-systems for propositional temporal logics. In *CSL '88: Proceedings of the 2nd Workshop on Computer Science Logic*, pages 240–253, London, UK, 1989. Springer-Verlag.

[15] Regimantas Pliuskevicius. Investigation of finitary calculus for a discrete linear time logic by means of infinitary calculus. In *Baltic Computer Science, Selected Papers*, pages 504–528, London, UK, 1991. Springer-Verlag.

[16] Stefan Schwendimann. A new one-pass tableau calculus for PLTL. In *TABLEAUX '98: Proceedings of the International Conference on Automated Reasoning with Analytic Tableaux and Related Methods*, pages 277–292, London, UK, 1998. Springer-Verlag.