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An Efficient Cluster Algorithm for CP(N-1) Models

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We construct an efficient cluster algorithm for ferromagnetic $SU(N)$-symmetric quantum spin systems. Such systems provide a new regularization for $CP(N-1)$ models in the framework of D-theory, which is an alternative non-perturbative approach to quantum field theory formulated in terms of discrete quantum variables instead of classical fields. Despite several attempts, no efficient cluster algorithm has been constructed for $CP(N-1)$ models in the standard formulation of lattice field theory. In fact, there is even a no-go theorem that prevents the construction of an efficient Wolff-type embedding algorithm. We present various simulations for different correlation lengths, couplings and lattice sizes. We have simulated correlation lengths up to 250 lattice spacings on lattices as large as 640×640 and we detect no evidence for critical slowing down.

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1. Standard Formulation of $CP(N-1)$ Models

The manifold $CP(N-1) = SU(N)/U(N-1)$ is a $(2N-2)$-dimensional coset space relevant in the context of the spontaneous breakdown of an $SU(N)$ symmetry to a $U(N-1)$ subgroup. In particular, in more than two space-time dimensions ($d > 2$) the corresponding Goldstone bosons are described by $N \times N$ matrix-valued fields $P(x) \in CP(N-1)$ which obey

$$P(x)^2 = P(x),\ P(x)^\dagger = P(x),\ \mathrm{Tr}P(x) = 1. \tag{1.1}$$

For $d = 2$ the Hohenberg-Mermin-Wagner-Coleman theorem implies that the $SU(N)$ symmetry cannot break spontaneously. Correspondingly, similar to 4-dimensional non-Abelian gauge theories, the fields $P(x)$ develop a mass-gap nonperturbatively. Motivated by these observations, D'Adda, Di Vecchia, and Lüscher [1] introduced $CP(N-1)$ models as interesting toy models for QCD. The corresponding Euclidean action is given by

$$S[P] = \int d^2x\ \frac{1}{g^2}\mathrm{Tr}[\partial_\mu P \partial_\mu P], \tag{1.2}$$

where g^2 is the dimensionless coupling constant. Note that this action is invariant under global $\Omega \in SU(N)$ transformations

$$P(x)' = \Omega P(x) \Omega^\dagger, \tag{1.3}$$

and under charge conjugation C which acts as ${}^C P(x) = P(x)^*$.

2. D-Theory Formulation of $CP(N-1)$ Models

In this section we describe an alternative formulation of field theory in which the 2-dimensional $CP(N-1)$ model emerges from the dimensional reduction of discrete variables — in this case $SU(N)$ quantum spins in $(2+1)$ space-time dimensions. The dimensional reduction of discrete variables is the key ingredient of D-theory, which provides an alternative nonperturbative regularization of field theory. In D-theory we start from a ferromagnetic system of $SU(N)$ quantum spins located at the sites x of a 2-dimensional periodic square lattice. The $SU(N)$ spins are represented by Hermitean operators $T_x^a = \frac{1}{2}\lambda_x^a$ (Gell-Mann matrices for the triplet representation of $SU(3)$) that generate the group $SU(N)$ and thus obey

$$[T_x^a, T_y^b] = i\delta_{xy} f_{abc} T_x^c,\ \mathrm{Tr}(T_x^a T_y^b) = \frac{1}{2}\delta_{xy}\delta_{ab}. \tag{2.1}$$

In principle, these generators can be taken in any irreducible representation of $SU(N)$. However, as we will see later, not all representations lead to spontaneous symmetry breaking from $SU(N)$ to $U(N-1)$ and thus to $CP(N-1)$ models. The Hamilton operator for an $SU(N)$ ferromagnet takes the form

$$H = -J\sum_{x,i} T_x^a T_{x+\hat{i}}^a, \tag{2.2}$$

where $J > 0$ is the exchange coupling. By construction, the Hamilton operator is invariant under the global $SU(N)$ symmetry, i.e. it commutes with the total spin given by

$$T^a = \sum_x T_x^a. \tag{2.3}$$

The Hamiltonian H describes the evolution of the quantum spin system in an extra dimension of finite extent β. In D-theory this extra dimension is not the Euclidean time of the target theory, which is part of the 2-dimensional lattice. Instead, it is an additional compactified dimension which ultimately disappears via dimensional reduction. The quantum partition function

$$Z = \mathrm{Tr}\exp(-\beta H) \tag{2.4}$$

(with the trace extending over the Hilbert space) gives rise to periodic boundary conditions in the extra dimension.

The ground state of the quantum spin system has a broken global $SU(N)$ symmetry. The choice of the $SU(N)$ representation determines the symmetry breaking pattern. We choose a totally symmetric $SU(N)$ representation corresponding to a Young tableau with a single row containing n boxes. It is easy to construct the ground states of the $SU(N)$ ferromagnet, and one finds spontaneous symmetry breaking from $SU(N)$ to $U(N-1)$. Consequently, there are $(N^2-1)-(N-1)^2 = 2N-2$ massless Goldstone bosons described by fields $P(x)$ in the coset space $SU(N)/U(N-1) = CP(N-1)$. In the leading order of chiral perturbation theory the Euclidean action for the Goldstone boson fields is given by

$$S[P] = \int_0^\beta dt \int d^2x\ \mathrm{Tr}[\rho_s \partial_\mu P \partial_\mu P - \frac{2n}{a^2}\int_0^1 d\tau\ P\partial_t P \partial_\tau P]. \tag{2.5}$$

Here ρ_s is the spin stiffness, which is analogous to the pion decay constant in QCD. The second term in eq.(2.5) is a Wess-Zumino-Witten term which involves an integral over an interpolation parameter τ.

For $\beta = \infty$ the system then has a spontaneously broken global symmetry and thus massless Goldstone bosons. However, as soon as β becomes finite, due to the Hohenberg-Mermin-Wagner-Coleman theorem, the symmetry can no longer be broken, and, consequently, the Goldstone bosons pick up a small mass m nonperturbatively. As a result, the corresponding correlation length $\xi = 1/m$ becomes finite and the $SU(N)$ symmetry is restored over that length scale. The question arises if ξ is bigger or smaller than the extent β of the extra dimension. When $\xi \gg \beta$ the Goldstone boson field is essentially constant along the extra dimension and the system undergoes dimensional reduction. Since the Wess-Zumino-Witten term vanishes for field constant in t, after dimensional reduction the action reduces to

$$S[P] = \beta\rho_s \int d^2x\ \mathrm{Tr}[\partial_\mu P \partial_\mu P], \tag{2.6}$$

which is just the action of the 2-d target $CP(N-1)$ model. The coupling constant of the 2-d model is determined by the extent of the extra dimension and is given by

$$\frac{1}{g^2} = \beta\rho_s. \tag{2.7}$$

Due to asymptotic freedom of the 2-d $CP(N-1)$ model, for small g^2 the correlation length is exponentially large, i.e.

$$\xi \propto \exp(4\pi\beta\rho_s/N). \tag{2.8}$$

Here $N/4\pi$ is the 1-loop coefficient of the perturbative β-function. Indeed, one sees that $\xi \gg \beta$ as long as β itself is sufficiently large. In particular, somewhat counter-intuitively, dimensional

reduction happens in the large β limit because ξ then grows exponentially. In D-theory one approaches the continuum limit not by varying a bare coupling constant but by increasing the extent β of the extra dimension. This mechanism of dimensional reduction of discrete variables is generic and occurs in all asymptotically free D-theory models [2, 3]. It should be noted that (just like in the standard approach) no fine-tuning is needed to approach the continuum limit.

3. Path Integral Representation of $SU(N)$ Quantum Spin Systems

Let us construct a path integral representation for the partition function Z of the $SU(N)$ quantum spin ferromagnet introduced above. In an intermediate step we introduce a lattice in the Euclidean time direction, using a Trotter decomposition of the Hamiltonian. However, since we are dealing with discrete variables, the path integral is completely well-defined even in continuous Euclidean time. Also the cluster algorithm to be described in the following section can operate directly in the Euclidean time continuum [4]. Hence, the final results are completely independent of the Trotter decomposition. In 2 spatial dimensions (with an even extent) we decompose the Hamilton operator into 4 terms

$$H = H_1 + H_2 + H_3 + H_4, \tag{3.1}$$

with

$$H_{1,2} = \sum_{\substack{x=(x_1,x_2) \\ x_i \text{even}}} h_{x,i}, \quad H_{3,4} = \sum_{\substack{x=(x_1,x_2) \\ x_i \text{odd}}} h_{x,i}. \tag{3.2}$$

The individual contributions

$$h_{x,i} = -J\, T^a_x T^a_{x+\hat{i}}, \tag{3.3}$$

to a given H_i commute with each other, but two different H_i do not commute. Using the Trotter formula, the partition function then takes the form

$$Z = \lim_{M\to\infty} \mathrm{Tr}\left\{\exp(-\varepsilon H_1)\exp(-\varepsilon H_2)\exp(-\varepsilon H_3)\exp(-\varepsilon H_4)\right\}^M. \tag{3.4}$$

We have introduced M Euclidean time-slices with $\varepsilon = \beta/M$ being the lattice spacing in the Euclidean time direction. Inserting complete sets of spin states $q \in \{u,d,s,...\}$ the partition function takes the form

$$Z = \sum_{[q]} \exp(-S[q]). \tag{3.5}$$

The sum extends over configurations $[q]$ of spins $q(x,t)$ on a $(2+1)$-dimensional space-time lattice of points (x,t). The Boltzmann factor is a product of space-time plaquette contributions with

$$\begin{aligned}
&\exp(-s[u,u,u,u]) = \exp(-s[d,d,d,d]) = 1, \\
&\exp(-s[u,d,u,d]) = \exp(-s[d,u,d,u]) = \frac{1}{2}[1+\exp(-\varepsilon J)], \\
&\exp(-s[u,d,d,u]) = \exp(-s[d,u,u,d]) = \frac{1}{2}[1-\exp(-\varepsilon J)].
\end{aligned} \tag{3.6}$$

In these expressions the flavors u and d can be permuted to other values. All the other Boltzmann factors are zero, which implies several constraints on allowed configurations.

4. Cluster Algorithm for $SU(N)$ Quantum Ferromagnets

Let us now discuss the cluster algorithm for the $SU(N)$ quantum ferromagnet. Just like the original $SU(2)$ loop-cluster algorithm [5, 6], the $SU(N)$ cluster algorithm builds a closed loop connecting neighboring lattice points with the spin in the same quantum state, and then changes the state of all those spins to a different randomly chosen common value. To begin cluster growth, an initial lattice point (x,t) is picked at random. The spin located at that point participates in two plaquette interactions, one before and one after t. One picks one interaction arbitrarily and considers the states of the other spins on that plaquette. One of the corners of this interaction plaquette will be the next point on the loop. For configurations $C_1 = [u,d,u,d]$ or $[d,u,d,u]$ the next point is the time-like neighbor of (x,t) on the plaquette, while for configurations $C_2 = [u,d,d,u]$ or $[d,u,u,d]$ the next point is the diagonal neighbor. If the states are all the same, i.e. for $C_3 = [u,u,u,u]$ or $[d,d,d,d]$, with probability

$$p = \frac{1}{2}[1+\exp(-\varepsilon J)] \tag{4.1}$$

the next point on the loop is again the time-like neighbor, and with probability $(1-p)$ it is the diagonal neighbor. The next point on the loop belongs to another interaction plaquette on which the same process is repeated. In this way the loop grows until it finally closes.

5. Critical slowing down in the continuum limit

In order to determine the efficiency of this algorithm one has to study its critical slowing down when one approaches the continuum limit. We have used a multi-cluster algorithm for an $SU(3)$ quantum ferromagnet which corresponds to a $CP(2)$ model. As an observable, we have chosen the uniform magnetization which gives the cleanest signal,

$$M = \sum_{x,t}(\delta_{q(x,t),u} - \delta_{q(x,t),d}). \tag{5.1}$$

The autocorrelation time τ of the magnetization is determined from the exponential fall-off of the autocorrelation function. The simulations have been performed at fixed ratio $\xi/L \approx 2.5$, for lattice sizes L/a = 20, 40, 80, 160, 320, 640 and the corresponding correlation lengths ξ/a = 8.87(1), 16.76(1), 32.26(3), 64.6(1), 123.4(2), 253(1). Remarkably, the autocorrelation time doesn't change when one varies the size of the system and the correlation length and stays close to $\tau \approx 1$ sweep. This is a strong indication for an almost perfect algorithm where the critical slowing down is completely eliminated.

6. Conclusions

Due to a no-go theorem [7], so far no efficient cluster algorithm has been developed for $CP(N-1)$ models in the usual Wilson formulation. In the D-theory formulation, one has been able to perform simulations using a multi-cluster algorithm for large correlation lengths and with a corresponding autocorrelation time of about one sweep. Remarkably, there is almost no variation

of the autocorrelation time when one spans a factor of about 30 in the correlation length. The critical slowing down of the algorithm in the continuum limit is hence completely eliminated. These results can be compared to the ones obtained with the efficient multigrid algorithm [8]. Our method has the advantage to obtain autocorrelation times more than 20 times smaller for similar correlation lengths, the multi-cluster algorithm is in addition straightforward to implement.
With the D-theory regularization, it has recently also been possible to simulate $CP(N-1)$ models at non-trivial θ-vacuum angle [9] which is normally impossible due to a severe sign problem.

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