Extremes (2009) 12:239–263
DOI 10.1007/s10687-008-0078-y

Conditional limit results for type I polar distributions

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Received: 27 June 2007 / Revised: 20 October 2008 /
Accepted: 3 November 2008 / Published online: 21 November 2008

Abstract Let $(S_1, S_2) = (R\cos(\Theta), R\sin(\Theta))$ be a bivariate random vector with associated random radius R which has distribution function F being further independent of the random angle Θ. In this paper we investigate the asymptotic behaviour of the conditional survivor probability $\overline{I}_{\rho,u}(y) := \boldsymbol{P}\left\{\rho S_1 + \sqrt{1-\rho^2} S_2 > y | S_1 > u\right\}, \rho \in (-1, 1), \in I\!R$ when u approaches the upper endpoint of F. On the density function of Θ we impose a certain local asymptotic behaviour at 0, whereas for F we require that it belongs to the Gumbel max-domain of attraction. The main result of this contribution is an asymptotic expansion of $\overline{I}_{\rho,u}$, which is then utilised to construct two estimators for the conditional distribution function $1 - \overline{I}_{\rho,u}$. Furthermore, we allow Θ to depend on u.

Keywords Polar distributions · Elliptical distributions · Gumbel max-domain of attraction · Conditional limit theorem · Tail asymptotics · Estimation of conditional distribution

AMS 2000 Subject Classifications 60F05 · 60G70

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1 Motivation

Let (S_1, S_2) be a spherical bivariate random vector with associated random radius $R > 0$ (almost surely) with distribution function F. The random vector (X, Y) with stochastic representation

$$(X, Y) \stackrel{d}{=} \left(S_1, \rho S_1 + \sqrt{1-\rho^2} S_2\right), \quad \rho \in (-1, 1)$$

is an elliptical random vector ($\stackrel{d}{=}$ stands for equality of the distribution functions). If F is in the Gumbel max-domain of attraction with positive scaling function w, i.e.,

$$\lim_{u \uparrow x_F} \frac{1 - F\big(u + x/w(u)\big)}{1 - F(u)} = \exp(-x), \quad \forall x \in I\!R, \tag{1}$$

where $x_F \in (0, \infty]$ is the upper endpoint of F, then Theorem 4.1 in Berman (1983) implies the following Gaussian approximation

$$\lim_{u \uparrow x_F} \boldsymbol{P}\left\{Z_{u,\rho} > \rho u + y\sqrt{u/w(u)}\right\} = \boldsymbol{P}\left\{Z > y/\sqrt{1-\rho^2}\right\}, \quad \forall y \in I\!R, \tag{2}$$

with $Z_{u,\rho} \stackrel{d}{=} Y|X > u$ and Z a standard Gaussian random variable (mean 0 and variance 1).

Berman's result shows that the Gumbel max-domain of attraction assumption is crucial for the derivation of Eq. 2. Conditional limit results for F in the Weibull max-domain of attraction and (X, Y) a bivariate elliptical random vector are obtained in Berman (1992), Hashorva (2007b). The case F is in the Fréchet max-domain of attraction is simpler to deal with, see Berman (1992).

As shown in Cambanis et al. (1981) we have the following stochastic representation

$$(S_1, S_2) \stackrel{d}{=} \big(R\cos(\Theta), R\sin(\Theta)\big), \tag{3}$$

with R independent of the random angle Θ which is uniformly distributed on $(-\pi, \pi)$, i.e., $(\cos(\Theta))^2$ possesses the Beta distribution with parameters $1/2, 1/2$.

When $(\cos(\Theta))^2$ is Beta distributed, then the random vector (S_1, S_2) is a generalised symmetrised Dirichlet random vector. Generalisation of Eq. 2 for such (S_1, S_2) is presented in Hashorva (2008b) with limit random variable Z being Gamma distributed (see below Example 1).

Three natural questions arise:

a) What is the adequate approximation of the conditional survivor function $\boldsymbol{P}\left\{Z_{u,\rho} > y\right\}$ if $\Theta \in (-\pi, \pi)$ is some general random angle with unknown distribution function?
b) What can be said about the limit random variable Z?
c) Does Z has a more general distribution if the random angle $\Theta = \Theta_u$ varies with u?

In this paper we show that if Θ_u possesses a positive density function h_u with a certain local asymptotic behaviour at 0, then we can answer both questions raised above. The generalisation of Eq. 2 for bivariate polar random vectors (see Definition 1 below) satisfying Eq. 1 is given in Section 3. Two applications of our results are presented in Section 4. The first one concerns the asymptotic behaviour of survivor functions of bivariate polar random vectors. In the second application we discuss the estimation of the conditional distribution function $\boldsymbol{P}\left\{Z_{u,\rho} \le y\right\}$. Proofs and related results are relegated to Section 5.

2 Preliminaries

We shall explain first the meaning of some notation, and then we introduce the class of bivariate polar random vectors. A set of assumptions needed to derive the main results of this paper concludes this section.

If X is a random variable with distribution function H this will be alternatively denoted by $X \sim H$. When H possesses the density function h we write $X \backsimeq h$.

In the following ψ is a positive measurable function such that for all $z \in (0, \infty)$

$$\psi(z) \le K \max\left(z^{\lambda_1}, z^{\lambda_2}\right), \quad K > 0, \lambda_i \in (-1/2, \infty), \quad i = 1, 2, \tag{4}$$

where $\boldsymbol{E}\{\psi(W^2/2)\} > 0$ with W a standard Gaussian random variable. Since $\boldsymbol{E}\{\psi(W^2/2)\} < \infty$ we can define a new distribution function Ψ on $\mathbb{R}$ by

$$\Psi(z) := \frac{\int_{-\infty}^{z} \exp\left(-s^2/2\right) \psi\left(s^2/2\right)\, ds}{\int_{-\infty}^{\infty} \exp\left(-s^2/2\right) \psi\left(s^2/2\right)\, ds}, \quad \forall z \in \mathbb{R}. \tag{5}$$

We denote by $\Psi_{\alpha,\beta}, \alpha, \beta > 0$ the Gamma distribution with density function $x^{\alpha-1} \exp(-\beta x)\beta^{\alpha}/\Gamma(\alpha)$, $x \in (0, \infty)$, where $\Gamma(\cdot)$ is the Gamma function.

Next, we introduce the class of bivariate polar random vectors. Throughout the paper R denotes a positive random radius with distribution function F independent of the random angle $\Theta \in (-\pi, \pi)$, and (S_1, S_2) is a bivariate random vector with representation (3). In the special case Θ is uniformly distributed on $(-\pi, \pi)$ for any two constants a_1, a_2 (see Lemma 6.1 in Berman 1983) we have

$$a_1 S_1 + a_2 S_2 \stackrel{d}{=} \sqrt{a_1^2 + a_2^2} S_1 \stackrel{d}{=} \sqrt{a_1^2 + a_2^2} S_2, \tag{6}$$

hence linear combinations of spherical random vectors (i.e. the elliptical random vectors) are very tractable.

If the random angle Θ is not uniformly distributed on $(-\pi, \pi)$, then Eq. 6 does not hold in general. In this paper we do not make specific distributional assumptions on Θ. We assume however that the random angle Θ possesses a positive density function h on $(-\pi, \pi)$.

Definition 1 A bivariate random vector (X, Y) is referred to as a bivariate polar random vector with coefficients $a_i, b_i, i = 1, 2$ if it has the stochastic representation

$$(X, Y) \stackrel{d}{=} \big(a_1 S_1 + a_2 S_2, b_1 S_1 + b_2 S_2\big), \quad (S_1, S_2) \stackrel{d}{=} \big(R\cos(\Theta), R\sin(\Theta)\big), \tag{7}$$

where $R \sim F$ and $R > 0$ (almost surely) being independent of the random angle $\Theta \in (-\pi, \pi)$.

Clearly, bivariate elliptical random vectors are included in the above class, which is defined in terms of three components, a) the distribution of the associated random radius R, b) the distribution function of the random angle Θ, and c) the deterministic coefficients a_1, a_2, b_1, b_2. In this paper we consider for simplicity the case

$$a_1 = 1, a_2 = 0, \text{ and } b_1 = \rho, b_2 = \sqrt{1-\rho^2}, \quad \rho \in (-1, 1).$$

We refer to ρ as the pseudo-correlation coefficient, and call (X, Y) simply a bivariate polar random vector with pseudo-correlation coefficient ρ. We have thus the stochastic representation

$$(X, Y) \stackrel{d}{=} \left(S_1, \rho S_1 + \sqrt{1-\rho^2} S_2\right), \quad (S_1, S_2) \stackrel{d}{=} \big(R\cos(\Theta), R\sin(\Theta)\big), \quad R \sim F, \tag{8}$$

with $R > 0$ independent of Θ.

We note in passing that S_1, S_2 are in general dependent random variables. If S_1 and S_2 are independent, for instance if R^2 is chi-squared distributed with 2 degrees of freedom and Θ is uniformly distributed on $(-\pi, \pi)$, then (X, Y) is a linear combination of independent Gaussian random variables.

Next, we formulate three assumptions needed in this paper:

A1. [Gumbel max-domain of attraction]
The distribution function F with upper endpoint x_F is in the Gumbel max-domain of attraction satisfying Eq. 1 with the scaling function w. Further, suppose that $F(0) = 0$ and $x_F \in (0, \infty]$.
We formulate next an assumption for the second order approximation in Eq. 1 initially suggested in Abdous et al. (2008).

A2. [Second order approximation of F]
Let F be a distribution function on $[0, \infty)$ satisfying Assumption A1. Suppose that there exist positive functions A, B such that

$$\left|\frac{1 - F\big(u + x/w(u)\big)}{1 - F(u)} - \exp(-x)\right| \le A(u)B(x) \tag{9}$$

holds for all $u < x_F$ large enough and any $x \in [0, \infty)$. Furthermore we assume $\lim_{u \uparrow x_F} A(u) = 0$, and B is locally bounded on finite intervals of $[0, \infty)$.

A3. [Local approximation of $h_n, n \ge 1$ along t_n]
Let $h_n : (-\pi, \pi) \to [0, \infty), n \ge 1$ be a sequence of density functions such that $h_n(\theta) = h_n(-\theta), \forall \theta \in [0, \pi/2)$, and let $t_n, n \ge 1$ be positive constants tending to ∞ as $n \to \infty$. Assume that for any sequence of positive measurable functions $\tau_n(s) = 1 + O(s/t_n), n \ge 1, s \ge 0$ for all large n we have

$$h_n\left(\tau_n(s)\sqrt{\frac{2z}{t_n}}\right) = h_n\big(1/\sqrt{t_n}\big)\psi_{\tau_n}\big(z\tau_n(s)\big), \quad \forall s, z \in [0, \infty), \tag{10}$$

where $\psi_{\tau_n}, n \ge 1$ are positive measurable functions such that

$$\psi_{\tau_n}(s) \to \psi(s), \quad n \to \infty$$

locally uniformly for $s, z \in [0, \infty)$ with ψ_{τ_n} satisfying Eq. 4 for all large n and all $s \in [0, \varepsilon t_n)$ with ε a fixed positive constant.
Our last assumption concerns the second order asymptotic behaviour of $h_n, n \ge 1$ at 0.

A4. [Second order approximation of $h_n, n \ge 1$ along t_n]
Suppose that Assumption A3 holds for some given sequence $t_n, n \ge 1$, and further for any sequence of functions $\tau_n(s) = 1 + O(s/t_n), n \ge 1, s \ge 0$ for all large n we have

$$\left| \frac{1}{h_n(1/\sqrt{t_n})} h_n\left(\tau_n(s)\sqrt{\frac{2z}{t_n}}\right) - \psi(z)\right| \le a(t_n) b_n(z), \quad \forall s, z \in [0, \infty), \tag{11}$$

where $a, b_n, n \ge 1$ are positive measurable functions such that

$$\lim_{n\to\infty} a(t_n) = 0, \quad \lim_{n\to\infty} b_n(s) = b(s),$$

and $b_n, n \ge 1$ satisfy Eq. 4 for all n large.

3 Main results

In this section we consider a bivariate polar random vector (X, Y) with pseudo-correlation ρ and representation (8). We are interested in the asymptotic behaviour of the conditional distribution $Y|X > u_n$ when u_n tends $(n \to \infty)$ to the upper endpoint x_F of F. Several authors have dealt with such conditional probabilities and their statistical estimation, see e.g., Gale (1980), Eddy and Gale (1981), Berman (1982, 1983, 1992), Heffernan and Tawn (2004), Abdous et al. (2005, 2008), Heffernan and Resnick (2007), and Hashorva (2006, 2008a, b). Statistical modelling of conditional distributions is treated in the excellent monograph Reiss and Thomas (2007).

The main restriction on F is that it satisfies Assumption A1 with the scaling function w. Such polar random vectors are referred to alternatively as Type I polar random vectors.

The scaling function w possesses two crucial asymptotic properties: a) uniformly on the compact sets of $\mathbb{R}$

$$\lim_{u\uparrow x_F} \frac{w\big(u + z/w(u)\big)}{w(u)} = 1, \tag{12}$$

and b)

$$\lim_{u\uparrow x_F} uw(u) = \infty, \quad \lim_{u\uparrow x_F} w(u)(x_F - u) = \infty \text{ if } x_F < \infty. \tag{13}$$

Refer to Galambos (1987), Reiss (1989), Embrechts et al. (1997), Falk et al. (2004), Kotz and Nadarajah (2005), De Haan and Ferreira (2006), or Resnick (2008) for more details on the Gumbel max-domain of attraction.

We derive in the next theorem the asymptotic behaviour of $R\cos(\Theta_n)$, with Θ_n a random angle depending on n.

Theorem 1 *Let R be a positive random radius with distribution function F independent of the random angle $\Theta_n \backsimeq h_n, n \geq 1$. Let $u_n, n \geq 1$ be constants such that $u_n < x_F, n \geq 1$ and $\lim_{n\to\infty} u_n = x_F$ with $x_F \in (0, \infty]$ the upper endpoint of F. If F satisfies Assumption* A1, *and the density functions $h_n, n \geq 1$ satisfy Assumption* A3 *along $t_n := u_n w(u_n), n \geq 1$ with $\psi, \psi_{\tau_n}, n \geq 1$, then we have*

$$\boldsymbol{P}\{R\cos(\Theta_n) > u_n\} = \big(1+o(1)\big)t_n^{-1/2}h_n\big(1/\sqrt{t_n}\big)\big[1-F(u_n)\big]$$

$$\times \int_{-\infty}^{\infty} \exp\left(-x^2/2\right)\psi\left(x^2/2\right)\,dx, \quad n \to \infty. \tag{14}$$

If $\Theta_n = \Theta, \forall n \geq 1$ not depending on n, then $R\cos(\Theta)$ has distribution function in the Gumbel max-domain of attraction with the scaling function w. Furthermore, the convergence in probability

$$q_n\big|R\cos(\Theta) - u_n\big|\Big|R\cos(\Theta) > u_n \overset{p}{\to} 0, \quad n \to \infty \tag{15}$$

holds for any sequence $q_n, n \geq 1$ such that $\lim_{n\to\infty} w(u_n)/q_n = \infty$.

We note in passing that Eq. 14 is obtained in Theorem 12.3.1 of Berman (1992) assuming that $(\cos(\Theta_n))^2$ is Beta distributed with positive parameters a, b. See also Tang (2006, 2008) for some important results on tail asymptotics of products of independent random variables.

We state now the main result of this section.

Theorem 2 *Let $(X_n, Y_n), n \geq 1$ be a bivariate polar random vector with representation (8), where $\rho \in (-1, 1)$, $R \sim F$ and $\Theta_n \backsimeq h_n, n \geq 1$. Let $u_n, n \geq 1$ be a positive sequence such that $u_n < x_F, n \geq 1$ and $\lim_{n\to\infty} u_n = x_F$. Suppose that F satisfies Assumption* A1 *and $h_n, n \geq 1$ satisfy Assumption* A3 *along $t_n := u_n w(u_n), n \geq 1$ with $\psi, \psi_{\tau_n}, n \geq 1$. If further $\limsup_{n\to\infty} h_n((1+o(1))/\sqrt{t_n})/h_n(1/\sqrt{t_n}) < \infty$, then for any $x > 0$, $y \in \mathbb{R}$ we have*

$$\lim_{n\to\infty} \boldsymbol{P}\Big\{Y_n \leq \rho u_n + y u_n/\sqrt{t_n}, X_n \leq u_n + x/w(u_n)\Big| X_n > u_n\Big\}$$

$$= \boldsymbol{P}\Big\{Z \leq y/\sqrt{1-\rho^2}, W \leq x\Big\}, \tag{16}$$

with $Z \sim \Psi$ being independent of $W \sim \Psi_{1,1}$, where Ψ is defined in Eq. 5.

Assumption A3 is somewhat cumbersome. If we consider random angles Θ_n not depending on n for all large n, a tractable condition on the local asymptotic behaviour of the density of Θ_n is imposed below.

Theorem 3 *Under the setup of Theorem* 2 *if* $\Theta_n = \Theta \backsimeq h, n \ge 1$ *and instead of Assumption* A3 *we suppose that the density function* h *of* Θ *is regularly varying at* 0 *with index* $2\delta \in (-1, \infty)$, *then for any sequence* $u_n < x_F, n \ge 1$ *such that* $\lim_{n\to\infty} u_n = x_F$ *we have*

$$\boldsymbol{P}\{X_n > u_n\} = \big(1 + o(1)\big)\frac{2^{\delta+1/2}}{\Gamma\big(\delta + 1/2\big)} t_n^{-1/2} h\big(\sqrt{1/t_n}\big)\big[1 - F(u_n)\big], \quad n \to \infty, \tag{17}$$

and X_1 *has distribution function in the max-domain of attraction of the Gumbel distribution with the scaling function* w. *Furthermore Eq.* 15 *holds for any sequence* $q_n, n \ge 1$ *such that* $\lim_{n\to\infty} w(u_n)/q_n = \infty$, *and for* $x > 0, y \in I\!R$ *given constants Eq.* 16 *is satisfied with* $Z^2 \sim \Psi_{\delta+1/2,1/2}$, *and* Z *symmetric about* 0 *independent of* $W \sim \Psi_{1,1}$.

We present next an illustrating example.

Example 1 (Kotz Type III Polar Random Vector) Let $R \sim F$ be a random radius with tail asymptotic behaviour

$$1 - F(u) = \big(1 + o(1)\big) K u^N \exp\big(-r u^\delta\big), \quad K > 0, \delta > 0, N \in I\!R, \quad u \to \infty. \tag{18}$$

If $\Theta \backsimeq h$ is a random angle independent of R we call (X, Y) with stochastic representation (8) a Kotz Type III polar random vector with pseudo-correlation $\rho \in (-1, 1)$. If we set $w(u) := r\delta u^{\delta-1}, u > 0$, then

$$\lim_{u\to\infty} \frac{\boldsymbol{P}\{R > u + x/w(u)\}}{\boldsymbol{P}\{R > u\}} = \exp(-x), \quad \forall x \in I\!R$$

implying that F is in the Gumbel max-domain of attraction with the scaling function w. Suppose that $h(\theta) = h(-\theta), \forall \theta \in [0, \pi/2)$, and further

$$h(\theta) = c_{a,b}|\sin(\theta)|^{2a-1}|\cos(\theta)|^{2b-1}, \quad \theta \in (-\varepsilon, \varepsilon), \quad \varepsilon \in (0, \pi),$$

where $a, b, c_{a,b}$ are positive constants. Note that when

$$\varepsilon = \pi, \quad c_{a,b} = \frac{1}{2}\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)},$$

then (X, Y) is a generalised symmetrised Dirichlet random vector (see Hashorva 2008b). It follows that Assumption A3 is satisfied with

$$h\big(1/\sqrt{t_n}\big) = \big(1 + o(1)\big) c_{a,b} t_n^{1/2-a}, \quad \psi(s) = (2s)^{a-1/2}, \quad s > 0, \quad t_n \to \infty$$

and h is regularly varying at 0 with index $2a-1$. By Eq. 14 for $u_n \to \infty$ we have

$$\boldsymbol{P}\{X > u_n\} = \big(1+o(1)\big)c_{a,b}\,K\big(2/(r\delta)\big)^a\Gamma(a)u_n^{N-a\delta}\exp\big(-ru_n^\delta\big).$$

Next, for any $x>0$, $y \in I\!R$ Theorem 2 implies

$$\lim_{n\to\infty}\boldsymbol{P}\Big\{Y \le \rho u_n + yu_n^{1-\delta/2},\, X \le u_n + xu_n^{1-\delta}\Big| X > u_n\Big\}$$
$$= \boldsymbol{P}\left\{Z \le y\sqrt{r\delta/(1-\rho^2)},\, W \le r\delta x\right\},$$

with Z symmetric about 0 independent of $W \sim \Psi_{1,1}$, and $Z^2 \sim \Psi_{a,1/2}$. Remark that if $a=1/2$, then Z is a standard Gaussian random vector. When also $b=1/2$, then (X,Y) is an elliptical random vector with pseudo-correlation ρ.

In the next theorem we show a second order correction for the conditional limit result obtained in Eq. 16 which is of some interest for statistical applications.

Theorem 4 *Under the assumptions and the notation of Theorem* 2, *if furthermore Assumptions* A2 *and* A4 *are satisfied where* $x_F=\infty$ *and* $\rho \in [0,1)$, *then we have locally uniformly for any* $z \in I\!R$ *(set* $z_{n,\rho} := \rho u_n + zu_n\sqrt{1-\rho^2}/\sqrt{t_n}$*)*

$$\boldsymbol{P}\Big\{Y_n > z_{n,\rho}\Big| X_n > u_n\Big\} = 1-\Psi(z) + \frac{1}{\sqrt{t_n}}\frac{\rho}{\sqrt{1-\rho^2}}\Psi'(z)$$
$$+ O\left(A(u_n)+a(t_n)+\frac{1}{t_n}\right), \quad n\to\infty, \quad (19)$$

provided that

$$\max\left(\int_0^\infty B(s)ds,\, \int_0^\infty B(s)\max(s^{\lambda_1}, s^{\lambda_2})ds\right) < \infty,$$

where $\lambda_i \in (-1/2,\infty)$, $i=1,2$ *are the constants related to Assumption* A3.

Remark 1

a) Abdous et al. (2008) show several examples of distribution functions F satisfying Assumption A2. The assumptions on h can be easily checked for common distribution functions using Taylor expansion.
b) If we assume h is regularly varying with index $2\delta \in (-1,\infty)$ instead of the Assumption A3 and modifying A4 accordingly, then Eq. 19 holds with $\Psi := \Psi_{\delta+1/2,1/2}$, provided that

$$\max\left(\int_0^\infty B(s)ds,\, \int_0^\infty B(s)s^\delta ds\right) < \infty.$$

c) Since $R > 0$ with distribution function in the Gumbel max-domain of attraction implies that also $R^p, p \in (0, \infty)$ has distribution function in he Gumbel max-domain of attraction our results can be easily extended when considering linear combinations of $(S_1, S_2) = (RI_1|\cos(\Theta_*)|^{1/p}, RI_2|\sin(\Theta_*)|^{1/p})$ with $R, \Theta_* \in [0, \pi/2)$, I_1, I_2 mutually independent and I_1, I_2 two random variables assuming values $-1, 1$ such that $\boldsymbol{P}\{I_1 = 1\}\,\boldsymbol{P}\{I_2 = 1\} > 0$ holds.

4 Applications

In this section we present two applications: a) we obtain an asymptotic expansion for the joint survivor probability of polar random vectors, and b) we discuss briefly the estimation of the conditional distributions of such vectors.

4.1 Tail asymptotics

Let (X, Y) be a bivariate polar random vector with pseudo-correlation coefficient $\rho \in (-1, 1)$. Assume that the distribution function F of the random radius R has an infinite upper endpoint. In various situations quantification of the asymptotics of the joint survivor probability $\boldsymbol{P}\{X > x, Y > y\}$ is of interest when x, y become large. Our asymptotic results in Section 3 imply an asymptotic expansion of this survivor probability, provided that (X, Y) is of Type I. Explicitly, under the assumptions of Theorem 3 we obtain for any $x > 0$, $y \in \mathbb{R}$ and u large (set $x_u := u + x/w(u)$, $y_{u,\rho} := \rho u + y\sqrt{u/w(u)}$, $u > 0$)

$$\boldsymbol{P}\left\{X > x_u, Y > y_{u,\rho}\right\}$$
$$= \big(1 + o(1)\big)\exp(-x)\big[1 - \Psi_{\delta+1/2,1/2}(y)\big]\boldsymbol{P}\{X > u\}, \quad u \to \infty.$$

In our asymptotic result the sequence $y_{u,\rho}$ increases like ρu since by (13)

$$y_{u,\rho} = \left(1 + \frac{y}{\rho\sqrt{uw(u)}}\right)\rho u = \big(1 + o(1)\big)\rho u, \quad u \to \infty.$$

It is of some interest to consider also constants $y_{u,\rho} = (1 + o(1))cu$, $u > 0$, with $c \in (\rho, 1]$. In view of Theorem 3 for any $c \in (-\infty, \rho)$ we have

$$\boldsymbol{P}\left\{X > x_u, Y > y_{u,c}\right\} = \big(1 + o(1)\big)\exp(-x)\boldsymbol{P}\{X > u\}, \quad u \to \infty.$$

When $c \in (\rho, 1]$ the joint survivor probability $\boldsymbol{P}\left\{X > x_u, Y > y_{u,c}\right\}$ diminishes faster than $\boldsymbol{P}\{X > u\}$, i.e.,

$$\lim_{n\to\infty} \frac{\boldsymbol{P}\left\{X > x_u, Y > y_{u,c}\right\}}{\boldsymbol{P}\{X > u\}} = 0.$$

If (X, Y) is a bivariate elliptical random vector we may write (see Hashorva 2007a)

$$\boldsymbol{P}\{X > u, Y > cu\} = \big(1 + o(1)\big)\frac{\alpha_{\rho,c}K_{\rho,c}}{2\pi}\frac{1 - F\big(\alpha_{\rho,c}u\big)}{uw\big(\alpha_{\rho,c}u\big)}, \quad u \to \infty \quad (20)$$

for any $c \in (\rho, 1]$ with

$$\alpha_{\rho,c} := \sqrt{\big(1 - 2c\rho + \rho^2\big)/\big(1 - \rho^2\big)} \in (1, \infty), \quad K_{\rho,c} := \frac{\big(1-\rho^2\big)^{3/2}}{\big(1-c\rho\big)\big(c-\rho\big)} \in (0, \infty).$$

In a forthcoming paper we extend Eq. 20 to the case of Type I bivariate polar random vectors.

4.2 Estimation of conditional distributions

Let $(X_i, Y_i), i \le n, n \ge 1$ be independent and identically distributed bivariate polar random vectors with pseudo-correlation coefficient $\rho \in (-1, 1)$ and random radius $R \sim F$ with $x_F = \infty$. Define the conditional distribution function

$$I_{\rho,x}(y) := \boldsymbol{P}\{Y_1 \le y | X_1 > x\}, \quad x, y \in I\!R.$$

For (X_1, Y_1) elliptically distributed Abdous et al. (2008) provide novel estimators of the the conditional distribution function $I_{\rho,x}$. Motivated by the aforementioned paper under the assumptions of Theorem 3 we have (set $t_u := uw(u), u > 0$ and suppose that F satisfies Assumption A1)

$$\sup_{y \in I\!R}\Big| I_{\rho,u}\big(u\big[\rho + y\sqrt{1/t_u}\big]\big) - \Psi_{\delta+1/2,1/2}\big(y/\sqrt{1-\rho^2}\big)\Big| \to 0, \quad u \to \infty, \quad (21)$$

where 2δ is the index of the regular variation of h at 0. Under Assumptions A2 and A4 we obtain additionally the second order asymptotic expansion

$$I_{\rho,u}\big(u\big[\rho + 1/t_u + y\sqrt{1/t_u}\big]\big) = \Psi_{\delta+1/2,1/2}\big(y/\sqrt{1-\rho^2}\big) + O\left(A(u) + a(u) + \frac{1}{t_u}\right), \quad u \to \infty. \quad (22)$$

These approximations motivate the following estimators of $I_{\rho,x}$ for x large and y positive, namely

$$\hat{I}^{(1)}_{\rho,x,n}(y) := \Psi_{\delta+1/2,1/2}\left(\frac{y - \hat{\rho}_n x}{\sqrt{\big(1 - \hat{\rho}_n^2\big)\, x/\hat{w}_n(x)}}\right), \quad n > 1,$$

and

$$\hat{I}^{(2)}_{\rho,x,n}(y) := \Psi_{\delta+1/2,1/2}\left(\frac{y - \hat{\rho}_n\left(x + 1/\hat{w}_n(x)\right)}{\sqrt{\big(1 - \hat{\rho}_n^2\big)\, x/\hat{w}_n(x)}}\right), \quad n > 1,$$

where $\hat{\rho}_n$ is an estimator of ρ, and $\hat{w}_n(\cdot)$ is an estimator of $w(\cdot)$.

An estimator of $\hat{\rho}_n$ can be constructed by considering the relation between ρ and the expectation $\boldsymbol{E}\{Y\}$, provided that the latter exists. Estimation of δ and w are difficult tasks. If the scaling function w (related to the Gumbel max-domain of attraction of F) is simple, say $w(u) = c\gamma u^{\gamma-1}, c, \gamma > 0, u > 0$, then an estimator $\hat{w}_n$ is constructed by estimating separately c and γ from $X_1, \ldots, X_n$ (recall X_1 has distribution function in the Gumbel max-domain of attraction with the scaling function w). See Abdous et al. (2008), Hashorva (unpublished manuscript) for more details.

In practical situations also the constant δ might be unknown and therefore has to be estimated. One possibility of estimating δ is to utilise Eq. 17.

We note that for elliptical random vectors both estimators $\hat{I}^{(1)}_{\rho,x,n}$ and $\hat{I}^{(2)}_{\rho,x,n}$ are suggested in Abdous et al. (2008). Since in this paper we consider estimation of c and γ based on the sample $X_1, \ldots, X_n$ and not from the observations related to the random radius R both estimators suggested have a different asymptotic behaviour as the original ones suggested in the aforementioned paper.

5 Related results and proofs

Set in the following

$$\alpha_\rho(x, y) := \sqrt{1 + \big((y/x) - \rho\big)^2/\big(1 - \rho^2\big)},$$
$$\alpha^*_\rho(x, y) := \alpha_\rho(x, y)x/y, \quad x, y \in \mathbb{R}, y \neq 0, \quad \rho \in (-1, 1). \tag{23}$$

For $1 \leq a < b \leq \infty$, $x > 0$ constants, and h, F two positive measurable functions we define

$$J\big(a, b, x, h\big) := \int_a^b \big[1 - F(xt)\big]h(t)\frac{1}{t\sqrt{t^2 - 1}}\,dt. \tag{24}$$

If $b = \infty$ write simply $J(a, x, h)$ suppressing the second argument. Write $\widetilde{h}(\cdot)$ and $\overline{h}_\rho(\cdot)$ instead of $h(\arccos(1/\cdot))$ and $h(\arcsin(1/\cdot) - \arcsin(\rho))$, respectively.

Next, we shall prove two lemmas, and then proceed with the proof of the main results. The first lemma is formulated for F with infinite upper endpoint. It generalises Lemma 5 in Hashorva (2008a) for bivariate elliptical random vectors. If F has a finite upper endpoint, say $x_F = 1$, then a similar result holds. Statement b) and c) should be reformulated requiring additionally that $x^2 + 2\rho xy + y^2 < 1 - \rho^2$ with $|x|, |y| \in [0, 1]$.

Lemma 5 *Let the random radius $R \sim F$ be independent of the random angle $\Theta \in (-\pi, \pi)$ and define a bivariate polar random vector (X, Y) with pseudo-correlation $\rho \in (-1, 1)$ via Eq. 8. If the upper endpoint x_F of F is infinite and*

Θ possesses a density function h such that $h(\theta) = h(-\theta), \theta \in [0, \pi/2)$, then we have:

a) *For any $x > 0$*

$$\boldsymbol{P}\{X > x\} = 2J\big(1, x, \widetilde{h}\big). \tag{25}$$

b) *For any $x > 0$, $y \in (0, x]$ such that $y/x > \rho$*

$$\boldsymbol{P}\{X > x, Y > y\} = J\big(\alpha_\rho(x, y), x, \widetilde{h}\big) + J\big(\alpha^*_\rho(x, y), y, \overline{h}_\rho\big). \tag{26}$$

c) *For any $x > 0$ and $y/x \in (0, \rho)$, $\rho > 0$*

$$\boldsymbol{P}\{X > x, Y > y\} = 2J\big(1, x, \widetilde{h}\big) - J\big(\alpha_\rho(x, y), x, \widetilde{h}\big) + J\big(\alpha^*_\rho(x, y), y, \overline{h}_\rho\big). \tag{27}$$

Proof Since the associated random radius R is almost surely positive being further independent of Θ and $h(-\theta) = h(\theta), \cos(-\theta) = \cos(\theta), \forall\theta \in [0, \pi/2)$ for any $x > 0$ we obtain

$$\begin{aligned}\boldsymbol{P}\{X > x\} &= 2\int_0^{\pi/2} \boldsymbol{P}\{R > x/\cos(\theta)\}\, h(\theta)\, d\theta \\ &= 2\int_1^{\infty} \big[1 - F(xs)\big]\frac{h(s)}{s\sqrt{s^2 - 1}}\, ds = 2J\big(1, x, \widetilde{h}\big).\end{aligned}$$

We prove next the second statement. By the assumptions $(X, Y) \stackrel{d}{=} (R\cos(\Theta), R\sin(\Theta + \arcsin(\rho)))$, consequently for $x > 0$, $y > 0$ two positive constants

$$\begin{aligned}&\boldsymbol{P}\left\{S_1 > x, \rho S_1 + \sqrt{1 - \rho^2} S_2 > y\right\} \\ &\quad = \boldsymbol{P}\{R\cos(\Theta) > x, R\sin(\Theta + \arcsin(\rho)) > y\}.\end{aligned}$$

Since $\sin(\arcsin(\rho) + \theta)/\cos(\theta)$ is strictly increasing in $\theta \in [-\arcsin(\rho), \pi/2]$ with inverse $\arctan((\cdot - \rho)/\sqrt{1 - \rho^2})$ (see Klüppelberg et al. 2007) we have

$$\begin{aligned}\boldsymbol{P}\{X > x, Y > y\} &= \int_{\arctan\left((y/x-\rho)/\sqrt{1-\rho^2}\right)}^{\pi/2} \boldsymbol{P}\{R > x/\cos(\theta)\}\, dQ(\theta) \\ &\quad + \int_{-\arcsin(\rho)}^{\arctan\left((y/x-\rho)/\sqrt{1-\rho^2}\right)} \boldsymbol{P}\{R > y/\sin(\theta + \arcsin(\rho))\}\, dQ(\theta),\end{aligned}$$

with Q the distribution function of Θ. Transforming the variables we obtain for $y/x > \rho$

$$\boldsymbol{P}\{X > x, Y > y\} = J\big(\alpha_\rho(x, y), x, \widetilde{h}\big) + J\big(\alpha^*_\rho(x, y), y, \overline{h}_\rho\big),$$

and if $y/x \le \rho$ with x, y positive

$$\boldsymbol{P}\{X > x, Y > y\} = 2J\big(1, x, \widetilde{h}\big) - J\big(\alpha_\rho(x, y), x, \widetilde{h}\big) + J\big(\alpha^*_\rho(x, y), y, \overline{h}_\rho\big),$$

hence the proof is complete. □

Lemma 6 *Let F be a distribution function with upper endpoint $x_F \in (0, \infty]$ satisfying further Eq. 1 with the scaling function w and let $1 \le a_n \le b_n, \gamma_n > 1$, $u_n \in (0, x_F), t_n := u_n w(\gamma_n u_n), n \ge 1$ be positive constants such that*

$$b_n u_n < x_F, n \ge 1, \quad \lim_{n\to\infty} \gamma_n = \gamma \in [1, \infty), \quad \lim_{n\to\infty} \gamma_n u_n = \lim_{n\to\infty} b_n u_n = x_F, \tag{28}$$

and further

$$\lim_{n\to\infty} t_n(a_n - \gamma_n) = \xi \in [0, \infty), \quad \lim_{n\to\infty} t_n(b_n - \gamma_n) = \eta \in [\xi, \infty]. \tag{29}$$

Let $h, r, \psi_n, n \ge 1$ be positive measurable functions. Assume that for some $\varepsilon > 0$

$$h(\gamma_n + s/t_n) = r(\gamma_n, t_n)\psi_n(s), \quad \forall s \in [0, \varepsilon t_n] \tag{30}$$

and

$$\psi_n(s) \to \psi(s) \in [0, \infty), \quad n \to \infty \tag{31}$$

locally uniformly with ψ_n satisfying Eq. 4 for all $n \ge 1, s \in [0, \varepsilon t_n]$ with λ_i, $i = 1, 2 \in (c, \infty)$. Suppose further $\int_{a_n}^{\infty} h(s)(s\sqrt{s^2-1})^{-1}\, ds < K < \infty, \forall n > 1$.

a) *If $\gamma \in (1, \infty)$ and $c = -1$, then we have*

$$J\big(a_n, b_n, u_n, h\big) = \big(1 + o(1)\big)\frac{r(\gamma_n, t_n)}{\gamma\sqrt{\gamma^2 - 1}} \frac{1 - F(\gamma_n u_n)}{t_n} \int_\xi^\eta \exp(-x)\psi(x)\, dx, \quad n \to \infty. \tag{32}$$

b) *When $\gamma = 1$ and $\lim_{n\to\infty} t_n(\gamma_n - 1) = \tau \in [0, \infty)$, then*

$$J\big(a_n, b_n, u_n, h\big) = \big(1+o(1)\big) r(\gamma_n, t_n)\frac{1-F(\gamma_n)}{\sqrt{t_n}} \int_\xi^\eta \exp(-x)\frac{1}{\sqrt{2x+2\tau}}\psi(x)\, dx \quad n \to \infty \tag{33}$$

holds provided that $c = -1/2$ if $\xi = \tau = 0$ and $c = -1$ otherwise.

Proof Set in the following for $n \geq 1$

$$u_n^* := \gamma_n u_n, \quad t_n := u_n w\big(u_n^*\big), \quad l_n(x) = \gamma_n + x/t_n,$$

$$\psi_n^*(x) := \frac{\gamma\sqrt{\gamma^2-1}}{l_n(x)\sqrt{l_n^2(x)-1}}\psi_n(x), \quad x \geq 0$$

and $\xi_n := t_n(a_n - \gamma_n), \eta_n := t_n(b_n - \gamma_n)$. Since $\lim_{n\to\infty} u_n^* = x_F$, then Eq. 13 implies

$$\lim_{n\to\infty} t_n = \infty, \quad \lim_{n\to\infty} w(u_n^*)(x_F - u_n^*) = \infty.$$

If c_1, c_2 are two arbitrary constants such that $c_2 > c_1 > 1$ for all n large we have

$$\int_{\gamma+c_2}^{\infty} \big[1 - F(u_n s)\big] h(s) \frac{1}{s\sqrt{s^2-1}}\, ds \leq \big[1 - F\big(u_n(\gamma + c_2)\big)\big] \int_{\gamma+c_2}^{\infty} h(s) \frac{1}{s\sqrt{s^2-1}}\, ds$$

and

$$\begin{aligned} J\big(a_n, b_n, u_n, h\big) &\geq \int_{a_n}^{b_n} \big[1 - F(u_n s)\big] h(s) \frac{1}{s\sqrt{s^2-1}}\, ds \\ &\geq \int_{a_n}^{\gamma+c_1} \big[1 - F(u_n s)\big] h(s) \frac{1}{s\sqrt{s^2-1}}\, ds \\ &\geq \big[1 - F\big(u_n(\gamma + c_1)\big)\big] \int_{a_n}^{\gamma+c_1} h(s) \frac{1}{s\sqrt{s^2-1}}\, ds. \end{aligned}$$

Assume that $x_F = \infty$. Since $1 - F$ is rapidly varying (see e.g., Resnick 2008) i.e.,

$$\lim_{n\to\infty} \frac{1 - F(u_n x)}{1 - F(u_n)} = 0, \quad \forall x > 1$$

for any $\varepsilon^* > 0$ we obtain

$$J\big(a_n, b_n, u_n, h\big) = \big(1 + o(1)\big) \int_{a_n}^{\gamma+\varepsilon^*} \big[1 - F(u_n s)\big] h(s) \frac{1}{s\sqrt{s^2-1}}\, ds.$$

If $\gamma \in (1, \infty)$, then

$$\psi_n^*(s) \to \psi(s), \quad n \to \infty$$

locally uniformly for $s \ge 0$ and ψ_n^* satisfying Eq. 4 for all $s \in [0, \varepsilon t_n)$, $\varepsilon > 0$. As in the Proof of Lemma 7 of Hashorva (2007a) as $n \to \infty$ we obtain

$$\begin{aligned}
&J\big(a_n, b_n, u_n, h\big)\\
&\quad = \big(1+o(1)\big) \int_{\gamma_n+\xi_n/t_n}^{\min(\gamma_n+\eta_n/t_n,\gamma+\varepsilon)} \big[1 - F(u_n s)\big] h(s) \frac{1}{s\sqrt{s^2-1}}\, ds\\
&\quad = \frac{\big(1+o(1)\big)}{t_n} \int_{\xi_n}^{\min(\eta_n,t_n(\gamma-\gamma_n+\varepsilon))} \big[1 - F\big(u_n^* + x/w\big(u_n^*\big)\big)\big]\\
&\qquad \times h\big(l_n(x)\big) \frac{1}{l_n(x)\sqrt{l_n^2(x)-1}}\, dx\\
&\quad = \frac{\big(1+o(1)\big)}{\gamma\sqrt{\gamma^2-1}} \frac{r\big(\gamma_n, t_n\big)}{t_n} \int_{\xi_n}^{\min\big(\eta_n,t_n\big(\gamma-\gamma_n+\varepsilon\big)\big)} \big[1 - F\big(u_n^* + x/w\big(u_n^*\big)\big)\big] \psi_n^*(x)\, dx\\
&\quad = \big(1+o(1)\big) \frac{1}{\gamma\sqrt{\gamma^2-1}} \frac{r\big(\gamma_n, t_n\big)}{t_n} \big[1 - F(u_n^*)\big] \int_{\xi}^{\eta} \exp(-x)\psi(x)\, dx.
\end{aligned}$$

Next, if $\gamma = 1$ redefine

$$\psi_n^*(s) := \frac{1}{\sqrt{t_n} l_n(s)\sqrt{l_n^2(s)-1}} \psi_n(s), \qquad n \ge 1, s \ge 0.$$

We have

$$\psi_n^*(s) \to \frac{\psi(s)}{\sqrt{2\tau + 2s}} =: \psi^*(s)$$

locally uniformly for $s \ge 0$. Hence as in the proof above for $\varepsilon > 0$ and $n \to \infty$ we obtain

$$\begin{aligned}
J\big(a_n, b_n, u_n, h\big) &= \big(1+o(1)\big) \frac{1}{\sqrt{t_n}} \int_{\xi_n}^{\min(\eta_n,\varepsilon t_n)} \big[1 - F\big(u_n^* + x/w(u_n^*)\big)\big] \psi_n^*(x)\, dx\\
&= \big(1+o(1)\big) r\big(\gamma_n, t_n\big) \frac{1 - F(u_n)}{\sqrt{t_n}} \int_{\xi}^{\eta} \exp(-x) \frac{1}{\sqrt{2x+2\tau}} \psi(x)\, dx.
\end{aligned}$$

Similarly, the asymptotic results follow when $x_F \in (0, \infty)$, hence the proof is complete. □

Proof of Theorem 1 We consider for simplicity only the case $x_F = \infty$. For all n large Eq. 25 implies

$$\boldsymbol{P}\{R\cos(\Theta_n) > u_n\} = 2\int_1^\infty \big[1 - F(u_n x)\big] h_n\big(\arccos(1/x)\big)\frac{1}{x}\frac{1}{\sqrt{x^2-1}}\,dx.$$

We have (set $t_n := u_n w(u_n), n \ge 1$)

$$\arccos\big(1/(1+s/t_n)\big) = \frac{\sqrt{2s}}{\sqrt{t_n}}\big(1 + O(s/t_n)\big) =: \sqrt{2s/t_n}\,\tau_n(s), \quad n \to \infty$$

locally uniformly for $s \ge 0$. Hence the Assumption A3 on h_n implies

$$h_n\big(\arccos\big(1/\big(1+s/t_n\big)\big)\big) = h_n\big(\tau_n(s)\sqrt{2s/t_n}\big) = h_n\big(1/\sqrt{t_n}\big)\psi_{\tau_n}\big(s\tau_n(s)\big), \quad s \ge 0.$$

Applying Lemma 6 with $\tau - 1 = \gamma = \gamma_n = a_n = 1, n \ge 1$ and $b_n = \infty, n \ge 1$ we obtain

$$\begin{aligned}&\boldsymbol{P}\{R\cos(\Theta_n) > u_n\}\\ &\quad = \big(1 + o(1)\big)h_n\big(1/\sqrt{t_n}\big)\frac{1 - F(u_n)}{\sqrt{t_n}}\int_{-\infty}^{\infty} \exp(-s^2/2)\psi(s^2/2)\,ds.\end{aligned}$$

If $h_n = h, n \ge 1$, then by the Assumption A3 we have $\lim_{n\to\infty} h(1/\sqrt{t_n})/h(y_n/\sqrt{t_n}) = 1$ for any sequence $y_n, n \ge 1$ such that $\lim_{n\to\infty} y_n = 1$. Consequently, the self-neglecting property of w in Eq. 12 implies

$$\begin{aligned}\lim_{n\to\infty}\frac{\boldsymbol{P}\{R\cos(\Theta) > u_n + x/w(u_n)\}}{\boldsymbol{P}\{R\cos(\Theta) > u_n\}} &= \lim_{n\to\infty}\frac{1 - F\big(u_n + x/w(u_n)\big)}{1 - F(u_n)}\\ &= \exp(-x), \quad \forall x \in I\!R.\end{aligned}$$

Hence for any $z > 0$

$$\begin{aligned}&\boldsymbol{P}\{q_n|R\cos(\Theta) - u_n| > z | R\cos(\Theta) > u_n\}\\ &\quad = \frac{\boldsymbol{P}\{R\cos(\Theta) > u_n + z/q_n\}}{\boldsymbol{P}\{R\cos(\Theta) > u_n\}}\\ &\quad = \frac{\boldsymbol{P}\big\{R\cos(\Theta) > u_n + \big(z/w(u_n)\big)\big(w(u_n)/q_n\big)\big\}}{\boldsymbol{P}\{R\cos(\Theta) > u_n\}}\\ &\quad \to 0, \quad n \to \infty,\end{aligned}$$

thus the result follows. □

Proof of Theorem 2 Set for $n \geq 1$ and $z \in I\!R$

$$v_n = z\sqrt{u_n/w(u_n)}, \quad \chi_n := v_n/u_n, \quad \alpha_n := \sqrt{1+(v_n/u_n)^2}, \quad t_n := u_n w(u_n), \quad n \geq 1$$

and write in the sequel $\widetilde{h}_n(\cdot)$ and $\overline{h}_{n,\rho}(\cdot)$ instead of $h_n(\arccos(1/\cdot))$ and $h_n(\arcsin(1/\cdot) - \arcsin(\rho))$, respectively.

Since $\lim_{n\to\infty} t_n = \infty$ by the assumptions on h making use of Eqs. 13 and 15 we retrieve the convergence in probability

$$\sqrt{w(u_n)/u_n}(X_n - u_n)\Big| X_n > u_n \stackrel{p}{\to} 0, \quad n \to \infty.$$

Consequently, it suffices to show the proof for $\rho = 0$. Next, we prove the convergence in distribution

$$\sqrt{w(u_n)/u_n} Y_n^* \stackrel{d}{\to} Z \sim \Psi, \quad n \to \infty,$$

with $Y_n^* \stackrel{d}{=} Y_n | X_n > u_n$ and Ψ defined in Eq. 5. Since $\chi_n = v_n/u_n > \rho = 0$ holds for all large n, we have in view of Lemma 5 for all large n

$$\boldsymbol{P}\{X_n > u_n, Y_n > v_n\} = J\left(\alpha_n, u_n, \widetilde{h}_n\right) + J\left(\chi_n^{-1}\alpha_n, v_n, \overline{h}_{n,\rho}\right),$$

where $\alpha_n = 1 + (1 + o(1))z^2/(2t_n), n \to \infty$. As in the Proof of Theorem 1 we obtain for the first term

$$J\left(\alpha_n, u_n, \widetilde{h}_n\right) = \frac{\big(1 + o(1)\big)h_n\big(1/\sqrt{t_n}\big)}{\sqrt{t_n}}\big[1 - F(u_n)\big]\int_z^{\infty} \exp\big(-x^2/2\big)\psi\big(x^2/2\big)\,dx.$$

Further, for any $s \geq 0$ (set $l_n(s) := \chi_n^{-1} + s/(v_n w(u_n))$ we have

$$\frac{1}{l_n(s)} = \frac{\chi_n}{1 + s/t_n} = \sqrt{z^2/t_n}\frac{1}{1 + s/t_n}, \quad n \to \infty.$$

Consequently, the assumption on h_n implies for all $s \geq 0$

$$\overline{h}_{n,\rho}\big(l_n(s)\big) = h_n\left(\frac{z}{\sqrt{t_n}}\tau_n(s)\right) = h_n\big(1/\sqrt{t_n}\big)\psi_{\tau_n}\left(\tau_n(s)z^2/2\right),$$

where $\tau_n(s) := 1 + O(s/t_n), s \geq 0, n \geq 1$. Hence

$$\frac{1}{l_n(s)\sqrt{\big(l_n(s)\big)^2 - 1}}\overline{h}_{n,\rho}\big(l_n(s)\big) = h_n\big(1/\sqrt{t_n}\big)\chi_n^{3/2}\psi_{\tau_n}\left(\tau_n(s)z^2/2\right), \quad n \to \infty.$$

As in the Proof of Lemma 6 we have thus

$$\begin{aligned}
J\left(\chi_n^{-1}\alpha_n, v_n, \overline{h}_{n,\rho}\right) &= \int_{\chi_n^{-1}\alpha_n}^{\infty} \left[1 - F(v_n t)\right]\overline{h}_{n,\rho}(t)\frac{1}{t\sqrt{t^2-1}}\,dt \\
&= h_n\big(1/\sqrt{t_n}\big)\frac{1 - F(u_n)}{v_n w(u_n)}\chi_n^{3/2} \\
&\quad \times \int_{t_n[\alpha_n-1]}^{\infty} \frac{1 - F\big(v_n l_n(s)\big)}{1 - F(u_n)}\psi_{\tau_n}\big(\tau_n(s)z^2/2\big)\,ds \\
&= \big(1 + o(1)\big)h_n(1/\sqrt{t_n})\frac{1 - F(u_n)}{v_n w(u_n)}\chi_n^{3/2}\psi(z^2/2) \\
&\quad \times \int_{z^2/2+o(1)}^{\infty} \frac{1 - F\big(u_n + s/w(u_n)\big)}{1 - F(u_n)}\,ds \\
&= o\left(J\left(1, u_n, \widetilde{h}_n\right)\right), \quad n \to \infty
\end{aligned}$$

implying

$$\begin{aligned}
\lim_{n\to\infty} \boldsymbol{P}\left\{Y_n^* > z\sqrt{u_n/w(u_n)}\right\} &= \lim_{n\to\infty} \frac{\boldsymbol{P}\left\{Y_n > z\sqrt{u_n/w(u_n)},\, X_n > u_n\right\}}{\boldsymbol{P}\left\{X > u_n\right\}} \\
&= \frac{\int_z^{\infty} \exp\left(-x^2/2\right)\psi\left(x^2/2\right)\,dx}{\int_{-\infty}^{\infty} \exp\left(-x^2/2\right)\psi\left(x^2/2\right)\,dx} = 1 - \Psi(z).
\end{aligned}$$

Thus the proof is complete. □

Proof of Theorem 3 By the assumption on h we have $h(s)=s^{2\delta}L(s)$ for all $s>0$ in a neighbourhood of 0 with $L(s)$ a positive slowly varying function such that $\lim_{t\to 0} L(ts)/L(t) = 1, \forall s > 0$. Furthermore, by Proposition B.1.10 in De Haan and Ferreira (2006) we have for any $\varepsilon > 0, \xi > 0$

$$\left|\frac{h(ts)}{h(t)} - s^{2\delta}\right| \le \varepsilon \max\left(s^{2\delta-\xi}, s^{2\delta+\xi}\right) \tag{34}$$

holds for any $s \in (0, t_0(\varepsilon, \xi)/t), t \in (0, 1)$ with $t_0(\varepsilon, \xi)$ some positive constant. Since for positive constants $t_n, n \ge 1$ such that $\lim_{n\to\infty} t_n = \infty$

$$\frac{h\left(\sqrt{2s/t_n}\right)}{h\left(\sqrt{1/t_n}\right)} = (2s)^{\delta} =: \psi(s), \quad \forall s > 0$$

the result follows along the lines of the Proof of Theorem 2 utilising further Eq. 34. □

Proof of Theorem 4 Set for $n \ge 1$, $z \in I\!R$ and $|\rho| < 1$

$$v_n := \rho u_n + z\sqrt{1-\rho^2}\sqrt{u_n/w(u_n)}, \quad \chi_n := v_n/u_n, \quad t_n := u_n w(u_n),$$
$$\alpha_n := \sqrt{1 + \big(1/\chi_n^2 - \rho\big)/\big(1-\rho^2\big)}.$$

In view of Lemma 5 for all large n we have $\boldsymbol{P}\{X_n > u_n\} = 2J(1, u_n, \widetilde{h}_n)$ and further for $\rho \ge 0$, $z \ge 0$

$$\boldsymbol{P}\{Y_n > v_n | X_n > u_n\} = \frac{1}{2J\left(1, u_n, \widetilde{h}_n\right)}\Big[J\big(\alpha_n, u_n, \widetilde{h}_n\big) + J\left(\chi_n^{-1}\alpha_n, v_n, \overline{h}_{n,\rho}\right)\Big].$$

In order to show the proof we need to approximate $J\left(1 + \frac{z^2}{2t_n}, u_n, \widetilde{h}_n\right)$ and $J\left(\chi_n^{-1}\alpha_n, v_n, \overline{h}_{n,\rho}\right)$. We have

$$\alpha_n = 1 + \frac{z^2}{2t_n} + O\big(1/t_n^2\big), \quad n \ge 1.$$

As in the Proof of Theorem 2 we obtain $n \to \infty$

$$J\left(\alpha_n, u_n, \widetilde{h}_n\right)$$
$$= \big(1+o(1)\big)h_n\big(1/\sqrt{t_n}\big)\frac{1-F(u_n)}{\sqrt{t_n}}\big[1-\Psi(z)\big]\int_{-\infty}^{\infty}\exp\left(-x^2/2\right)\psi\big(x^2/2\big)\,dx.$$

Assumptions A3 and A4 imply $\psi(x) \le \max(x^{\lambda_1}, x^{\lambda_2})$ and $b(x) \le \max(x^{\lambda_1^*}, x^{\lambda_2^*})$ for some $\lambda_i, \lambda_i^* \in (-1/2, \infty)$. Consequently, the assumptions on B and b yield

$$\int_0^{\infty}\exp(-x)\psi(x)\sqrt{x}\,dx < \infty, \quad \int_0^{\infty}B(x)\psi(x)\frac{1}{\sqrt{2x}}\,dx < \infty,$$
$$\int_0^{\infty}\exp(-x)b(x)\frac{1}{\sqrt{2x}}\,dx < \infty.$$

Define for all $x \ge 0$ and $n \ge 1$

$$g_n(x) := \frac{1}{(1+x/t_n)\sqrt{t_n(1+x/t_n)^2 - t_n}}.$$

For all $x > 0$, $n \ge 1$ we have $g_n(x) \le \frac{1}{\sqrt{2x}}$, and further

$$\Big|\frac{1}{\sqrt{2x}} - g_n(x)\Big| \le \frac{1}{\sqrt{2x}}\Big|1 - \frac{1}{1+x/t_n}\Big| + \frac{1}{1+x/t_n}\Big|\frac{1}{\sqrt{2x}} - \frac{1}{\sqrt{2x + x^2/t_n}}\Big| \le 5/4\sqrt{x}/(t_n\sqrt{2}). \tag{35}$$

Consequently, for $\zeta \geq 0$ and n large we obtain

$$
\begin{aligned}
&\Big|\frac{\sqrt{t_n}}{h_n(1/\sqrt{t_n})[1-F(u_n)]} J\left(1+\zeta/t_n, u_n, \widetilde{h}_n\right) - \int_\zeta^\infty \exp(-x)\psi(x)\frac{1}{\sqrt{2x}}\,dx\Big| \\
&\quad = \Big|\int_{1+\zeta/t_n}^\infty \frac{1-F(u_n t)}{1-F(u_n)}\frac{\sqrt{t_n}}{h_n(1/\sqrt{t_n})}\widetilde{h}_n(t)\frac{1}{t\sqrt{t^2-1}}\,dt - \int_\zeta^\infty \exp(-x)\psi(x)\frac{1}{\sqrt{2x}}\,dx\Big| \\
&\quad = \Bigg|\int_\zeta^\infty \frac{1-F\big(u_n + x/w(u_n)\big)}{1-F(u_n)}\frac{\widetilde{h}_n(1+x/t_n)}{h_n(1/\sqrt{t_n})} g_n(x)\,dx \\
&\qquad - \int_\zeta^\infty \exp(-x)\psi(x)\frac{1}{\sqrt{2x}}\,dx\Bigg| \\
&\quad \leq \int_\zeta^\infty \Big|\frac{1-F\big(u_n + x/w(u_n)\big)}{1-F(u_n)} - \exp(-x)\Big|\frac{\widetilde{h}_n(1+x/t_n)}{h_n(1/\sqrt{t_n})} g_n(x)\,dx \\
&\qquad + \int_\zeta^\infty \exp(-x)\Big|\frac{\widetilde{h}(1+x/t_n)}{h_n(1/\sqrt{t_n})} - \psi(x)\frac{1}{\sqrt{2x}}\Big|\,dx \\
&\quad \leq A(u_n)\int_\zeta^\infty B(x)\psi_n(x)\frac{1}{\sqrt{2x}}\,dx \\
&\qquad + \int_\zeta^\infty \exp(-x) g_n(x)\Big|\frac{\widetilde{h}(1+x/t_n)}{h_n(1/\sqrt{t_n})} - \psi(x)\Big|\,dx \\
&\qquad + \int_\zeta^\infty \exp(-x)\psi(x)\Big|g_n(x) - \frac{1}{\sqrt{2x}}\Big|\,dx \\
&\quad \leq A(u_n)\int_\zeta^\infty B(x)\psi(x)\frac{1}{\sqrt{2x}}\,dx + a(t_n)\int_\zeta^\infty \exp(-x)\frac{1}{\sqrt{2x}} b_n(x)\,dx \\
&\qquad + \frac{5\sqrt{2}}{8t_n}\int_\zeta^\infty \exp(-x)\psi(x)\sqrt{x}\,dx \\
&\quad = O\Big(A(u_n) + a(t_n) + 1/t_n\Big) =: R_n(u_n).
\end{aligned}
$$

Since

$$
\begin{aligned}
&\Big|\int_{z^2/2+O(1/t_n)}^\infty \exp(-x)\psi(x)\frac{1}{\sqrt{2x}}\,dx - \int_{z^2/2}^\infty \exp(-x)\psi(x)\frac{1}{\sqrt{2x}}\,dx\Big| \\
&\quad = O(1/t_n), \quad n \to \infty
\end{aligned}
$$

we have

$$\left|\frac{\sqrt{t_n}}{h_n(1/\sqrt{t_n})[1-F(u_n)]} J(\alpha_n, u_n, \widetilde{h}_n) - \int_{z^2/2}^{\infty} \exp(-x)\psi(x)\frac{1}{\sqrt{2x}}\, dx\right| = R_n(u_n).$$

Assume for simplicity in the following that $\rho > 0$ and $z \geq 0$. The other case can be established as in the Proof of Theorem 2. We obtain the first order asymptotic expansion (set $l_n(s) := \chi_n^{-1} + s/(v_n w(u_n))$, $s > 0, n > 1$)

$$\begin{aligned}
&J\left(\chi_n^{-1}\alpha_n, v_n, \overline{h}_{n,\rho}\right)\\
&\quad = \int_{\alpha_n u_n/v_n}^{\infty} \big[1 - F(v_n t)\big]\overline{h}_{n,\rho}(t)\frac{1}{t\sqrt{t^2-1}}\, dt\\
&\quad = \frac{1-F(u_n)}{v_n w(u_n)} \int_{v_n w(u_n)[\chi_n^{-1}\alpha_n - \chi_n^{-1}]}^{\infty} \frac{1-F\big(u_n + s/w(u_n)\big)}{1-F(u_n)}\overline{h}_{n,\rho}\big(l_n(s)\big)\\
&\qquad \times \frac{1}{l_n(s)\sqrt{l_n^2(s)-1}}\, ds\\
&\quad = \frac{\rho^2}{\sqrt{1-\rho^2}} h_n(1/\sqrt{t_n})\frac{1-F(u_n)}{v_n w(u_n)}\\
&\qquad \times \int_{t_n[\alpha_n - 1]}^{\infty} \frac{1-F\big(u_n + s/w(u_n)\big)}{1-F(u_n)}\psi_{\tau_n}\big(z^2/2\big(1+o(1)\big)\big)\, ds\\
&\quad = \frac{\rho^2}{\sqrt{1-\rho^2}} h_n(1/\sqrt{t_n})\frac{1-F(u_n)}{v_n w(u_n)}\\
&\qquad \times \int_{z^2/2+O(1/t_n)}^{\infty} \frac{1-F\big(u_n + s/w(u_n)\big)}{1-F(u_n)}\psi_{\tau_n}\big(z^2/2\big(1+o(1)\big)\big)\, ds\\
&\quad = \big(1+o(1)\big)\frac{\rho}{\sqrt{1-\rho^2}}\psi(z^2/2) h_n(1/\sqrt{t_n})\frac{1-F(u_n)}{t_n}\\
&\qquad \times \int_{z^2/2}^{\infty} \exp(-s)\, ds, \quad n \to \infty.
\end{aligned}$$

Define next for $n \geq 1$ and $s \geq 0$

$$g_n(s) := \frac{1}{l_n(s)\sqrt{l_n^2(s)-1}}.$$

We have for all $s \ge 0, n \ge 1$

$$\Big|g_n(s) - \widetilde{\rho}\Big| < \left(\frac{z}{\sqrt{t_n}} + s/t_n\right) K, \quad g_n(s) < \widetilde{\rho}K, \quad \widetilde{\rho} := \rho^2/\sqrt{1-\rho^2},$$

with $K > 1$ a positive constant. Next, for any $\zeta \ge 0$ we may write

$$\begin{aligned}
&\Big|\frac{v_n w(u_n)}{h_n(1/\sqrt{t_n})[1-F(u_n)]} J(l_n(\zeta), v_n, \overline{h}_{n,\rho}) - \widetilde{\rho}\psi(\zeta)\exp(-\zeta)\Big| \\
&= \Big|\int_\zeta^\infty \frac{1-F\big(v_n l_n(s)\big)}{1-F(u_n)} \frac{\overline{h}_{n,\rho}(l_n(s))}{h_n(1/\sqrt{t_n})} g_n(s)\, ds - \tilde{\rho}\int_\zeta^\infty \psi(\zeta)\exp(-s)\, ds\Big| \\
&\le \int_\zeta^\infty \Big|\frac{1-F\big(u_n + s/w(u_n)\big)}{1-F(u_n)} \frac{\overline{h}_{n,\rho}\big(l_n(s)\big)}{h_n(1/\sqrt{t_n})} g_n(s)\, ds - \widetilde{\rho}\psi(\zeta)\exp(-s)\Big|\, ds \\
&\le \int_\zeta^\infty \Big|\frac{1-F\big(u_n + s/w(u_n)\big)}{1-F(u_n)} - \exp(-s)\Big| \frac{\overline{h}_{n,\rho}(l_n(s))}{h_n(1/\sqrt{t_n})} g_n(s)\, ds \\
&\quad + \int_\zeta^\infty \exp(-s)\Big|\frac{\overline{h}_{n,\rho}(l_n(s))}{h_n(1/\sqrt{t_n})} g_n(s) - \widetilde{\rho}\psi(\zeta)\Big|\, ds \\
&\le A(u_n)K\widetilde{\rho}\psi(\zeta)\int_\zeta^\infty B(s)\, ds + \widetilde{\rho}\int_\zeta^\infty \exp(-s)\Big|\frac{\overline{h}_{n,\rho}(l_n(s))}{h_n(1/\sqrt{t_n})} - \psi(\zeta)\Big|\, ds \\
&\quad + \int_\zeta^\infty \exp(-s)\Big|g_n(s) - \widetilde{\rho}\Big| \frac{\overline{h}_{n,\rho}(l_n(s))}{h_n(1/\sqrt{t_n})}\, ds \\
&\le A(u_n)K\widetilde{\rho}\psi(\zeta)\int_\zeta^\infty B(s)\, ds + \widetilde{\rho}a(t_n)b_n(\zeta)\exp(-\zeta) \\
&\quad + \psi(\zeta)\int_\zeta^\infty \exp(-s)\Big|g_n(s) - \widetilde{\rho}\Big|\, ds \\
&= R_n(u_n).
\end{aligned}$$

Hence for all n large since $\alpha_n = 1 + z^2/(2t_n) + O(1/t_n^2)$ we may write

$$\Big|\frac{v_n w(u_n)}{h_n\big(1/\sqrt{t_n}\big)\big(1-F(u_n)\big)} J\big(\chi_n^{-1}\alpha_n, v_n, \overline{h}_{n,\rho}\big) - \widetilde{\rho}\psi\big(z^2/2\big)\exp\big(-z^2/2\big)\Big| = R_n(u_n).$$

Consequently, as $n \to \infty$

$$J\left(\alpha_n, u_n, \widetilde{h}_n\right) = h_n(1/\sqrt{t_n}) \frac{1 - F(u_n)}{\sqrt{t_n}} \left[[1 - \Psi(z)] \int_{-\infty}^{\infty} \exp(-x^2/2)\psi(x^2/2)\, dx + R_n(u_n) \right]$$

and

$$J(\chi_n^{-1}\alpha_n, v_n, \overline{h}_{n,\rho}) = \frac{1}{\sqrt{t_n}} h_n(1/\sqrt{t_n}) \frac{1 - F(u_n)}{\sqrt{t_n}} \left[\frac{\rho}{\sqrt{1-\rho^2}} \psi(z^2/2) \exp(-z^2/2) + R_n(u_n) \right]$$

implying

$$\boldsymbol{P}\{Y_n > v_n | X_n > u_n\} = 1 - \Psi(z) + \frac{1}{\sqrt{t_n}} \frac{\rho}{\sqrt{1-\rho^2}} \frac{\exp(-z^2/2)\psi(z^2/2)}{\int_{-\infty}^{\infty} \exp(-x^2/2)\psi(x^2/2)\, dx} + R_n(u_n), \quad n \to \infty.$$

The proof for $z \le 0$ follows with similar calculations, hence the result. □

Acknowledgements I would like to thank two referees for careful reading of the manuscript, several suggestions and corrections.

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