A THESIS ON

RANKING AND SELECTION IN THE METHOD OF PAIRED COMPARISON

with best compliments
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TO

MY PARENTS

CERTIFICATE

This is to certify that the thesis entitled "RANKING AND SELECTION IN THE METHOD OF PAIRED COMPARISON" being submitted by K.S. SCARIAH to the Indian Institute of Technology, Delhi for the award of Doctor of Philosophy in Statistics is a record of bonafide research work carried out by him. Shri K.S. SCARIAH has worked under our guidance and supervision and has fulfilled the requirements for the submission of this thesis, which to the best of our knowledge, has reached the requisite standard.

The results contained in this thesis have not been submitted in part or in full, to any other University or Institute for the award of any degree or diploma.

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C O N T E N T S

S Y N O P S I S

Experimental situations where the subjective judgements or appraisals of the individuals, lead to qualitative, comparative responses or situations where quantifications through measurements are difficult or illusary, the method of paired comparison has a significant role to play. Interest in paired comparison has been generated through the paired nature of competitions of many kinds, through the experimental simplicity of sensory comparisons of items observed in pairs, and through combinatorial properties associated with the construction of experimental designs or tournaments.

The method of paired comparisons has had a long and honourable history in psycological experiments. During the past three decades this method has attracted the attention of people from a wide spectrum of interest. In general, the method would be applicable whenever the objects can be presented to judges to assess their non-measurable quality in pairs either simulteniously or in succession.

The basic goal in the method of paired comparison is to discriminate between $t > 2$ treatments, on the basis of preference data obtained by presenting the treatments in pairs in a given order to a set of judges according to a specified paired comparison design. In such experimental situations the method has led to a surprising amount of model building to provide simple stochastic representation. The main statistical problem with these models have generally been that of estimation of the true worths of the treatments, testing of goodness of fit of the model, testing the hypothesis of equality of treatments etc, based on the score vector obtained by using a suitable economical paired comparison design. Another type of problem that is of interest is to select the best treatment with at least a specified high probability, using a definite selection rule. Selection rules are defined in terms of the score vector that is obtained by using the given design. A related problem is that of choosing a subset of given treatments which includes the best treatment with at least a specified high probability. These problems are respectively special cases of the two basic formulations referred to as Indifference zone formulation and Subset selection formulation in the area of Ranking and Selection procedures for statistical populations.

The majority of the work in the method of paired comparison has been in the model building and its fitting to the experimental data from full paired comparison design. The only work in context of ranking and selection procedures in this area has been due to Trawinski and David (1963) and Trawinski (1969). These authors only considered the selection of the best treatment under both the formulations viz :- indifferences zone and subset selection when the data are only by full paired comparison design with equal number of replications for each pair of treatments. The main subject of interest in the present thesis is the further development of selection and ranking procedures in the method of paired comparison. Besides this, the modification incorporating multiplicative order effect for Bradley - Terry model and additive order effect for Thurstone - Mosteller model have also been considered.

In regard to the construction of ranking and selection procedures, the following three types of goals have been considered under indifferences zone formulation

<u>Goal 1:</u> Selection of the best treatment.

<u>Goal 2:</u> Selection of the k best treatments.

<u>Goal 3:</u> Selection of all treatments better than control,

while under the subset selection the only goal considered is Goal 1 above, For defining the selection rules for these goals we have used data from full paired comparison design or from symmetrical paired comparison design or from both and whenever feasible a comparative performance of the rules under these designs have been studied. An additional feature of the work in this thesis is that the extension of ranking and selection procedures has been done in case ties are also permissible in the paired comparison. Since the selection rules used were based on the scores of the individual treatments obtained by using a given design, the development of selection procedures necessitated us to consider a relatively tedious problem of deriving the distribution of scores and the distribution of order statistics of scores. This in turn needed generation of score vectors on computers. Because exact distribution may at times becomes cumbersome,it was found desirable to develop asymptotic procedures.

The thesis consists of six Chapters. The following is the Chapter-wise description of the thesis in brief.

Chapter I: This chapter is an introductory chapter which reviews the literature and contains definitions that are used in the following chapters.

Chapter II: It deals with the selection problems for Goal I under indifference zone as well as subset selection approach, when the data are obtained by symmetrical paired comparison design with equal number of replications per pair. It is assumed that no ties are permissible in paired comparison of treatments. Tables for implementation of two selection rules have been presented. The comparison of the average number of observations by using symmetrical paired comparison design against full paired comparison design for a given pre-assigned probability of correct selection using the same selection rule has been numerically presented for the asymptotic case. An asymptotic expression of the expected subset size under subset selection formulation has also been given.

Chapter III: As in Chapter II, this chapter also considers Goal 1 under both formulations of ranking and selection. But now it is assumed that ties are also permissible in the paired comparison of treatments and full paired comparison design with equal number of replications for each pair is used for obtaining the data. The incorporation of ties leads to modifications in score vector which in turn leads to the developments of new distribution theory. Tables for implementation of selection rules based on the modified score vector have been provided.

Chapter IV: Results obtained in chapter III for ful paired comparison design have been modified when data are obtained from symmetrical paired comparison design with provision for ties in paired comparison. Tables have been computed for implementing the selection rules and numerical comparisons of the two desings viz; full paired comparison and symmetrical paired comparison designs with provision for ties have been made asymptotically.

Chapter V: All the previous Chapters considered Goal I. In this Chapter other goals have also been considered. It is assumed no ties are permitted and data are from full paired comparison design with the same number of replications for each pair. The following three problems have been discussed.

(a) Selection rule for Goal 2 have been considered under the indifference zone set-up. All details regarding implementation of the selection procedures have been provided.

(b) Selection rule for Goal 3 and its implementation has been worked out under indifference zone set-up.

(c) Using Thurstone - Mosteller model for paired comparison we have given selection rules for Goal I under both the formulations i.e. indifference zone and subset selection. All results have been developed on asymptotic basis and the problem is seen to be similar to that of ranking means in multivariate normal population.

Chapter VI: This Chapter deals with the problems of order effect in paired comparisons. We have extended Bradley - Terry and Thurstone - Mosteller Models for paired comparison experiments to incorporate order effect due to treatment specified characteristics. The order effects are assumed additive for Thurstone - Mosteller model and multiplicative for Bradley - Terry model. The estimation of worth parameters and order effect parameters is considered. Also the testing procedures for goodness of fit of the models have been given.

CHAPTER I

INTRODUCTION AND REVIEW

1.1 Experimental situations where the subjective judgements or appraisals of the individuals leads to qualitative comparative responses or situations where quantifications through measurements are difficult or illusory, the method of paired comparisons has a significant role to play. In such experimental situations the method has led to a surprising amount of model building to provide simple stochastic representation. Interest in paired comparisons has been generated through the paired nature of the computations of many kinds, through the experimental simplicity of sensory comparisons of items observed in pairs, and through combinatorial properties associated with the construction of experimental designs or tournaments.

The method of paired comparisons has had a long and honourable history in psychological experiments. During the past three decades the method of paired comparisons has attracted the attention of people from a wide spectrum of interest:- statistics, quality testing of agricultural products, marketing research, preference measurement, multi dimensional scaling, sports competition and many others. The method of paired comparisons can be used for assessment of non measurable quality of treatments. In all the above areas, and in general, it would be applicable whenever the objects can be presented in pairs either simultaneously or in succession.

The basic goal in the method of paired comparisons is to discriminate between $t > 2$ treatments, on the basis of preference data obtained by presenting the treatments in pairs, in a given orders to a set of judges, according to a specified paired comparisonsdesign. Let $T_1, T_2, \ldots T_t$ be the t treatments. The treatment T_i (i=1, 2,....t) has a true merit or a true parametric value S_i, when judged with respect to some real characteristic Y. We assume that the t merits $S_1, S_2, \ldots S_t$ can be represented by t points on a merit scale.

We write $T_i > T_j$ if T_i is preferred to T_j and define

$$T_i > T_j \quad \text{iff} \quad S_i > S_j$$

while $$T_i = T_j \quad \text{iff} \quad S_i = S_j$$

In any comparison between T_i and T_j the data in a single replication are of the form of preference, no preference for one of the treatments or a tie. Certain scoring systems can be adopted for this kind of observations. Eg. score 1 for preference '0' for no preference and ½ for a tie. Some times judges are not allowed to declare a tie, i.e. in all cases they are forced to give decision favouring one or the other treatment in the pair. In such situations, we have only 0, 1, scoring system.

A paired comparison design in which every possible $\binom{t}{2}$ pairs of treatments is compared is called a full paired (FP) comparison design or Round Robin paired comparison design.

A FP design involves huge amount of experimentation. Hence designs using fractional sets of symmetrical and standard paired comparisons were developed. According to this one uses a suitable design in which he compares a fraction of all pairs of treatments and uses the scores from such comparisons to rank the set of given t treatments. This method is very useful for assessment of quality of different treatments at a reduced cost. Two simplest kinds of fractional pair designs are the symmetrical pairs (SP) design and the standard paired (SCP) comparison design.

An SP design consists of exactly t paired comparisons such that each of the t treatment pairs with exactly two other treatments. Eg. one set of paired comparison under SP design involving treatments $T_1, T_2, \ldots T_t$ would be (T_1, T_2), (T_2, T_3), $(T_3, T_4) \ldots (T_{t-1}, T_t)$, (T_t, T_1), which is one among the $\frac{(t-1)!}{2}$. possible sets of paired comparisons under SP design.

In any of these designs, the pairs of treatments can be compared in a number of independent replications. Let n_{ij} be the number of independent replications on the comparison between the pair (T_i, T_j). If every treatment is compared a fixed number of times, the design is called a balanced paired comparison design.

1.2 Mathematical Model

In order to formulate the basic model for the method of paired comparisons, one proceeds as follows. At the moment we shall assume that the judges are not allowed to declare a tie and order effect are not present. Consider 't' treatments T_1, $T_2, \ldots T_t$ with merits $S_1, S_2 \ldots S_t$ respectively. The observed merit of T_i will vary from observation to observation and may also be represented on the same scale as S_i by the continuous random variable Y_i $(-\infty \leq Y_i \leq \infty)$. Then in a paired comparison of T_i and T_j if we declare $T_i > T_j$ then

$$Y_i > Y_j \quad \text{and} \quad T_i < T_j \quad \text{then} \quad Y_i < Y_j$$

By assumption $P_r(Y_i = Y_j) = 0$

and let

$$\pi_{ij} = P_r(Y_i > Y_j) \qquad (1.1)$$

$$i \neq j = 1, 2, \ldots, t,$$

The probability π_{ij} is called the preference probability of T_i over T_j. Note that

$$\pi_{ij} + \pi_{ji} = 1$$

and $\pi_{ij} = 1/2$ when $T_i = T_j$

The preference probabilities π_{ij} are said to satisfy a linear model if $\pi_{ij} = H(S_i - S_j)$ (1.2)

where H is the c. d. f. of a symmetrical random variable with mean 0 i.e. $H(-x) = 1 - H(x)$. The following two special linear models have been widely used in the past.

1. T.M. Model

In this model it is assumed that H is the c. d. f. of the standard normal distribution (denoted by ϕ). Thus preference probabilities are given by $\pi_{ij} = \phi(S_i - S_j)$

$$= \int_{-(S_i - S_j)}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-1/2\, y^2} \, dy \qquad (1.3)$$

This model is known as Thurstone Mosteller Model (1951a).

2. B.T. Model

Let $S_i = \ln \pi_i$, $i=1, \ldots, t$ and assume without loss of generality that the merit scale is transformed so that

$$\pi_i \geq 0 \ , \ \Sigma \pi_i = 1 \tag{1.4}$$

Now if we suppose that H is the c. d. f. of a logistic distribution i.e. $H(x) = \frac{1}{1+e^{-x}}$ then preference probabilities are given by

$$\pi_{ij} = \pi_i / (\pi_i + \pi_j) \tag{1.5}$$

This model is known as Bradely - Terry model (1952).

Consider a paired comparison design D in which the pair (T_i, T_j), $i > j$, is compared n_{ij}, $(i, j = 1, 2, \ldots t)$ times where n_{ij} is any integer ≥ 0. Let x_{ijr} be a characteristic random variable corresponding to the comparison of T_i and T_j in the r^{th} replication.

$$x_{ijr} = \begin{cases} 1 & \text{if } T_i > T_j, \ (i \neq j = 1, 2 \ldots t, \ r = 1, \ldots n_{ij}) \\ 0 & \text{if } T_j > T_i \end{cases} \tag{1.6}$$

We assume that there is no replication effect, that all $\binom{t}{2}$ comparisons are independent. Then we have

$$\begin{aligned} P_r (x_{ijr} = 1) &= \pi_{ij} \\ P_r (x_{ijr} = 0) &= \pi_{ji} = 1 - \pi_{ij} \end{aligned} \tag{1.7}$$

The score a_i of treatment T_i is given by

$$a_i = \sum_{\substack{j=1 \\ j \neq i}}^{t} \sum_{r=0}^{n_{ij}} x_{ijr} \tag{1.8}$$

The vector $\underline{a} = (a_1, a_2, \ldots a_t)$ will be called the score vector.

The above definitions are originally due to Kendall, who considered the particular case with $n_{ij}=n$.

The main statistical problems connected with these models and their generalisations incorporating ties and order effects have generally been that of estimation of true worth S_i, testing of goodness of fit of the model, testing the hypothesis of equality of treatments ratings, test of significance of tie, or order effect parameters, based on the relevent scores obtained by a suitable paired comparison design D.

A more general formulation for the method of paired comparisons could be to assume that the preference probabilities π_{ij}, $i > j$, $i, j = 1, 2, \ldots t$ constitutes the parameters of the model and if $T_i > T_j$ then $\pi_{ij} > \frac{1}{2}$ whereas if $T_i = T_j$ then $\pi_{ij} = \frac{1}{2}$. Then we try to define a feasible true ranking in terms of the probabilities π_{ij}. One way to achieve this is to compute

$$\pi_{i.} = \sum_j \pi_{ij}, \; i = 1, 2, \ldots\ldots t \tag{1.9}$$

the row-sum probabilties corresponding to treatment T_i and rank treatments according to desending order of $\pi_{i.}$'s

In the formulation of mathematical model we assumed that judges are not allowed to declare a tie. Even if the judges are efficient, there is a certain threshold with in which they will not be able to detect the difference between two treatments. In such cases, they have to declare a tie instead of giving a forced decision in one way or the other. Thus it is postulated that there exists an interval of length 2β centered at the origin of the distribution of $Y_i - Y_j$ with in which the judge can not distinguish between Y_i and Y_j and will declare a tie. Accordingly in a paired comparison between T_i and T_j, we shall declare $T_i > T_j$ if $Y_i > Y_j + \beta$ $T_i < T_j$ if $Y_i < Y_j - \beta$ and $T_i = T_j$ if $|Y_i - Y_j| \leq \beta$. Now the preference probability of T_i over T_j would be given by

$$\pi_{i.ij} = P((Y_i - Y_j) > \beta) \quad (1.10)$$

and that of T_j over T_i by

$$\pi_{j.ij} = P((Y_j - Y_i > \beta) \quad (1.11)$$

We now also have a +ve probability for declaring a tie which is given by

$$\pi_{0.ij} = P(|Y_i - Y_j| \leq \beta) \quad (1.12)$$

clearly $\pi_{i.ij} + \pi_{j.ij} + \pi_{0.ij} = 1$

The definition of linear model now generalises to the following. The preference probabilities are said to satisfy a linear model if

$$\pi_{i.ij} = H(S_i - S_j - \beta) \quad (1.13)$$

where H is the c. d. f. of a symmetrical random variable with mean 0. Note that

$$\pi_{j.ij} = 1 - H(S_i - S_j + \beta) \quad (1.14)$$

and $\pi_{0.ij} = H(S_i - S_j + \beta) - H(S_i - S_j - \beta)$ (1.15)

Another type of problem that is of interest is selecting the best treatment with at least a specified high probability P* on the basis of a definite selection rule defined in terms of the score vector $\underline{a}$ that is obtained by using a convenient and economical design D. A related problem is that of choosing a subset of t given treatments which includes the best treatment with at least a specified probability P* close to 1. These problems are respectively the special cases of the two basic formulations referred to as Indifference zone formulation and Subset selection formulation in the area of Ranking and Selection procedures for statistical populations. The next section briefly discusses these two procedures.

1.3 Ranking and Selection procedures:

Ranking and selection procedures are statistical techniques for comparing a number t of populations denoted by $T_1, T_2 \ldots\ldots T_t$. Assume that the populations are not all the same and can be ordered in some meaningful way from worst to best or from smallest to largest. Let population T_i be characterised by a parameter θ_i, $i = 1, 2, \ldots\ldots t$, then we say that $T_i > T_j$ if $\theta_i > \theta_j$. The population corresponding to maximum θ_i is called the best. For example in an agricultural experiment the populations may be the different varieties of wheat seeds and θ_i may be the true yield of the i^{th} variety.

The classical approach in the preceeding situation is to test the hypothesis $H_0: \theta_i = \theta_j$ or the joint hypothesis $H_0: \theta_1 = \theta_2 = \ldots\ldots = \theta_t$ for the normal populations $N(\theta_i, \sigma^2)$, $(i=1, \ldots\ldots t)$. These hypothesis can be tested by using the analysis of variance methods. For other types of populations one can use the Neyman - Pearson theory. But if the null hypothesis is rejected, it means the parameters differ significantly, then one would like to check which population has highest or lowest parameter. Previously, a less efficient method of least significant differences based on t-tests had been used to find the difference between the parameters and thus choose the best one. Now the more realistic approach of ranking and selection under indifference zone and subset selection set up have taken this place.

The parametric space Ω is the space of all parametric configurations $(\theta_1, \theta_2, \ldots\ldots \theta_t)$. Let $\theta_{(1)} \leq \theta_{(2)} \leq \ldots\ldots \leq \theta_{(t)}$ be the ordered values of θ_i, i=1, 2,......t. Hence the ordered populations are $T_{(1)} \leq T_{(2)} \leq \ldots \leq T_{(t)}$. We assume that it is not known which of the populations T_i corresponds to $T_{(t)}$. The parametric configuration of θ_i is such that

$$\{\theta_{(1)} = \theta_{(2)} = \ldots\ldots = \theta_{(t-1)} = \theta,\ \theta_{(t)} = \alpha\theta\},\ \alpha \geq 1$$

or $\{\theta_{(1)} = \theta\ ,\ \theta_{(2)} = \theta_{(3)} = \ldots\ldots\ldots\ldots = \theta_{(t)} = \alpha\theta\},\ \alpha \geq 1$

is called slippage parametric configuration. The population $T_{(1)}$, $T_{(2)}, \ldots\ldots\ldots, T_{(k)}$ (or $T_{(t-k+1)}, T_{(t-k+2)}, \ldots\ldots\ldots T_{(t)}$) corresponding the parameters $\theta_{(1)}, \ldots\ldots\ldots \theta_{(k)}$ (or $\theta_{(t-k+1)}, \theta_{(t-k+2)} \ldots\ldots\ldots \theta_{(t)}$) respectively are called k-best populations. Populations T_i's i=1, ...t for which parameters $\theta_i \geq \theta_0$ ($\theta_i \leq \theta_0$), i=1, 2,.....t are called better than (worse than) control or standard population T_0, where θ_0 is the parameter corresponding to a control or standard population T_0. In a particular problem our goal may be any one of these i.e. selection of the best or k-best or better than control.

1.3.1 <u>Indifference zone approach:</u>

The pioneer of indifference zone approach is Bechhofer (1954). The following definitions would be needed for the discussion of this approach. Let α^* be a specified constant. Define the sub space $\Omega\alpha^*$ of Ω by $\Omega\alpha^* = \{(\theta_1, \theta_2, \ldots\ldots \theta_k) : \theta_{(k)} - \theta_{(k-s)} \geq \alpha^*\}$

$\Omega\alpha^*$ is called the preference zone. The complement of $\Omega\alpha^*$ is called the indifference zone. For the population T_i, let X_i be an appropriate statistic for θ_i, based on observations using the given design D. A criterion based on statistics X_i, i=1, 2,.....t which is used to select the population meeting our goal is called the selection rule (R). By Correct Selection (CS) we mean the selection of the population meeting our goal by the selection rule R. A selection rule R is said to satisfy P** condition iff $P(CS/R) \geq P^{**}$ for all $\underline{\theta} \in \Omega_\alpha^*$ (1.16)

The configuration $\underline{\theta}_{LF}$ at which the probability of correct selection attains its infimum in Ω_α^* is called least favourable configuration (LFC). For a given P** one usually fixes the sample size n (the number of observations per population) so that the P** condition (1.16) is satisfied by the selection rule R. To achieve this, one solves for n the following equation

$$\inf_{\Omega_\alpha^*} P(CS/R) = P^{**} \tag{1.17}$$

If distribution of X_i is statistically increasing in θ_i and the goal is that of selection of the best population one would usually define the selection rule R which selects the population corresponding to $\max_{1 \leq i \leq t} X_i$ as the best population.

1.3.2 <u>Subset selection approach</u>:

Subset selection approach in ranking and selection was initiated by Gupta (1956). According to this procedure, we select a subset of random size s of t given populations that includes all the populations meeting our goal with a high pre-assigned probability P*. The following definitions would be needed. A criterion based on the statistics X_i, i=1,2,.....t, which tells us whether or not to include any T_i in the subset is called the selection rule (R). The Correct Selection (CS) means that the subset selected according to rule R includes all populations meeting our goal.

A subset selection rule R is said to satisfy P* condition iff $P(CS/R) \geq P^*$ for all $\underline{\Theta} \in \Omega$ (1.18)

where $\frac{1}{t} < P^* < 1$, a pre-assigned number. Then again a parametric configuration $\underline{\Theta}_{LF}$ at which the probability of correct selection attains its infimum in Ω is called least favourable configuration (LFC). The constants involved in the selection rule R are called selection constants. The selection constants of any rule R are chosen so that selection rule R satisfies the P* condition. (1.18).

In order to choose selection constants one generally solves the following equation.

$$\inf_{\Omega} P(CS/R) = P^* \qquad (1.19)$$

If distribution of X_i is stocastically increasing in Θ_i, then the rules proposed for the specific goal of selection of the best population may be one of the following two types

R_1 : Select T_i iff

$$X_i \geq \max_{1 \leq j \leq t} X_j - d , \quad d \geq 0 \qquad (1.20)$$

R_2 : Select T_j iff

$$X_i \geq c \max_{1 \leq j \leq t} X_j , \quad 0 \leq c \leq 1 \qquad (1.21)$$

where d and c are selection constants. For other goals, these rules can be modified.

1.4 Review of Literature:

Historically the method of paired comparisons was introduced in psycological experiments beginning with researches of Witmer and Cohn (1894), Titchner (1901) described the method in detail in one of the earliest text books on psycological experiments. But the actual experiment was conducted by Thurstone (1927) for the purpose of estimating the relative strengths of treatment stimuli through subjective testing. He postulated that a subjective continuum over which sensations are jointly normally distributed with equal standard deviations and zero co-relations between pairs.

Kendall and Smith (1939) proposed paired comparison method of combinatorial type. Moran (1947) calculated the moments of distribution of circular triads and proved that the distribution tends to normality when the number of objects being compared increases.

Mosteller (1951 a, b, c) demostrated that we need not assume that the correlation between responses are zero as assumed by Thurstone (1927). He modified the Thurstone model under the assumption of equal correlations between the responses. This modified model came to be known as TM model. In his paper Mosteller dealt with the problem of estimation of true worth S_i of the treatments T_i. Harris (1957) has modified Mosteller's work to account for order effects. Glen and David (1960) introduced tie effect in TM model and replaced the normal distribution by an angular distribution and developed least square estimates for S_i and tie parameter.

The second model of paired comparison as stated in previous section was given by Bradely and Terry (1952) and Terry, Bradely and Davis (1952). A mathematical model defined as BT model in the last section was postulated and tested for equality of parameters π_i of the model.

Scheffe (1952) gave an analysis of variance model for paired comparison experiments, in which the judges preferences are expressed on a 7 or a 9 point scale. Procedures for testing the appropriateness of the model, developed by Bradely and Terry (1952), their applications, were presented by Bradely (1954). Again Bradely (1955) examined some of the large sample properties of his method of paired comparisons. Wei (1952) and Kendall (1955) considered the problem of reducing the number of pairs required, by the use of balanced experimental designs. Dykstra (1956) presented a quick and easy method of obtaining first estimates of the rating $\pi_i'^s$ in the case of BT model. David (1959) developed some exact test procedures for paired comparison experiments, assuming preference probabilities as the parameters of the model. Dykstra (1960) extended BT model and generalised their results to the case of unequal number of repetitions on pairs. David (1963) used cyclic designs developed by Kemptorne to get fractional sets of paired comparisons for reasons of economy. A good deal of work in the references stated above was beautifully summed in the monograph entitled "The Method of Paired Comparisons" by David 1963. Buhlmann and Huber (1963) studied several possibilities for defining the correct ranking of treatments in terms of the under lying preference probability matrices. They have shown that ranking in descending order of the row-sum is optimal, in a very strong sense, for BT model.

Trawinski and David (1963) employed the techniques of ranking and selection of statistical populations in the method of paired comparisons. They proposed two selection rules for selecting the best and the subset containing the best treatment when observations came from FP design. Trawinski (1969) developed an asymptotic approximation of the expected size of the selected subset under the selection rule developed by Trawinski and David (1963).

Thomson Jr. and Russel Remage Jr. (1964) studied the theoretical aspect of the problem of obtaining rankings from paired comparisons, from the point of view of graph theory. They have introduced a mathematical model based on the concept of weak stochastic sensityvity of imformation theory. Bradely (1965) showed that BT model can be derived from a variety of diffierent initial assumptions about the nature of experiment. Rao and Kupper (1967) modified the BT model by permitting ties and defining the same by a threshold parameter. Davidson (1970) developed an alternative model by modifying BT model to incorporate tied observations. Davidson and Bradely (1969) extended the BT model to the case when the pair is judged with respect to more than one character. Sadasivan (1970) introduced the concept of SCP design in paired comparison with a view to reduce the tedium of experimentation. Sadasivan et al (1971) have modified the BT model with and without ties when observations are from SP design. Davidson and Beaver (1977) extended both Rao-Kupper(1967) and Davidson (1970) models to allow for with in pair order effects.

Some other papers in the field of paired comparison are by Bose (1956), Thomson and Singh (1967). Singh and Thomson (1968), Singh (1973, 1976), Beaver and Gokhle (1975), Fluck and Korsh (1975), Finberg and Larntz (1975), El-Helbay and Bradley (1976), Davidson and Farguber. (1976), Sadasivan and Sundaram (1977), El-Helbay and Bradley (1977), Little and Boyett (1977), Bauor (1978), Singh and Gupta (1975,1978), Latta (1979), Finberg (1979), Kousgaard (1980 ab), and Sadasivan (1981).

Next we review some of the papers cited above confining ourselves mainly to the work in which we are interested in this thesis.

<u>Mosteller (1951a)</u>

TM model as introduced in section 3 was first discussed in this paper. From equation (1.3) it is clear that $Y_i - Y_j$ are normally distributed with 0 mean and variance 1. Y_i may be correlated with equal correlation among the pairs.

Suppose n independent observations are made on each pair (T_i, T_j) either by a single judge or a group of judges having equal discriminatory powers relative to the treatments concerned. Let N_{ij} be the number of times $T_i > T_j$ in those n comparisons.

$p_{ij} = N_{ij}/n$ = proportion of preferences for T_i

$p_{ji} = N_{ji}/n$ = proportion of preferences for T_j

and $N_{ij} + N_{ji} = n$

Replacing the parameters by their estimates p_{ij} in (1.3) we have

$$p_{ij} = \phi\,(S_i' - S_j')$$
$$p_{ji} = \phi\,(S_j' - S_i') \qquad (1.22)$$

where $S_i' - S_j'$ are respectively the experimental estimates of $S_i - S_j$ resulting from the comparison of T_i and T_j. so that

$$S_i' - S_j' = \overset{-1}{\phi}(p_{ij}) = D_{ij}'$$
$$S_j' - S_i' = \overset{-1}{\phi}(p_{ji}) \qquad (1.23)$$

where $\overset{-1}{\phi}(p_{ij})$ and $\overset{-1}{\phi}(p_{ji})$ are discrete variates which may be called pseudo - normal deviates exceeded with probabilities $(1-p_{ij})$ and $(1-p_{ji})$ respectively. Given the data in this form for each pair (i, j=1.....t) least square estimates were obtained by minimising

$$\sum_{i,j}[D_{ij}' - (S_i' - S_j')]^2$$

with $S_1' = 0.$

Mosteller (1951c): A test of the assumption underlying Thurstone's method of paired comparisons was developed and illustrated in this paper.

Harries (1957): has modified TM model of comparitive judgement to account for temporal or spatial order of presentation of pairs of treatments. Harries's model interpretted with in the frame work of TM model assumes that the expected value of the differences between the responses to two treatments is normally distributed with mean say $(S_i - S_j)$ and variance, $2\sigma^2(1-\rho^2)$, while the order of presentation introduces a bias in the mean so that for the ordered presentation (T_i, T_j) the expected difference in response is $(S_i - S_j + \alpha)$ and expected difference in response for the ordered presentation (T_j, T_i) is $(S_i - S_j - \alpha)$. Following Mosteller, least squares estimates of parameters are obtained and a goodness of fit test of the proposed model has been discussed.

Glenn and David (1960): In this paper a modification of TM model to incorporate tied observation in paired comparison experiments has been discussed. Consequently they assumed that the preference probabilities are satisfying the linear model and are given by (1.13), (1.14), (1.15) with H replaced by ϕ .

clearly $$\pi_{i.ij} + \pi_{0.ij} = \phi\,(S_i - S_j + \beta) \qquad (1.24)$$

$$\pi_{j.ij} + \pi_{0.ij} = \phi\,(\beta - S_i + S_j)$$

Replacing the paramerters $\pi_{i.ij}$, $\pi_{j.ij}$ and $\pi_{0.ij}$ by their estimates which are taken to be corresponding sample proportions $p_{i.ij}$, $p_{j.ij}$ and $p_{0.ij}$ respectively in n comparisons of (T_i, T_j), they got the relations

$$\begin{aligned} p_{i.ij} + p_{0.ij} &= \phi\,(\beta'_{(ij)} + S'_i - S'_j) \\ p_{j.ij} + p_{0.ij} &= \phi\,(\beta'_{(ij)} - S'_i + S'_j) \end{aligned} \qquad (1.25)$$

where $\beta'_{(ij)}$ and $S'_i - S'_j$ are respectively the estimated values of β_{ij} and $S_i - S_j$ resulting from the comparison of T_i and T_j,

writing $p_{i.ij} + p_{0.ij} = a_{ij}$ and

$$p_{j.ij} + p_{o.ij} = a_{ji} \qquad (1.26)$$

they obtained the relations

$$\beta'_{(ij)} + S'_i - S'_j = \overset{-1}{\phi}(a_{ij})$$

$$\beta'_{(ij)} + S'_j - S'_i = \overset{-1}{\phi}(a_{ji}) \qquad (1.27)$$

Solving for $\beta'_{(ij)}$ and $S'_i - S'_j$ they had

$$\beta'_{(ij)} = 1/2\ [\overset{-1}{\phi}(a_{ij}) + \overset{-1}{\phi}(a_{ji})] \text{ and}$$
$$S'_i - S'_j = 1/2\ [\overset{-1}{\phi}(a_{ij}) - \overset{-1}{\phi}(a_{ji})] \qquad (1.28)$$

Because of lack of independence due to tied observations it was necessary to introduce arcsin transformation, which yielded

(1) $\beta'_{(ij)} = 1/2\ [\overset{-1}{\sin}(2a_{ij}-1) + \overset{-1}{\sin}(2a_{ji}-1)] = G_{ij}$ (1.29)

(2) $S'_i - S'_j = 1/2\ [\overset{-1}{\sin}(2a_{ij}-1) - \overset{-1}{\sin}(2a_{ji}-1)] = H_{ij}$ (1.30)

(3) $\text{Var}\ [\overset{-1}{\sin}(2a_{ij}-1)] = \text{Var}\ [\overset{-1}{\sin}(2a_{ji}-1)] = 1/n$

(4) Correlation between $\overset{-1}{\sin}(2a_{ij}-1)$ and $\overset{-1}{\sin}(2a_{ji}-1)$ denoted by

$$\rho_{ij} = -\sqrt{\frac{\pi_{i.ij}\ \pi_{j.ij}}{(1-\pi_{i.ij})(1-\pi_{j.ij})}} \qquad (1.31)$$

(5) $\text{Var}\ (\beta'_{(ij)}) = (1 + \rho_{ij})/2n$ (1.32)

and

(6) $\text{Var}\ (S'_i - S'_j) = (1 - \rho_{ij})/2n$ (1.33)

They have obtained the least squres estimates of β and S_i as

$$\beta^* = \frac{2}{t(t-1)} \sum_{i<j}^{t} G_{ij} \quad \text{and}$$

$$\begin{bmatrix} S_2^* \\ S_3^* \\ \cdot \\ \cdot \\ S_t^* \end{bmatrix} = \frac{1}{t} \begin{bmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 1 \\ \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdots & 2 \end{bmatrix} \begin{bmatrix} \sum_{j=1}^{t} H_{2j} \\ \sum_{j=1}^{t} H_{3j} \\ \cdots \\ \sum_{j=1}^{t} H_{tj} \end{bmatrix} \tag{1.34}$$

with $S_1^* = 0$

Using (1.29) and (1.30) in (1.32) and (1.33) they obtained

$$\begin{aligned} \text{Var } (G_{ij}) &= (1 + \rho_{ij})/2n \\ \text{Var } (H_{ij}) &= (1 - \rho_{ij})/2n \end{aligned} \tag{1.35}$$

In order to allow for hetrogenity in these variances, they have weighted each G_{ij}, H_{ij} in proportion to the inverse of its variance. Using the initial estimates of β^* and S_i^* as obtained in (1.34), they first got the estimates r_{ij} of ρ_{ij}

$$r_{ij} = -\sqrt{\frac{(1-\hat{a}_{ij})(1-\hat{a}_{ji})}{\hat{a}_{ij} \quad \hat{a}_{ji}}} \tag{1.36}$$

where $\hat{a}_{ij} = 1/2\ [1+ \sin\ (\beta^* + S_i^* - S_j^*)]$

$\hat{a}_{ji} = 1/2\ [1+ \sin\ (\beta^* - S_i^* + S_j^*)]$

and then estimated var (G_{ij}) by

$(1+ r_{ij})/2n$, and var (H_{ij})

by $(1 - r_{ij})/2n$

Next defining V_{ij} as the weight associated with G_{ij} and W_{ij} that with H_{ij}, they have taken

$$V_{ij} = 1/(1+r_{ij}) \quad \text{and}$$

$$W_{ij} = 1/(1-r_{ij}) \text{ for all } \binom{t}{2} \text{ pairs.}$$

Denoteing β^{**} and $S_i^{**}(i=1,\ldots.t)$ with $S_1^* = 0$ as estimates of β and S_i obtained from the weighted analysis, they got

$$\beta^{**} = \sum_{i<j}^{t} V_{ij} G_{ij} \Big/ \sum_{i<j}^{t} V_{ij}$$

and

$$B^{**} = \begin{bmatrix} S_2^{**} \\ S_3^{**} \\ \cdot \\ \cdot \\ S_t^{**} \end{bmatrix} = (X'WX)^{-1}(X'WY) \qquad (1.37)$$

where

$$X'WX = \begin{bmatrix} \sum_{j=1}^{t} w_{2j} & -w_{23} & \cdots\cdots & -w_{2t} \\ -w_{23} & \sum_{j=1}^{t} w_{3j} & \cdots\cdots & -w_{3t} \\ \cdots & \cdots & \cdots & \cdots \\ -w_{2t} & -w_{3t} & \cdots\cdots & \sum_{j=1}^{t} w_{tj} \end{bmatrix}$$

and

$$X'WY = \begin{bmatrix} \sum_{j=1}^{t} w_{2j} H_{2j} \\ \sum_{j=1}^{t} w_{3j} H_{3j} \\ \cdots\cdots \\ \sum_{j=1}^{t} w_{tj} H_{tj} \end{bmatrix}$$

with $W_{ii}=0$

They also obtained estimated variance covariance matrix associated with vector B** as

$$\sum_{B^{**}} = (X' W X)^{-1}/(2 n) \quad (1.38)$$

Finally they have discussed a test for goodness of fit for their model.

Bradely R.A. and Terry M.E. (1952)

BT model as introduced in section 3 was discussed in this paper. In a paired comparison of the pair (T_i, T_j), they have postulated that

$$P(T_i > T_j) = \pi_{ij} = \pi_i/(\pi_i + \pi_j)$$

where π_i is the intrinsic worth corresponding to the treatment T_i, i=1, 2......t on a scale satisfying $\pi_i \geq 0$, $\Sigma \pi_i = 1$. Ties are not permitted in this model. They obtained MLE for π_i and developed a likelihood ratio test for testing the equality of treatments. These are discussed briefly in the following.

The normal equations which maximise the likelihood can be written as

$$\frac{a_i}{p_i} - \sum_{i \neq j} \frac{n_{ij}}{(p_i+p_j)} = 0, \; i=1,2...t. \quad (1.39)$$

and $\sum p_i = 1$ where p_i is the estimator for π_i.

Solution of equations (1.39) for p_i has been obtained iteratively. To test $H_0 : \pi_1 = \pi_2 = ...\pi_t = 1/t$ against the alternative $H_0 : \pi_i \neq \pi_j$ for some i, j, = 1, 2,......t, $i \neq j$, the likelihood ratio statistic to be used is

$$-2\ln \lambda_1 = 2\,N \ln 2 - 2\,B_1 \ln 10,\; N = \sum_{i<j} n_{ij} \qquad (1.40)$$

where $B_1 = \sum_{i<j} n_{ij} \ln (p_i + p_j) - \sum_i a_i \ln p_i$

For large n_{ij}, $-2\ln \lambda_1$ has the central chi-square distribution with (t-1) degrees of freedom under H_0.

Bradley R.A. (1955): As continuation of the work in Bradley and Terry (1952), in this paper, the distribution of variance covariance matrix of the ML estimates p_i of π_i has been done. He obtained the relative efficiency of the method of paired comparison in comparison with analysis of variance, from the limiting ratio of sample sizes required for equal powers, as $t/\pi(t-1)$ where $\pi = 22/7$. He has shown that BT Model behaves better than a multi-binomial test. Asymtotic powers of these two tests have been plotted for t=3 and 4 along with similar values for analysis of variance procedure.

Rao and Kupper (1967): In this paper BT model has been modified to accommodate ties in paired comparisons. A tie parameter θ has been introduced to adjust the preference probabilities and tie probability associated with the comparison between T_i, T_j as follows:

$$\pi_{l.ij} = \frac{\pi_l}{\theta\,(\pi_i + \pi_j) + (1-\theta)\,\pi_l} \qquad l = i,\, j$$

$$\pi_{0.ij} = 1 - \pi_{i.ij} - \pi_{j.ij} \text{ for } i \neq j,\; j = 1,2,\ldots t \qquad (1.41)$$

Note that these preference probabilities can be obtained from (1.13), (1.14) and (1.15) by replacing H by the c.d.f. of logastic distribution and setting $\pi_i = \ln S_i$ and $\theta = \ln \beta$.

The ML estimators for θ and π_i's have been obtained and their asymptotic joint distribution also.

Davidson (1970): In this paper another model with basically similar consideration as Rao and Kupper (1967) was introduced. The preference probabilities in the new model have been expressed by

$$\pi_{l.ij} = \frac{\pi_l}{\pi_i + \pi_j + v\sqrt{\pi_i \pi_j}} \qquad l = i, j \quad \text{and}$$

tie probability by (1.42)

$\pi_{o.ij} = 1 - \pi_{i.ij} - \pi_{j.ij}$ with $1/v$ taken as an index of discrimination between T_i, T_j. ML estimators of π_i's and v have been obtained.

Beaver and Gokhale (1975): In this paper the following extension of BT Model without ties to account for a possible order of presentation effect within pairs was proposed.

Let $\pi_{i.ij}$ and $\pi_{i.ji}$ represent the probability of preference for T_i when the orders of presentation are (T_i, T_j) and (T_j, T_i) respectively. They assumed the existence of a parameter α_{ij} associated with pair (i, j) such that the preference probabilities for the ordered pair (T_i, T_j) are

$$\begin{aligned} \pi_{i.ij} &= \frac{\pi_i + \alpha_{ij}}{\pi_i + \pi_j} \\ \pi_{j.ij} &= \frac{\pi_j - \alpha_{ij}}{\pi_i + \pi_j} \end{aligned} \qquad (1.43)$$

while the preference probabilities for the ordered pair (T_j, T_i) are

$$\pi_{i.ji} = \frac{\pi_i - \alpha_{ij}}{\pi_i + \pi_j} \qquad (1.44)$$
$$\pi_{j.ji} = \frac{\pi_j + \alpha_{ij}}{\pi_i + \pi_j}$$

with the restriction that $|\alpha_{ij}| \leq$ minimum (π_i, π_j).

Singh and Gupta (1975): Developed a new paired comparison model using psychophysical ideas of sensory preception. Two hypothesis were advanced. One of them postulated that the number of signals registered at the brain during time t as a result of sensory receptors activated by a stimulus is governed by a stochastic process. Thus in a paired comparison task a bivariate process would govern the number of signals corresponding to a pair of stimuli. The second hypothesis pertains to the probability of reporting one stimulais grater than the other in terms of the joint waiting time associated with the bivariate process. An advantage of this approach is that the physical interpretations of the parameters involved in the model presents itself naturally. BT Model is a special case of this model.

Singh and Gupta (1978): This paper deals with the statistical inference aspect of the model presented above. Based upon the results of Bhat and Nagaur (1965), a locally most powerful stringent test was developed for testing the hypothesis of equal preference from paired comparison data. A numerical example was used to illustrate the result.

Davidson and Beaver (1977): This paper deals with modified BT Model of Rao and Kupper (1967) and Davidson (1970) given earlier, to allow for within paire order effect. The extended model allows multiplicative order effect β, independent of T_i and T_j.

For the order of presentation (i,j) as an extension of Rao and Kupper model (1967) the preference probabilities were given by

$$\begin{aligned} \pi_{i.ij} &= \pi_i / (\pi_i + \beta\,\theta\,\pi_j) \\ \pi_{j.ij} &= \beta\pi_j / (\theta\,\pi_i + \beta\,\pi_j) \\ \pi_{0.ij} &= (\theta^2-1)\,\beta\,\pi_i\,\pi_j / (\pi_i + \beta\theta\,\pi_j)(\theta\pi_i + \beta\,\pi_j) \end{aligned} \quad (1.45)$$

and that for Davidson model (1970)

$$\begin{aligned} \pi_{i.ij} &= \pi_i / [\pi_i + \beta\,\pi_j + v\,(\pi_i\,\pi_j)^{1/2}] \\ \pi_{j.ij} &= \beta\,\pi_j / [\pi_i + \beta\,\pi_j + v\,(\pi_i\,\pi_j)^{1/2}] \\ \pi_{0.ij} &= v\,(\pi_i\,\pi_j)^{1/2} / [\pi_i + \beta\,\pi_j + v\,(\pi_i\,\pi_j)^{1/2}] \end{aligned} \quad (1.46)$$

The method of ML has been used to obtain the estimates of the parameters in the above models. The weighted least squares analysis has also been used for analysing the data from the paired comparison with order effect in the presence of ties.

Sadasivan (1970): The author has compared the efficiency of some experimental designs in which standard pairs and triads are tested an equal number of times by different judges. A brief discussion as to how the data from the designs can be analysed using BT model or combinatorial method was given.

Maitri (1982): The chapter VI and VII of Maitri's Ph.D. dissertation contains the problem of selection of best treatment using paired comparison data. In chapter VI a modified knock out design called $T_2(1,c)$ knock out pair comparison design has been defined. The problem of selection of best treatment using the data from this design under indifference zone formulation has been attempted. Optimal $T_2(1,c)$ knock out paired comparison design with minimum expected number of comparisons for selecting the best treatment has been given. Also $T_2(1,c)$ design has been compared with round robin design.

In chapter VII Maitri discussed the problem of choosing the subset containing the best treatment when the data is obtained by using $\overset{c}{SP}$ design. He achieved the required goal by reducing this problem to the problem of selecting the best Binomial parameter in a reduced set of parametric configuration.

Buhlmann and Huber (1963): This paper is concerned with the following ranking problems: $t>3$ treatments are compared pairvise. The results of all comparisons can be summerised in a preference matrix $X=(x_{ij})$ where $x_{ij} = 1, 0,$ or $\frac{1}{2}$ respectively according as $T_i > T_j$, $T_j > T_i$ over $T_i = T_j$. The authors are concerned with choosing all treatments in the order of their preference when X is known. For the case of no ties they have assumed for all $i \neq j$

(i) $(x_{ij}=1)$ with probability π_{ij}

(ii) $(x_{ij}=0)$ ith probability π_{ji} and

(iii) $\pi_{ij} + \pi_{ji} = 1$

If the results of all paired comparisons are independent then the probability matrix $P=(\pi_{ij})$ describes the complete underlying probability structure of the preference matrix X. In terms of p matrix they have considered the following different methods for ranking t treatments

(a) Define $m_i = \min\limits_{j \neq i} p_{ij}$, $j=1.....t$ and rank treatments according to decending order of m_i

(b) Define $p_i = \sum_j \pi_{ij}$ and rank the treatments according to desending order of p_i

(c) Define $P^{(N)} = P^{(N-1)}p$ and rank in desending order of $p_i^{(N)}$ where $p = (p_1, p_2, \ldots\ldots p_t)$ as defined in (b) and $p^{(N)} = (p_1^{(N)}, p_2^{(N)}, \ldots\ldots p_t^{(N)})$. They have shown that ranking in desending order of p_i is optimum if and only if $P = (\pi_{ij})$ satisfying BT model

i.e. $\pi_{ij} = H(S_i - S_j)$ where (S_i) are constants and H is the logistic function. They have stated that the result is true for fractional pairs as well.

Huber (1963): This is an extension of the above paper when general scoring system is adopted (this in particular includes tie case also) in the paired comparison design. It has been shown that ranking in descending order of the scores

$$a_i = \sum_{j=1}^{t} x_{ij}$$

uniformly minimises the risk among all permutation invariant procedures and for all reasonable loss functions, provided the x_{ij} $(i < j)$ are independent random variables distributed according to an exponential type distribution as given below.

$$P(x_{ij} \leq 1) = C\,(S_i - S_j) \int_{-\infty}^{1} e^{(S_i - S_j)T} \mu\,(\delta T) \qquad (1.47)$$

where μ is a symmetric probability measure on the real line.

Trawinski and David (1963): The authors investigated the problem of selection of the best treatment and a subset of treatments containing the best treatment using FP design with equal number 'n' of replications.

It was assumed that there are no ties, no order effect, no replication effect and preference probabilities satisfy a linear model (1.2). The ranking of the treatments was done with respect to the intrinsic rating S_i for treatment T_i. Two problems were considered the first being to select the treatment associated with $\max_{1 \leq i \leq t} S_i$, called the best treatment using indifference zone approach in ranking and selection.

For this problem theselection rule R was as follows

Rule R: Select the treatment corresponding to a_t as the best treatment. In case m scores tie for the last place, choose any of the corresponding treatments as the best with probability 1/m

Where a_t is the maximum score for $T_1, \ldots\ldots T_t$, score of a treatment T_i being defined as in (1.8) with $n_{ij} = n$.

They have obtained the probability function of the vector of scores $\underline{a} = (a_1, a_2, \ldots\ldots a_t)$ as

$$f(\underline{a}, C(\pi_{ij})) = \sum_{P_n} \prod_{r>s}^{t} \binom{n}{A_{rs}} \pi_{rs}^{A_{rs}} \times \pi_{sr}^{(n-A_{rs})} \qquad (1.48)$$

where $C(\pi_{ij})$ is the configuration of preference probabilities π_{ij}'s, given by

$$C(\pi_{ij}) = \{0 \leq \pi_{ij} \leq 1,\ i,\ j = 1,2,\ldots\ldots t,\ \pi_{ij}+\pi_{ji}=1\}. \qquad (1.48A)$$

A_{rs} is the number of times T_r is prefered to T_s and P_n is the restriction of the scores imposed by the following expressions

$$a_t = A_{t1} + A_{t2} + \ldots\ldots\ldots + A_{t,t-1}$$

$$a_{t-1} = A_{t-1,1} + A_{t-1,2} \ldots\ldots + A_{t-1,\ t-2} + (n-A_{t,t-1})$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_1 = (n-A_{21}) + (n-A_{31}) \ldots\ldots\ldots + (n-A_{t1})$$

They have established that

$$g\ (\underline{a},\ n) = \sum_{\mathrm{Pn}} \prod_{r>s}^{t} \binom{n}{A_{rs}} \qquad (1.48B)$$

is symmetric in a_i. From g ($\underline{a}$; n) they have obtained the partion function G(a,n) giving the number of permissible partitions of $\frac{1}{2}$ nt(t-1) into t scores $a_1, a_2, \ldots\ldots a_t$ irrespective of order. They have established the relation $G(\underline{a}, n) = (t!/ \prod_k m_k!) \times$

$$g\ (\underline{a};\ n) \qquad (1.49)$$

Tables of G(a, n) for different combinations of (t,n) have been tabulated by Trawinski (1961).

For convenience, let $S_1 \leq S_2 \leq \ldots\ldots\ldots\ldots \leq S_t$. and we write a_i for the score of T_i. In case there is a single best treatment among the treatments, others being equal, the preference probabilities must satisfy

$$\begin{aligned} \pi_{tj} &= \pi > 1/2, \quad j=1,2,\ldots\ldots\ldots, t-1 \\ \pi_{ij} &= 1/2, \quad j=1,2,\ldots\ldots\ldots\ldots, t-1, i \neq j \end{aligned} \qquad (1.50)$$

This is called the slippage configuration. Under this configuration they have given the expression of the probability of correct selection P as

$$P = 2^{-n\binom{t-1}{2}} \sum_{a_t=C}^{n(t-1)} \pi^{a_t} (1-\pi)^{n(t-1)-a_t} \times$$

$$\Sigma\, 1/t\ \ G(\underline{a}\ ;\ n) \qquad (1.51)$$

where the last summation extends over

$$\sum_{i=1}^{t-1} a_i = n\binom{t}{2} - a_t$$

and C is the smallest integer greater than or equal to $\frac{1}{2}n(t-1)$.
They have also obtained asymptotic approximation to this probability

of correct selection. They have tabulated the smallest 'n' needed to guarantee that $P \geq P^*$, a preassigned probability under the assumption (1.49) for $t=2(1)10(2)20$, $\pi = .55(0.05).95$ and $P^* = .75, 0.95, 0.9$.

The authors have not been able to locate the least favourable configuration in the preference zone which can be defined by

$$\{(s_1, s_2, \ldots\ldots\ldots\ldots s_{t-1}, s_t),\ \ s_{t-1} < s_t\}$$

In terms of preference probabilities, they have given an illustration to contradict that (1.48) which essentially means

$$\{s_1 = s_2 = \ldots\ldots\ldots\ldots = s_{t-1} < s_t\} \quad \text{is}$$

least favourable configuration.

The second problem dealt with is that of selecting a small subset s of 't' treatments containing the best treatment. A standard subset selection procedure has been adopted for this problem. The selection rule R_1 has been defined as follows

Rule R_1 : Include T_i in the selected subset if $a_i \geq a_t - \vartheta$ where ϑ a non-negative integer is the selection constant. The constant $\vartheta = \vartheta(t, n, P^*)$ is to be chosen such that the P^* condition (1.13) for rule R_1 is satisfied. They have shown that equality configuration $s_1 = s_2 = \ldots\ldots\ldots\ldots = s_t$ is least favourable. Equivalently in terms of preference probabilities least favourable configuration can be put as

$$\pi_{ij} = 1/2,\ i,j = 1,2,\ldots\ldots,t,\ i \neq j.$$

The last configuration is referred to as equality configuration. They have obtained

$$\inf_{(S_1,\, S_2,\, \ldots\ldots\ldots,\, S_t)} P(CS/R_1) = t^{-1}\, 2^{-n(t-1)} \,\Sigma\, M(a;\vartheta)\, G(\underline{a}\,;n) \qquad (1.52)$$

Where $M(\underline{a},\vartheta)$ is the number of a's in $\underline{a}$ which excced or equal $a_t - \vartheta$ for a given n, t and ϑ

Its asymptotic expression has been given in the paper as

$$\inf P(CS/R) \simeq \int_{-\infty}^{\infty} \phi^{k-1}(Z+W)\, \phi\, (Z) \qquad (1.53)$$

where $W = (\vartheta + 1)(nt)^{-\frac{1}{2}}$

Tabulated values of

ϑ for t = 2(1)20, n= 1(1)20 (5)50(10) 100 and

P* = .75, 0.90, .95, 0.975, 0.99

have been presented in this paper.

Trawinski (1969): This work is in continuation of Trawinski and David's (1963) work. The expected subset size for the rule R_1 corresponding to slippage and equality configuration have been obtained in this paper. Let

$$\alpha = nt\ (\pi - \tfrac{1}{2})$$

$$\sigma^2 = n\,\{2\pi(1-\pi) + 1/2\ (t-1)\}$$

$$\sigma_t^2 = n\,\{(t+2)\,\pi\,(1-\pi) + 1/4\ (t-2)\}$$

$$\Delta = \frac{(\vartheta - 1/2)}{\sigma}$$

$$\Delta_{t-1} = \frac{(\vartheta + 1/2 - \alpha)}{\sigma t}$$

$$\Delta_t = \frac{(\vartheta + 1/2 + \alpha)}{\sigma t}$$

$$C_t = n\{(t+1)\,\pi\,(1-\pi) - 1/4\}$$

$$\rho = \frac{\sigma}{2\sigma t}$$

$$\rho_t = \frac{C_t}{\sigma_t^2}$$

$$Z_1 = Z + \sqrt{2}\Delta$$

$$Z_2 = \sqrt{2}\,(\rho/\sqrt{\rho_t}) + \frac{\Delta_{t-1}}{\sqrt{\rho_t}}$$

$$Z_3 = \frac{\{\sqrt{\rho_t}\, Z + \Delta_t\}}{\sqrt{1-\rho_t}}$$

It has been shown that asymptotic expression for the expected subset size corresponding to the slippage configuration (1.50) is given by

$$E\,(S/R) \simeq \int_{-\infty}^{\infty} [(t-1)\,\phi^{(t-2)}(Z_1)\,\phi\,(Z_2) + \phi^{t-1}\,(Z_3)]\,d\,\phi\,(Z) \tag{1.54}$$

His evaluation of the expression assumes C_t to be positive. This is equivalent to the restriction that $\rho_t \geq 0$. Hence the asymptotic approximation (1.54) holds whenever

$$\pi < 1/2 + 1/2\,\{t/(t+1)\}^{1/2}.$$ For $\pi = 1$

$$E\,(S/R) \simeq (t-1)\,\phi\,(u) + 1$$

where $u = (2n\Upsilon + 1 - nt)/\{n(t-2)\}^{1/2}$

The expected proportion of the population included, that is $t^{-1}\,E(S)$ has been tabulated for $P^* = 0.90$, $t=3(1)10(2)20$, $n= 2(1)6, 8, 10(5)\ 25$ and $\pi = .6(0.1)1$

1.5. <u>Outline of the thesis:</u>

As is apparent from the proceeding discussion and reviews of the literature, majority of the work in the method of paired comparisons has been, model building and its fitting to the experimental data from FP design. The only work in the context of ranking and selection procedures in this area has been due to Trawinski and David (1963) and Trawinski (1969). These authors have only considered the selection of the best treatment under both the formulation viz. indifference zone and subset selection when the data are by FP design with equal number of replications for each pair of treatments. The main subject of interest in the present thesis is the further development of the selection and ranking procedures in the method of paired comparisons. Besides this, modification of the existing models viz. BT model incorporating multiplicative order effect and TM model incorporating additive order effects have also been considered.

In regard to the development of ranking and selection procedures, the following three types of goals have been considered under indifference zone formulation

Goal 1: Selection of the best treatment

Goal 2: Selection of the k best treatments

Goal 3: Selection of all treatments better than control

While under subset selection the only goal considered is Goal 1 above. For defining selection rules for the above goals we have used the data from FP design or SP design or from both and whenever feasible have compared performances of the rules under these designs. An SP design is specially suited to the situations where one postulates the presents of a single superior treatment among several other treatments which differ only slightly in their merits. Keeping in view of this application, a reduced parametric space for preference probabilities have been chosen while using SP design for achieving above goal. An additional feature of the work in this thesis is that the extension of the ranking and selection procedures has been done in case ties are also permissible in the paired comparisons. Since the selection rules used were based on the scores of the individual treatments obtained by using a given design, the development of selection procedures necessitiated us to consider a relatively tedious problem of deriving the distribution of scores and the distribution of order statistics of scores. This in turn needed generation of score vectors on computers. Because exact distribution may at times become cumbersome, it was found desirable to develop asymptotic procedures.

The thesis consists of five more chapters besides the present chapter I on introduction. The chapterwise contents of the remaining chapters are briefly as follows.

Chapter II: In this chapter it is assumed that the 't' treatments to be compaired has a single outlier while the other (t-1) treatments differ only slightly in merits. As remarked above here SP design is most suitable. This chapter deals with the selection problem for Goal 1 under indifference zone, as well as subset selection approach, when data are obtained by SP design with equal number of replications per pair. It is assumed that no ties are permissible in a paired comparison of treatments. Tables for implementation of two selection rules has been presented. The comparison of the average number of observations by using SP design against FP design for a given preassigned probability of correct selection of at least P* using the same selection rule have been numerically presented for asymptotic case. An asymptotic expression for the expected subset size under subset selection formulation has been also given.

Chapter III: As in chapter II, this chapter also considers Goal 1 under both formulation of ranking and selection. But now it is assumed that the ties are also permissible in the paired comparison of treatments and FP design with equal number of replications for each pair is used for obtaining the data. This incorporation of ties leads to modifications in score vectors and hence new distribution theory have to be developed. Tables for implementation of selection rules based on these score vectors have been provided. It has been observed that the provision of ties in the data from FP design improves the probability of correct selection in comparison to the selection rule of Trawinski and David (1963) based on data which does not have such provision.

Chapter IV: Results obtained in Chapter III for FP design have been modified when data are obtained from SP design with provision for ties in the paired comparison. Here again we assume the existence of a single outlier and consider the reduced parametric space for preference probabilities. Tables have been computed for implementing the selection rules and numerical comparisons of two designs viz. FP and SP design with provision for ties, have been done asymptotically.

Chapter V: All previous chapters considerded Goal 1. In this chapter other goals have also been considered. It is assumed no ties are permitted and data are from FP design with the same number of replications for each pair. The following three problems have been discussed.

(a) Selection rule for Goal 2 has been considered under indifference zone set up. All details regarding implementation of the selection procedures have been provided.

(b) Selection rule for Goal 3 and its implementation has been worked out under indifference zone set up.

(c) Using Thurstone - Mosteller model for paired comparisons we have given selection rules for Goal 1, under both the formulations i.e. indifference zone and subset selection. All developments have been done on asymptotic basis and the problem is seen to be similar to that of ranking means in multivariate normal populations.

Chapter VI: Deals with the problem of order effect in paired comparisons. We have extended BT and TM models for paired comparison experiments to incorporate order effects due to treatment specified characteristics. The order effects are assumed additive for TM model and multiplicative for BT model. Estimations of worth parameters and order effect parameters are considered. Also the testing procedures for goodness of fit of the model have been given.

CHAPTER II

SELECTION OF BEST TREATMENT USING SYMMETRICAL PAIRED COMPARISONS

2.1 Introduction

The aim of this chapter is to develop selection procedures under indifference zone as well as subset selection formulation for Goal 1 as defined in section 5 of chapter I, based on the scores a_i's obtained by using SP design with equal number 'n' of replications. The problems of interest here are similar to one we cited below.

Monosodium glutamate (MSG) is frequently used in different concentrations as taste stimulant or flavor enhancer. A certain concentration will produce the preferred flavor, while the other concentrations may not have the desired effect. The problem is to determine which concentration produces the preferred flavor of a particular preserved fruit.

Among several processes, techniques or therapies of approximately equal cost, suppose there exists one which is distinctly superior to the others of almost same level as measured on some merit scale. Sometimes it may not be possible to know in advance as to which of the processes, techniques or therapies is the superior one though its existence has been postulated, then the problem would be to select this.

In view of the preceeding discussion it is assumed in this chapter that 't' treatments to be compared have a single outlier while other (t-1) treatments differ only slightly. The Goal 1 in this case is equivalent to the selection of this outlier.

Because of the special nature of the problem considered here, we shall assume a simplified structure for the preference probabilities, which is given in the following section. We shall assume that no tie effect or judge effect or replication effect or order effect is present.

2.2 A model for preference probabilities

For the purposes of defining the model it is convenient to suppose here in this section that the t treatments T_1, T_2, T_3,........T_t be such that $T_1 \leq T_2 \leq T_3 \leq \ldots\ldots\ldots \leq T_t$ and T_t is the outlier. We assume that the preference probabilities have the following structure.

$$\{\pi_{tj} = p+(t-j)\varepsilon', \; j=1,2,\ldots\ldots t-1$$
$$\pi_{ij}=1/2+(i-j)\varepsilon \;\; i>j, \; i,j = 1,2,\ldots\ldots t-1\} \qquad (2.1)$$

where p, ε, ε' are parameters of the model. For the model to be meaningful, p, ε, ε' must satisfy the following restrictions.

i) $p \geq 1/2$

ii) $0 \leq \varepsilon \leq \varepsilon'$

iii) $0 \leq \varepsilon \leq \frac{p-1/2 + \varepsilon'}{(t-2)}$ (2.2)

iv) $0 \leq \varepsilon' \leq \frac{1-p}{(t-1)}$

where ε and ε' are small quantities. Parameter ε' gives the amount of increase in the preference probability when outlier is compared with a treatment which is next worst to the one being compared with the outlier at present i.e. $\pi_{t,j-1} = \pi_{tj} + \varepsilon'$ Likewise parameter ε signfies the amount of increase in the preference probability when any other treatment (apart from the outlier) is compared with the next worst treatment to the one being compared with it at present.

i.e. $\pi_{i\ j-1} = \pi_{ij} + \varepsilon \quad i>j,\ i=j=1,2,\ldots\ldots t-1$

Throughout this chapter we assume that preference probabilities satisfy the model (2.1) with restriction (2.2). The restriction (i) in (2.2) is obvious. Further since the perturbations are liable to be more, when an outlier is compared with other treatments than when equal treatments are compared, we have the restriction (ii) as above. By our assumption T_t being the outlier the minimum probability of T_t being preferred over the other treatments should be greater than the maximum preference probability for any other treatment. In other words $\pi + \varepsilon' \geq 1/2 + (t-2)\,\varepsilon$ which yields the restriction (iii). Restriction (iv) follows from the fact that the maximum probability of T_t being preferred over the other treatments is not greater than 1.

In the next section we develop some theorems on Kendall's row-sum procedures, showing there by that SP design can be used for outlier selection provided the proposed model (2.1) holds.

2.3 SOME THEOREMS ON KENDALL'S ROW-SUM PROCEDURES

Theorem I: When preference probabilities are known, even if theres are inconsistencies, rank order can be uniquely determined.

Illustration: Consider the case $t=3$. Let $\pi_{12} = 0.6$, $\pi_{23} = 0.7$, $\pi_{31} = 0.9$. Using Kendall's row-sum procedure the ranking is $T_3 > T_2 > T_1$, even though it is a case of circular triad.

A simple proof of Theorem I on the basis of the row-sum procedure is obvious. Hence the row-sum procedure breaks the circular triads.

Theorem II: The rank orders obtained by the row-sum procedure in equal and unequal numbers of replications of paired comparisons will be different. The null hypothesis of equal ranks is not necessarily satisfied in this case even if each $\pi_{ij} = 1/2$.

Proof: The rank order in the case of equal number 'n' of replications in paired comparison for T_i is determined by considering

$$R_i = n \sum_{\substack{j=1 \\ j \neq i}}^{t} \pi_{ij}, \quad i = 1, 2, \ldots\ldots t$$ and in the

case of unequal number n_{ij} of replications

$$R_i' = \sum_{\substack{j=1 \\ j \neq i}}^{t} n_{ij} \pi_{ij}, \quad i=1, 2, \ldots\ldots\ldots t$$

Now R_i is not necessarily equal to R_i'. Hence rank orders need not necessarily be the same in the two cases.

Further when $\pi_{ij} = 1/2, \quad i, j = 1, 2, \ldots\ldots t, \; i \neq j$

$$R_i = n \sum_{\substack{j=1 \\ j \neq i}}^{t} \pi_{ij} = n\,(t-1)/2$$

$$R_i' = \sum_{\substack{j=1 \\ j \neq i}}^{t} n_{ij} \pi_{ij} = 1/2 \sum_{\substack{j=1 \\ j \neq i}}^{t} n_{ij}$$

Hence R_i not necessarily $= R_i'$, proving the latter part of the theorem.

Theorem III: A single outlier can be detected correctly in FP design with equal number of replications in paired comparison experiments. The same is possible in case of SP design with equal number of replications on the pairs under model (2.1) with restriction (2.2).

Proof: Using the row-sum procedure for full pairs with equal number n of replications, the row-sum for the outlier T_t is $n\sum_{j=1}^{t-1} \pi_{tj}$ and for any other treatment T_1 is $n\sum_{\substack{j=1\\ j\neq 1}}^{t} \pi_{1j}$

Then it is obvious that

$$\sum_{j=1}^{t-1} \pi_{tj} > \sum_{\substack{j=1\\ j\neq 1}}^{t} \pi_{1j}$$

Thus a single outlier can be detected in this case. Now in SP the corresponding sum for outlier is $n(\pi_{ti}+\pi_{tj})$ while for other treatment T_1 is $n(\pi_{1k}+\pi_{1m})$. Clearly $n(\pi_{ti}+\pi_{tj}) > n(\pi_{1k}+\pi_{1m})$ whenever $\min\limits_{1\leq j\leq t-1} \pi_{tj} > \max\limits_{1\leq i,j\leq t-1} \pi_{ij}$

Since this condition is satisfied by our proposed model (2.1) with restriction (2.2), the outlier will always be detected in SP design under this model. This completes the proof of the Theorem.

It may be noted that in unequal number of replications if n_{ij} is increased widely in pairs not containing the outlier, the outlier will not be detected in any model.

In the development of ranking and selection procedure for the goal of selection of best treatment using SP design, the distribution of scores of different treatments under zero-one scoring system would be needed. The next section is devoted to these distributions.

2.4 DISTRIBUTION THEORY

Consider an SP comparison experiment consisting of 'n' replications of each of the pair (T_1, T_2), (T_2, T_3),....... (T_{t-1}, T_t), (T_t, T_1) among the treatments T_i $(i=1,2,......t)$ which is one among the $(t-1)!/2$ possible sets of paired comparisons under SP design. Let x_{ijr} be a characteristic random variable corresponding to the comparison of T_i and T_j in the r^{th} replication. Then as in (1.6), we now define

$$x_{ijr} = \begin{cases} 1 \text{ if } T_i > T_j, & i = 1,2,......t \\ & j = \left.\begin{matrix} i-1 \\ \text{or} \\ i+1 \end{matrix}\right\} \bmod t \\ 0 \text{ if } T_j > T_i, & (r=1,2,.......n) \end{cases} \tag{2.3}$$

we assume that (i) there are no ties (ii) the ratings are not affected by replications (iii) replications of the same pair are independent in the probabilistic sense and (iv) the trials of different pairs are independent in the probabilistic sense.

Then clearly
$$P_r\,(x_{ijr} = 1) = \pi_{ij} \tag{2.4}$$
$$P_r\,(x_{ijr} = 0) = \pi_{ji} = 1-\pi_{ij}$$

The score a_i of treatment T_i is given by $a_i = \sum_{r=1}^{n} a_{ir}$ where a_{ir} is the score for T_i from the r^{th} replication and is given by

$$a_{ir} = \sum_{j = \left.\begin{matrix} i-1 \\ \text{or} \\ i+1 \end{matrix}\right\} \bmod t} x_{ijr} \tag{2.5}$$

also
$$\sum_{i=1}^{t} a_{ir} = t, \quad \sum_{i=1}^{t} a_i = nt.$$

If $j = \left.\begin{matrix} i-1 \\ \text{or} \\ i+1 \end{matrix}\right\} \bmod t$, a_{ir} and a_{jr}

are correlated as are a_i and a_j. Other correlation among them is zero.

Exact joint distribution of scores

Let A_{rs} $(r > s)$ be the number of times T_r is preferred to T_s in n comparisons $(r=1, 2, \ldots\ldots t, \; s = \left.\begin{matrix} r-1 \\ \text{or} \\ r+1 \end{matrix}\right\} \bmod t)$. In view of our assumptions, the A_{rs} are independent and their joint distribution is therefore given by

$$F(A_{t,t-1}, A_{t-1,\,t-2}, \ldots\ldots A_{21}, A_{1t}) = \prod_{\substack{r=1 \\ s = \left.\begin{smallmatrix} r-1 \text{ or} \\ r+1 \end{smallmatrix}\right\} \bmod t}}^{t} \binom{n}{A_{rs}} \pi_{rs}^{A_{rs}} (1-\pi_{rs})^{(n-A_{rs})} \tag{2.6}$$

Scores of the 't' treatments can be expressed as

$$\begin{aligned} a_t &= A_{t1} + A_{t,t-1} \\ a_i &= A_{i,i+1} + A_{i,i-1}, \quad i = 2, 3 \ldots\ldots t-1 \\ a_1 &= A_{12} + A_{1t} \end{aligned} \tag{2.7}$$

It follows that the joint distribution of any $u\,(\leq t)$ scores is given by summing (2.6) subject to the restriction on the scores imposed by (2.7), in particular if $u=t$, the probability function of the vector of scores $\underline{a} = (a_1, a_2, \ldots\ldots a_t)$ may be written as

$$F\,[\underline{a}, C(\pi_{ij})] = \sum_{R_n} \prod_{\substack{r=1 \\ s = \left.\begin{smallmatrix} r-1 \\ \text{or} \\ r+1 \end{smallmatrix}\right\} \bmod t}}^{t} \binom{n}{A_{rs}} \pi_{rs}^{A_{rs}} (1-\pi_{rs})^{(n-A_{rs})} \tag{2.8}$$

where $C(\pi_{ij}) = \{0 \leq \pi_{ij} \leq 1, \; i > j = 1, \ldots t\}$ is the set of all parametric configurations, $\sum_{R_n}$ denotes the summation under the restriction (2.7).

In the following we derive some distributions under the assumption that there is a single outlier T_t while other treatments are of equal worth, i.e. when the parametric configurations in $C(\pi_{ij})$ satisfy

$$H_0 : \pi_{tj} = \pi > 1/2, \; j = 1,2,\ldots\ldots,t-1$$
$$\pi_{ij} = 1/2, \; i>j = 1,2,\ldots\ldots\ldots,t-1 \qquad (2.9)$$

The set of configuration so obtained is a subset of $C(\pi_{ij})$ denoted by $C(\pi)$. In this case, the joint distribution of scores is

$$F[\underline{a}, C(\pi)] = 2^{-n(t-2)} \sum_{R_n} \prod_{\substack{r=1 \\ s = r-1 \text{ or } r+1 \} \bmod t}}^{t} \binom{n}{A_{rs}} \pi^{a_t} (1-\pi)^{2n - a_t} \qquad (2.10)$$

From here, the marginal distribution of (a_1, a_t) can be written as

$$f(a_1, a_t) = 2^{-n} \Sigma \binom{n}{x_1} \binom{n}{x_2} \binom{n}{x_3} \pi^{a_t} (1-\pi)^{2n - a_t}$$

$$\left\{ \begin{array}{l} 0 \leq x_1, x_2, x_3 \leq n \\ a_1 = x_1 + x_2 \\ a_t = x_3 + n - x_2 \end{array} \right\} \qquad (2.11)$$

The distribution of $d_1 = a_1 - a_t$ can be obtained from (2.11) as

$$f(d_1) = 2^{-n} \sum_{a_t=0}^{2n} \Sigma \left[\binom{n}{x_1} \binom{n}{x_2} \binom{n}{x_3} \pi^{a_t} (1-\pi)^{2n-a_t} \right. \qquad (2.12)$$

$$\left\{ \begin{array}{l} 0 \leq x_1, x_2, x_3 \leq n \\ a_t + d_1 = x_1 + x_2 \\ a_t = x_3 + n - x_2 \end{array} \right\}$$

$(d_1 = -2n, -2n+1, \ldots\ldots\ldots -2, -1, 0, 1, \ldots\ldots\ldots 2n)$

Note that $d_{t-1} = a_{t-1} - a_t$ also have probability distribution identical to (2.12). This distribution can be conveniently expressed in terms of generating function.

Let $g_1(s)$ be the probability generating function of the random variable d_1, then

$$2^{n} g_1(s) = \sum_{d_1=-2n}^{2n} \Big[\sum_{a_t=0}^{2n} \Sigma \binom{n}{x_1}\binom{n}{x_2}\binom{n}{x_3} \pi^{a_t}(1-\pi)^{2n-a_t}\Big] s^{d_1}$$

$$\left\{ \begin{array}{l} 0 \leq x_1,\ x_2, x_3 \leq n \\ a_t + d_1 = x_1 + x_2 \\ a_t = x_3 + n - x_2 \end{array} \right\}$$

$$= \Sigma \binom{n}{x_1}\binom{n}{x_2}\binom{n}{x_3} s^{(x_1+x_2)} s^{(x_2-x_3-n)} \pi^{(n+x_3-x_2)} (1-\pi)^{(n+x_2-x_3)}$$

$$0 \leq x_1,\ x_2,\ x_3 \leq n$$

$$= [\pi(1-\pi)/s]^n (1+s)^n [1+(1-\pi)s^2/\pi]^n [1+\pi/(1-\pi)s]^n$$

Thus $f(d_1) = 2^{-n}$ x coefficient of s^{d_1} in the expansion of $[\pi(1-\pi)/s]^n(1+s)^n [1+(1-\pi)s^2/\pi]^n [1+\pi/(1-\pi)s]^n$.

Similarly we obtain the joint distribution of a_i and a_t from (2.10) as

$$f(a_i, a_t) = 2^{-2n} \Sigma \binom{n}{x_1}\binom{n}{x_2}\binom{n}{x_3}\binom{n}{x_4} \pi^{a_t}(1-\pi)^{2n-a_t}$$

$$\left\{ \begin{array}{l} 0 \leq x_1,\ x_2,\ x_3,\ x_4 \leq n \\ a_i = x_1 + x_2 \\ a_t = x_3 + x_4 \end{array} \right\}$$

$$= 2^{-2n} \binom{2n}{a_i}\binom{2n}{a_t} \pi^{a_t} (1-\pi)^{2n-a_t} \qquad (2.13)$$

for i= 2, 3, ...t-2, which shows that a_i and a_t have independent binomial distributions with parameters n, ½ and n, π.

Now distribution of $d_i = a_i - a_t$, i=2, 3, ...t-2 can be obtained from (2.13) as

$$f(d_i) = 2^{-2n} \sum_{a_t=0}^{2n} \binom{2n}{x_1+x_2} \binom{2n}{x_3+x_4} \pi^{a_t}(1-\pi)^{2n-a_t}$$

$$\left\{ \begin{array}{l} 0 \leq x_1, x_2, x_3, x_4 \leq n \\ a_t + d_i = x_1 + x_2 \\ a_t = x_3 + x_4 \end{array} \right\}$$

$d_i = (-2n, -2n+1, ...-2, -1, 0, 1, 2, ... 2n)$

Finally the last distribution can also be conveniently expressed in terms of generating function $g_2(s)$ of d_i. In fact

$$2^{2n} g_2(s) = \sum_{d_i=-2n}^{2n} \left[\sum_{a_t=0}^{2n} \binom{2n}{x_1+x_2} \binom{2n}{x_3+x_4} \pi^{a_t} (1-\pi)^{2n-a_t} \right] s^{d_i}$$

$$\left\{ \begin{array}{l} 0 \leq x_1, x_2, x_3, x_4 \leq n \\ a_t + d_i = x_1 + x_2 \\ a_t = x_3 + x_4 \end{array} \right\}$$

$$= \binom{2n}{x_1+x_2} \binom{2n}{x_3+x_4} s^{(x_1+x_2)} s^{-(x_3+x_4)} \pi^{(x_3+x_4)} (1-\pi)^{2n-x_3-x_4}$$

$$0 \leq x_1, x_2, x_3, x_4 \leq n$$

$$= (1+s)^{2n} [1+\pi/s(1-\pi)]^{2n} (1-\pi)^{2n} \qquad (2.14)$$

and hence

$$f(d_i) = 2^{-2n} \times \text{ coeeficient of } s^{di} \text{ in the expansion of } (1+s)^{2n}(1-\pi)^{2n}\,[1+\pi/(1-\pi)s]^{2n}$$

Asymptotic distribution of difference of scores

From (2.4) we have $E(x_{ijr}) = \pi_{ij}$

$$Var(x_{ijr}) = \pi_{ij}\,\pi_{ji}$$

Denoted by $\underline{ar} = (a_{1r}, a_{2r}, \ldots\ldots a_{tr})$, $E(\underline{ar}) = \underline{A}$
$\underline{A} = (A_1, A_2, \ldots\ldots\ldots A_t)$ and $\sigma = (\sigma_{ij})$ the variance covariance matrix of $\underline{ar}$ for any r, then

$$A_i = \pi_{i,i-1} + \pi_{i,i+1}$$

$$\sigma_{ii} = \pi_{i,i-1}\pi_{i-1,i} + \pi_{i,i+1}\,\pi_{i+1,i} \qquad (2.15)$$

$$\sigma_{ij} = Cov(x_{ijr}, x_{jir}) = -\pi_{ij}\,\pi_{ji} \text{ if } j = \left.\begin{matrix} i-1 \\ i+1 \end{matrix}\right\} \text{ or mod } t$$

$$= 0 \text{ if } j \neq i \neq i-1 \text{ or } i+1\} \text{ mod } t$$

Let $\underline{a} = \sum_{r=1}^{n} \underline{ar}$ be the score vector, then clearly $E(\underline{a}) = n\underline{A}$ and since the replications are independent, variance-covariance matrix for $\underline{a}$ is $n\sigma$. Distribution of the vector of differences among scores, defined by $\underline{d} = (d_1, d_2, \ldots\ldots\ldots d_{t-1})$, where $d_i = a_i - a_t$, $i = 1, 2, \ldots\ldots t-1$ is of special interest. If $d_{ir} = a_{ir} - a_{tr}$, $r = 1, 2, \ldots\ldots n$, then $d_i = \sum_{r=1}^{n} d_{ir}$, $i = 1, 2, \ldots\ldots\ldots t-1$. Further, if the vector $\underline{dr} = (d_{1r}, \ldots\ldots d_{t-1r})$ has mean $\underline{\alpha} = (\alpha_1, \alpha_2, \ldots\ldots\ldots \alpha_{t-1})$ and variance covariance matrix $\Sigma = (\Sigma_{ij})$ for any r, then since the replications are independent, we have $E(\underline{d}) = n\,\underline{\alpha}$ and variance-covariance matrix of $\underline{d}$ is $n\Sigma$.

The variates d_{ir} corresponding to the rth replications are of two types, viz. (i) $(d_{1r}, d_{t-1,r})$ and (ii) $(d_{2r}, d_{3r}, \ldots\ldots d_{t-2,r})$ with respect to mean and variance structure.

Thus for instance

$$\alpha_1 = (A_1 - A_t)$$

$$= \pi_{12} + 2\pi_{1t} - \pi_{t,\, t-1} - 1$$

and $$\Sigma_{11} = \sigma_{11} + \sigma_{tt} - 2\sigma_{1t} \qquad (2.16)$$

using (2.15)

Similarly $\alpha_2 = \pi_{21} + \pi_{23} - \pi_{t1} - \pi_{t,t-1}$

and $\Sigma_{22} = \pi_{21}\pi_{12} + \pi_{23}\pi_{32} + \pi_{t1}\pi_{1t} + \pi_{t,t-1}\pi_{t-1,t}$

However, d_{ir}'s fall in five categories with respect to covariance structure as shown in the following table

Table 2.1

Sl. No.	Type	No of terms	Σ_{ij} Formulation in terms of covariance	Σ_{ij} Formulation in terms of π_{ij}
1	d_{1r}, d_{2r}	2	$\sigma_{12} + \sigma_{tt} - \sigma_{1t}$	$2\pi_{1t}\pi_{t1} + \pi_{t,t-1}\pi_{t-1,t} - \pi_{12}\pi_{21}$
2	d_{1r}, d_{3r}	2(t-4)	$\sigma_{tt} - \sigma_{1t}$	$2\pi_{1t}\pi_{t1} + \pi_{t,t-1}\pi_{t-1,t}$
3	d_{1r}, d_{t-1r}	1	$\sigma_{tt} - \sigma_{1t} - \sigma_{t-1,t}$	$\pi_{1t}\pi_{t1} + \pi_{t,t-1}\pi_{t-1,t}$
4	d_{2r}, d_{3r}	(t-4)	$\sigma_{23} + \sigma_{tt}$	$\pi_{1t}\pi_{t1} + \pi_{t,t-1}\pi_{t-1,t} - \pi_{23}\pi_{32}$
5	d_{2r}, d_{4r}	$\frac{(t-4)(t-5)}{2}$	σ_{tt}	$\pi_{1t}\pi_{t1} + \pi_{t,t-1}\pi_{t-1,t}$

In case there is a single outlier T_t and parametric configuration satisfy H_0 (2.9), it may be noted that the two categories of variances and five categoties of covariances become equal within each category. In fact under H_0; means of d_{ir} are

$$\alpha_i = \begin{cases} \alpha = -2(\pi - 1/2), & i = 2, \ldots\ldots t-2 \\ \alpha' = -3(\pi - 1/2), & i = 1, t-1 \end{cases} \quad (2.17)$$

and their variances are

$$\sigma_{ii} = \begin{cases} \sigma_d^2 = 2[\pi(1-\pi) + 1/4], & i=2, \ldots\ldots t-2 \\ \sigma_d^{2\prime} = 5\pi(1-\pi) + 1/4, & i=1, t-1 \end{cases} \quad (2.18)$$

Let $\rho = (\rho_{ij})$ be the correlation matrix correspinding to the variance covariance matrix $\sum_n$ for any general configuration in $C(\pi_{ij})$, then since $d = \sum_{i=1}^{n} dr$, clearly by multivariate central limit theorem, the vector $\sqrt{n}\,[(d_1/n - \alpha_1)/\sqrt{\sigma_{11}}, \ldots\ldots (d_{t-1}/n - \alpha_{t-1})/\sqrt{\sigma_{t-1,t-1}}$ converges in law to the multivariate random variable with distribution $IN(0, \rho)$. In particular under the null hypothesis H_0 of single outlier T_t, ρ takes the following form because of our remarks about equality of covariances as above.

$$\rho = \begin{bmatrix} 1 & \rho_1 & \rho_2 & \rho_2 \cdots\cdots \rho_2 & \rho_3 \\ \rho_1 & 1 & \rho_4 & \rho_5 \; \rho_5 \cdots\cdots & \rho_2 \\ \rho_2 & \rho_4 & 1 & \rho_4 \; \rho_5 \cdots\cdots & \rho_2 \\ \rho_2 & \rho_5 & \rho_4 & 1 \; \rho_4 \; \rho_5 \cdots & \rho_2 \\ \rho_2 & \rho_5 & \rho_5 & \rho_4 \; 1 \; \rho_4 \cdots & \rho_2 \\ \cdots & \cdots & \cdots & \cdots\cdots\cdots\cdots & \rho_1 \\ \rho_3 & \rho_2 & \rho_2 \cdots\cdots\cdots\cdots & \rho_1 & 1 \end{bmatrix}$$

Having obtained the distribution of scores when the data is obtained with the help of S.P. design, we are in a position to discuss the selection procedures for the best treatment using SP design. This is done in the next section.

2.5 SELECTION OF THE BEST TREATMENT

Selection rule for best treatment

In many experiments designed to compare 't' treatments the primary interest lies in the detection of the best treatment. Further, if any one of the 't' treatments is strictly better than all the other (t-1) treatments and if 'n' the number of replications per pair is larger enough, then it is expected that the best treatment would get the highest score with the probability close to 1. It is, therefore, reasonable to adopt the following procedure R_1 for selecting the best treatment.

Rule R_1: Obtain $(a_1, a_2, \ldots\ldots a_t)$ the scores of 't' treatments $T_1, T_2, \ldots\ldots T_t$ respectively by using an SP design with n replications per pair and declare the treatment with the score $\max\limits_{1\leq i\leq t} a_i$ as the best treatment.

If 'm' score tie for the first place, then one of the corresponding 'm' treatments is chosen at random for the best treatment.

Without loss of generality, we can assume that T_t is the outlier. In that case the parametric configuration would belong to the set $C^*(\pi_{ij})=\{\pi_{tj}>1/2, 0\leq\pi_{ij}\leq 1, i>j=1,2,\ldots..t\}$

Call $C^*_\pi(\pi_{ij})=\{\pi_{tj}\geq\pi, 0\leq \pi_{ij}\leq 1, i>j=1,2,\ldots..t\}$ which is a subset of $C^*(\pi_{ij})$ as the preference zone where $\frac{1}{2} < \pi\leq 1$ is a preassigned number. As stated earlier, we are assuming model (2.1) for the preference probabilities in this chapter. Under the model, the complete parameter space denoted by $C^*(p,\epsilon,\epsilon')$ would be defined only in terms of three parameters p, ϵ and ϵ' and is given by restriction (2.2), while preference zone would become $C^*_\pi(p, \varepsilon, \varepsilon') = \{p \geq 1/2,\ 0 \leq \varepsilon \leq \varepsilon',$

$$0 \leq \varepsilon \leq \frac{p - 1/2 + \varepsilon'}{(t-2)},\ 0 \leq \varepsilon' \leq \frac{1-p}{(t-1)},\ p+\varepsilon' \geq \pi\}$$

Also the slippage configuration H_o defined in (2.9) with reference to $C^*_\pi(p, \epsilon, \epsilon')$ is

$$H_o': \quad p = \pi\ , \quad \epsilon = \epsilon' = 0 \tag{2.19}$$

Let P denote the probability of correct selection that the treatment T_t is in fact declared as the best treatment by Rule R_1. For implementing the selection procedure R_1, the number 'n' of replication per paired comparison in SP design is chosen such that

$$P > P^* \tag{2.20}$$

whenever π_{ij} are in C^*_π (p, ϵ, ϵ'), where $0 \leq P^* \leq 1$ is usually chosen to be close to 1.

Least favourable configuration for the probability of correct selection

The expression for P under general configuration in $C^*(P, \epsilon, \epsilon')$ is hard to write and is not attempted here. However, we conjucture that the infimum $P = P_\pi$ (say) in preference zone C^*_π (p, ϵ, ϵ') occurs at the configuration given by the hypothesis H_o' of single outlier defined in (2.19). Our belief in this conjucture is based on numerical varifications, on the particular cases involving 3, 4 and 5 treatments with single replication. In all these cases the least favourable configuration, in the preference zone turns out to be at the configuration given by H_o'.

For eg. when t=4, the possible distinct sets of symmetrical pairs are

(i) $(T_1, T_2), (T_2, T_3), (T_3, T_4), (T_4, T_1)$

(ii) $(T_1, T_2), (T_3, T_1), (T_4, T_3), (T_2, T_4)$ and

(iii) $(T_2, T_3), (T_3, T_1), (T_1, T_4), (T_4, T_2)$ (2.21)

The corresponding expressions for the probability of correct selection when n=1 are respectively

$$P_1 = (\pi_{12}\pi_{23}\pi_{34}\pi_{41} + \pi_{14}\pi_{43}\pi_{32}\pi_{21})/4 + \pi_{41}\pi_{43}(1-\pi_{21}\pi_{23}/2)$$

$$P_2 = (\pi_{12}\pi_{31}\pi_{43}\pi_{24} + \pi_{42}\pi_{34}\pi_{13}\pi_{21})/4 + \pi_{43}\pi_{42}(1-\pi_{12}\pi_{13}/2)$$

$$P_3 = (\pi_{23}P_{31}\pi_{14}\pi_{42} + \pi_{32}\pi_{13}\pi_{41}\pi_{24})/4 + \pi_{41}\pi_{42}(1-\pi_{31}\pi_{32}/2)$$

Each of the probability of correct selection as given by P_1, P_2 and P_3 have been extensively computed for different values p, θ and θ' in C^* (p, θ, θ') assuming the preference probabilities satisfy model (2.1) (Table 13). Note that we assume $T_1 \leq T_2 \leq T_3 < T_4$. It has been observed that P_2 which is based on the SP design involving pairs (T_1, T_2), (T_3, T_1), (T_2, T_4), (T_3, T_4) is uniformly minimum among P_1, P_2, P_3 for all p, θ and θ' in C^* (p, θ, θ'). Further each of P_1, P_2, P_3 attains same infimum and at the same configuration as given by H_o'. This varifies our conjucture in this particular case. We have not presented the numerical computations of P_1, P_2 and P_3 here only to save the space.

Similarly when t=5, the possible sets of pairs under SP design are

(1) $(T_1. T_2), (T_2, T_3), (T_3, T_4), (T_4, T_5), (T_5, T_1)$

(2) $(T_1, T_2), (T_2, T_5), (T_5, T_3), (T_3, T_4), (T_4, T_1)$

(3) $(T_5, T_2), (T_2, T_4), (T_4, T_1), (T_1, T_3), (T_3, T_5)$

(4) $(T_5, T_3), (T_3, T_2), (T_2, T_1), (T_1, T_4), (T_4, T_5)$ (2.22)

(5) $(T_1, T_5), (T_5, T_2), (T_2, T_4), (T_4, T_3), (T_3, T_1)$

(6) $(T_1, T_5), (T_5, T_3), (T_3, T_4), (T_4, T_2), (T_2, T_1)$

(7) $(T_2, T_5), (T_5, T_4), (T_4, T_1), (T_1, T_3), (T_3, T_2)$

(8) $(T_1, T_2), (T_2, T_4), (T_4, T_5), (T_5, T_3), (T_3, T_1)$

(9) $(T_2, T_1), (T_1, T_3), (T_3, T_4), (T_4, T_5), (T_5, T_2)$

(10) $(T_3, T_1), (T_1, T_5), (T_5, T_4), (T_4, T_2), (T_2, T_3)$

(11) $(T_4, T_1), (T_1, T_5), (T_5, T_3), (T_3, T_2), (T_2, T_4)$

(12) $(T_4, T_1), (T_1, T_5), (T_5, T_2), (T_3, T_2), (T_3, T_4)$ and

In case of SP design with single replicate the corresponding probability of correct selections respectively are

$$P_1' = (\pi_{12}\pi_{23}\pi_{34}\pi_{45}\pi_{51}+\pi_{15}\pi_{54}\pi_{43}\pi_{32}\pi_{21})/5+\pi_{54}\pi_{51}(1-\pi_{21}\pi_{23}/2)$$

$$P_2' = (\pi_{12}\pi_{25}\pi_{34}\pi_{41}\pi_{53}+\pi_{21}\pi_{52}\pi_{43}\pi_{14}\pi_{35})/5+\pi_{52}\pi_{53}(1-\pi_{43}\pi_{41}/2)$$

$$P_3' = (\pi_{25}\pi_{42}\pi_{14}\pi_{31}\pi_{53}+\pi_{52}\pi_{24}\pi_{41}\pi_{13}\pi_{35})/5+\pi_{52}\pi_{53}(1-\pi_{42}\pi_{41}/2)$$

$$P_4' = (\pi_{53}\pi_{32}\pi_{21}\pi_{14}\pi_{45}+\pi_{35}\pi_{23}\pi_{12}\pi_{41}\pi_{54})/5+\pi_{53}\pi_{54}(1-\pi_{12}\pi_{14}/2)$$

$$P_5' = (\pi_{15}\pi_{52}\pi_{43}\pi_{24}\pi_{31}+\pi_{51}\pi_{25}\pi_{34}\pi_{42}\pi_{13})/5+\pi_{51}\pi_{52}(1-\pi_{43}\pi_{42}/2)$$

$$P_6' = (\pi_{51}\pi_{35}\pi_{24}\pi_{12}\pi_{43}+\pi_{15}\pi_{53}\pi_{42}\pi_{21}\pi_{34})/5+\pi_{51}\pi_{53}(1-\pi_{43}\pi_{42}/2)$$

$$P_7' = (\pi_{25}\pi_{54}\pi_{41}\pi_{13}\pi_{32}+\pi_{52}\pi_{45}\pi_{14}\pi_{31}\pi_{23})/5+\pi_{52}\pi_{54}(1-\pi_{32}\pi_{31}/2)$$

$$P_8' = (\pi_{12}\pi_{24}\pi_{45}\pi_{53}\pi_{31}+\pi_{21}\pi_{42}\pi_{54}\pi_{35}\pi_{13})/5+\pi_{54}\pi_{53}(1-\pi_{12}\pi_{14}/2)$$

$P_9' = (\pi_{12}\pi_{31}\pi_{25}\pi_{43}\pi_{54}+\pi_{21}\pi_{13}\pi_{52}\pi_{34}\pi_{45})/5+\pi_{52}\pi_{54}(1-\pi_{31}\pi_{34}/2)$

$P_{10}' = (\pi_{13}\pi_{51}\pi_{32}\pi_{24}\pi_{45}+\pi_{31}\pi_{15}\pi_{23}\pi_{42}\pi_{54})/5+\pi_{51}\pi_{54}(1-\pi_{23}\pi_{24}/2)$

$P_{11}' = (\pi_{14}\pi_{51}\pi_{42}\pi_{23}\pi_{35}+\pi_{41}\pi_{15}\pi_{24}\pi_{32}\pi_{53})/5+\pi_{53}\pi_{51}(1-\pi_{41}\pi_{42}/2)$ and

$P_{12}' = (\pi_{14}\pi_{51}\pi_{43}\pi_{32}\pi_{23}+\pi_{41}\pi_{15}\pi_{34}\pi_{23}\pi_{52})/5+\pi_{51}\pi_{52}(1-\pi_{31}\pi_{34}/2)$

Here also we assume $T_1 \leq T_2 \leq \ldots\ldots \leq T_5$. Numerical computations show that the probability of correct selection P_4' based on SP design involving the pairs (T_5, T_4), (T_5, T_3), (T_4, T_1), (T_3, T_2), (T_2, T_1) yields uniformly smaller value from amongest $(P_1', P_2', \ldots\ldots P_{12}')$ for all p, σ and σ' belonging to C_π^* (p, σ, σ') under the model (2.1) and further each of the 12 P_i's have the same infimum in the preference zone C_π^* (p, σ, σ') which is attained at the same configuration as given by H_0'.

Above numerical results suggest that in the general case of SP design with t-treatments, the design involving symmetrical pairs (T_t, T_{t-1}), (T_t, T_{t-2}), (T_{t-1}, T_1) among t of its pairs would be less favourable to the outlier than the designs which do not involve all these three pairs.

Further all $\frac{(t-1)!}{2}$. SP designs involving t treatments have same infimum for probability of correct selection in the preference zone C_π^* (p, σ, σ') and at the same configuration as given in H_0'.

<u>Probability of correct selection under least favourable configuration</u>

Let P_π denote the probability of correct selection under the least favourable configuration. Then we know from the general theory of ranking and selection that in order to satisfy the requirement (2.20), the value of n is to be determined from the equation

$$P_\pi = P^* \qquad (2.23)$$

Assuming the validity of the above conjucture about the least favourable configuration, the basic problem of choosing 'n' reduces to that of computing P_π under H_0' and solving the equation (2.20) for n. Further since under model (2.1) the probability of correct selection using any set of t symmetrical pairs from amongst $\frac{(t-1)!}{2}$ possible sets of symmetrical pairs yields the same infimum with the same least favourable configuration given by H_0', the value of P_π without loss of generality, can be computed using any one of the $\frac{(t-1)!}{2}$ possible sets of t symmetrical pairs. For convenience we shall compute P_π using SP designs involving pairs $(T_1, T_2), (T_2, T_3) \ldots\ldots (T_{t-1}, T_t), (T_t, T_1)$.

From (2.10), the joint distribution of the scores under H_0' can be written as

$$f_\pi(\underline{a}) = 2^{-n(t-2)} g(\underline{a}, n) \pi^{a_t} (1-\pi)^{2n-a_t} \qquad (2.24)$$

where $$g(\underline{a}, n) = \sum_{Rn} \prod_{r=1}^{t} \binom{n}{A_{rs}}$$

$$s = r-1 \text{ or } r+1\} \bmod t.$$

is the frequency with which the given vector $\underline{a}$ can occur. This is in the same form as the corresponding distribution in the case of F.P. design discussed by Trawinski and David (1963). Following the arguments similar to theirs, we can write the probability to correct selection P_π under H_0' as

$$P_\pi = 2^{-n(t-2)} \sum_{a_t=n}^{2n} \pi^{a_t} (1-\pi)^{2n-a_t} \sum_{g}, (1/t)\, G(\underline{a}, n) \qquad (2.25)$$

where the last summation extends over the constraints

$$g' = \sum_{i=1}^{t-1} a_i = nt - a_t$$

$G(\underline{a}, n)$ in (2.25) is the number of permissible partitions of nt into t scores $a_1, a_2, \ldots\ldots, a_t$ irrespective of order and can be written as

$$G(\underline{a}, n) = (t!/\prod_k m_k!)\, g(\underline{a}, n) \tag{2.26}$$

where m_k is the number of scores of magnitude a_k. In particular when n=1, the expression of P_π in (2.23) takes the form

$$2^{-(t-2)}[\pi(1-\pi)G(1,1,\ldots\ldots 1;1)/t+\pi^2 \sum_{a_1+a_2\ldots\ldots+a_{t-1}=t-2} G(\underline{a};1)/t]$$

$$= 2^{-(t-2)}[\pi(1-\pi)2/t+\pi^2(2^t-2)/t]$$

$$= [\pi + (2^{t-1}-2)\,\pi^2]/(2^{t-3}\times t) \quad \text{(using table 1)} \tag{2.27}$$

P_π from (2.27) has been computed for different values of π =0.55(.05) 0.95 and t upto 20. Also for n=2, and t upto 7, (2.25) has been used to compute P_π for π=.55(.05) .95. We are unable to compute P_π for $n \geq 3$ for lack of computer facility. However, in the following, we shall develop an asymptotic approximation to P_π for large n, which seems to compare favourably even for smaller n.

Asymptotic approximation to infimum of probability of correct selection

Since the probability of ties for the maximum score in paired comparisons of t treatments, with n replications per paired comparisons, tends to zero as $n \to \infty$, the probability of correct selection under H_0' is asymptotically given by

$$P_{\pi A} = \lim_{n\to\infty} P_r\,(a_t \geq \max_{1\leq i\leq t-1} a_i)$$

$$= \lim_{n\to\infty} P_r\,\{d_i \leq 0,\ i=1,2,\ldots\ldots t-1\} \tag{2.28}$$

$$= \lim_{n\to\infty} P_r\,\{v_i \leq \Delta_i,\ i=1,2,\ldots\ldots t-1\}$$

where $v_i = [d_i - n\alpha]/\sqrt{n}\,\sigma_d$ and

$\Delta_i = -\sqrt{n}\,\alpha/\sigma_d = \Delta$, $i=2,3,\ldots\ldots.t-2$

$v_i = [d_i - n\,\alpha']/\sqrt{n}\,\sigma_d'$ and

$\Delta_i = -\sqrt{n}\,\alpha'/\sigma_d' = \Delta'$, $i=1$ and $t-1$

and d_i, α, α', σ_d and σ_d' are the same as defined in section (2.4). From the limiting multivariate normality of v_i, we have

$$P_{\pi A} = (2\pi)^{\frac{-1}{2}(t-1)} |\rho|^{-1/2} \int_{-\infty}^{\Delta'} \int_{-\infty}^{\Delta} \cdots \int_{-\infty}^{\Delta} \int_{-\infty}^{\Delta'} \exp\left[-\frac{1}{2} v' \rho^{-1} v\right] dv_1 dv_2 \ldots\ldots dv_{t-1} \tag{2.29}$$

The exact evaluation of $P_{\pi A}$ through (2.29) is hard. We can use approximation developed by E.J. Dudewiz (1969) to find the value of n satisfying $P_{\pi_A} \geq P^*$, for given large P^*. In fact since $\Delta < \Delta'$

$$P_{\pi A} \geq (2\pi)^{-(t-1)/2} |\rho|^{-1/2} \int_{-\infty}^{\Delta} \cdots\cdots \int_{-\infty}^{\Delta} \exp\left[-\frac{1}{2} v' \rho^{-1} v\right] \times dv_1 \ldots dv_{t-1} \tag{2.30}$$

Thus by Theorm (Section I, Dudewiz (1969), the minimum value of n needed to make the lower bound in (2.30) equal to P^* (close to 1) say n_A, is given by

$$n_A = \frac{\pi(1-\pi)+1/4}{2(\pi-1/2)^2} \times -2\ln(1-P^*) \tag{2.31}$$

Though the evaluation of n using this approximation is computationally very simple, but has the draw back that it is valid only for large P^*. Further more, it does not depend on the number of treatments t, but in fact P_{π_A} as given by (2.29) varies substantially with variation in t. Thus the agreement between approximate n_A calculated from (2.31) and correct least n satisfying $P_{\pi_A} \geq P^*$ is expected to hold good only for a limited range of values of t.

Another lower bound for P_{π_A} can be obtained with the help of the inequality for distribution function of multivariate normal proved by Sidak (1967), which gives

$$P_{\pi A} \geq \prod_{2 \leq i \leq t-2} P(|v_i| < \Delta) \prod_{i=1,t-1} P(|v_i| < \Delta')$$

$$= [2\phi(\Delta)-1]^{t-3} [2\phi(\Delta')-1]^{2} \qquad (2.32)$$

for large n. One can now compute n_B, the least n for which the lower bound in (2.32) exceeds or equal to P*. This approximation to n is applicable to any P* and is also dependent on t. In table 2 we have presented the values of lower bound of P_{π_A} given by (2.32), for some given set of values of t, n and π. In the same table 2, the values super suffixed as * were obtained with the help of exact expression (2.25). This table can be used for finding n_B for given t, π and P*. For numerical comparison of n_A and n_B we have prepared table 3. There seems to be a reasonable agreement between these two values for $5 \leq t \leq 15$, for large P* for most values of $\pi > 1/2$.

Comparison of FP and SP designs:

For any paired comparison design that uses the selection rule R_1 for choosing the best treatment, we shall take the infimum probability of correct selection over the preference zone $C^*_\pi(\pi_{ij})$ for given t and the size of the experiment, expressed in terms of number of paired comparisons N used in the design, as an index of the efficiency of that design.

In situations where there is reason to believe that there exists an outlier among the treatments and others are more or less equal, and when the problem is to detect this outlier, then SP design is more suited than FP design. In order to compare the asymmototic efficiency of SP design, when the problem is of detection of the outlier, with the FP design, we have constructed the charts 1 and 2. Actually for SP design. the lower bound of infimum as given by (2.32), while for FP design, the probability of correct selection under configuration H_0', (which is not necessarily least favourable) is plotted againt π for given t(14, 20) and N (182, 330) in these charts. Even then, the curve given by SP design is above the corresponding curve for FP design for any $\pi > .85$. This provides a good evidence that SP design is asymptotically more efficient than FP design in case t and π are large. Note that for constructing these chart, table 2 of this chapter for SP design and table 1 of Trawinski and David (1963) for FP design were used. These tables are also used to construct table 4 which indicates minimum reduction in the size of experiment for SP design against FP design for large t, large π and P*, where P* fixes the lower bound given in (2.31) for SP design, while it fixes the probability of correct selection under H_0', as in the previous charts. In fact, the computations have shown that SP design requires smaller experimental size even if either t is large or π is close to 1.

Next in chart 3 and 4 we have ploted the probability of correct selection P_π against π for different values of t, given N = 1 and 10 respectively. For n = 1 P_π was computed from (2.27) while for n = 10, the lower bound given by (2.32) was used for plotting. Analysing similar chart for FP design Trawinski and David (1963) (Fig IA and IB), constructed an example revealing that the single outlier configuration is not least favourable.

This was made possible as the curves on their charts for increasing values of t cross the curves corresponding to preceding values of t. But in contrast, in our charts 3 and 4 for SP design the curves corresponding to increasing values of t lie entirely below the curves corresponding to preceding values of t. Thus the contradiction against our conjucture of section (2.5) that the configuration H_0' of single outlier being least favourable for SP design is not evident from these charts.

In the next section we will give a subset selection rule for selection of the best treatment.

2.6 <u>SELECTION OF A SUBSET CONTAINING THE BEST TREATMENT:</u>

Our aim is to select a subset 's' of random size between 1 and t of the t given treatments which include the best treatment T_t with a high preassigned probability of atleast P*.

The subset selection rule R_2 is defined below:

<u>Rule R_2:</u> Retain treatment T_i (i=1, 2,.......t) in the subset 's' if its score $a_i \geq \max_{1 \leq i \leq t} a_i - \gamma$

where $(a_1, a_2, \ldots\ldots\ldots a_t)$ are scores corresponding to treatments $(T_1, T_2, \ldots\ldots\ldots T_t)$ respectively, obtained by using an SP design with n replications and $\gamma \geq 0$, is an integer called selection constant for rule R_2. This constant γ, for given n and t is the smallest positive integer chosen so as to satisfy the probability requirement given below

$$P_s \geq P^* \qquad (2.33)$$

for all configurations in $C^*(p, \varepsilon, \varepsilon')$ where P_s is the probability of correct selection. The same rule as R_2 was applied for FP design by Trawinski and David (1963).

Assuming T_t is the outlier, using the rule R_2, clearly we have

$$P_s = P(a_t \geq \max_{1 \leq i \leq t-1} a_i - \tau)$$

The expression of P_s is difficult to write in general and we shall directly proceed to find its infimum in $C^*(p, \varepsilon, \varepsilon')$.

We shall proceed in the same manner as for the previous rule R_1 and conjucture that the infimum of probability of correct selection under rule R_2 in the parameter space $C^*(p, \varepsilon, \varepsilon')$ is attained at the configuration implying all treatments are equally good. To accomplish this the detailed expressions for P_s in terms of $\{\pi_{ij}\}$ were written for 4 and 5 treatments under all possible sets of symmetrical pairs given by (2.21) and (2.22) respectively. For example for t=5 and SP design including pairs given by (6) in (2.22), the expression for P_{cs} with $\tau = 1$ is

$$\begin{aligned} P'_{6s} = {} & \pi_{34}\pi_{35}\pi_{51}+\pi_{12}\pi_{15}\pi_{53}+\pi_{21}\pi_{24}(\pi_{53}\pi_{15}+\pi_{51}\pi_{43} \times \\ & \pi_{35})+\pi_{42}\pi_{43}(\pi_{51}\pi_{35}+\pi_{53}\pi_{15}\pi_{21}) + (\pi_{51}\pi_{35} \times \\ & \pi_{24}\pi_{12}\pi_{43}+\pi_{15}\pi_{53}\pi_{42}\pi_{21}\pi_{34})/5 + \pi_{51}\pi_{53} \times \\ & (\pi_{24}\pi_{12}+\pi_{42}\pi_{34})+\pi_{51}\pi_{53}(\pi_{24}\pi_{21}+\pi_{43}\pi_{42})/2 \end{aligned}$$

For the sake of bravity, we have not given the expressions for P_{cs} for all the sets of SP designs in (2.22). Numerical computations were done for all these expressions for various values of p, ε, ε' in $C^*(p, \varepsilon, \varepsilon')$ assuming model (2.1) holds. These computations suggested almost similar conclusions as for rule R_1, which are following.

In the general case of SP design with t-treatments, the design involving symmetrical pairs (T_t, T_{t-1}), (T_t, T_{t-2}), (T_{t-1}, T_1)

among t of its pairs would be less favourable to the outlier than the designs which do not involve all these three pairs.

Further all $\frac{(t-1)!}{2}$ SP designs involving t treatments have same infimum for P_{cs} in the preference zone C^*_{π} (p, ε, ε′), which is attained at the configuration C* (1/2, 0, 0). Thus, the probability requirement (2.32) is equivalent to choosing least $\mathcal{V}$ such that

$$P_{cs}(\tfrac{1}{2}) \geq P^* \qquad (2.34)$$

where $P_{cs}(\tfrac{1}{2})$ is the value of P_{cs} at the least favourable configuration C* (½, 0, 0). The implementation of the procedure R_2 would therefore, require the expression for $P_{cs}(\tfrac{1}{2})$.

The expression for $P_{cs}(\tfrac{1}{2})$ can be expressed in the following form.

$$P_{cs}(\tfrac{1}{2}) = [2^{-nt} \Sigma M(\underline{a}, \mathcal{V})\, G(\underline{a}, n)]/t \qquad (2.35)$$

where $M(\underline{a}, \mathcal{V})$ isthe number of scores a_i in $\underline{a}$ which satisfy

$$a_i \geq \max_{1 \leq i \leq t} a_i - \mathcal{V} \quad \text{and } G(\underline{a}, n)$$

the partition function defined in (2.26). The summation in (2.35) is over all distinct partitions $\underline{a}$ of nt. The derivation of (2.35) is similar to the corresponding derivation for FP design. (Trawinski and David (1963) Section 4.2). Some exact evaluations of $\mathcal{V}$ using (2.35) in (2.34) appear in table 5 along with assymptotic values as derived in the next section.

Asymptotic approximation to $\mathcal{V}$:

The asymptotic probability of correct selection under rule R_2 at the least favourable configuration may be written as

$$P_{As}(1/2) = \lim_{n\to\infty} P_r \{a_{max} - a_t < \eta\} \quad (2.36)$$

$$= \lim_{n\to\infty} P_r \{d_i < \eta, \; i=1,2,\ldots\ldots t-1\}$$

$$= \lim P_r \{v_i < \theta, \; i=2,3,\ldots\ldots t-2, \; v_i < \theta', \; i=1 \text{ and } t-1\}$$

where $d_i = a_i - a_t, \; i=1,2,\ldots\ldots,t-1$

$v_i = d_i/\sqrt{n}\,\sigma d, \; i=2,3,\ldots\ldots,t-2$

$= d_i/\sqrt{n}\,\sigma d', \; i=1$ and $t-1$

and $\theta = \eta/\sqrt{n}\,\sigma d, \; i=2,3,\ldots\ldots t-2$

$\theta' = \eta/\sqrt{n}\,\sigma d', \; i=1$ and $t-1$

Note that when $\pi = 1/2$

$\sigma_d^2 = 1, \; \sigma_{d'}^2 = 3/2$ from (2.17).

using the same argument as for the expression (2.28), a lower bound for $P_{As}(\frac{1}{2})$ is given by

$$P_{As}(\tfrac{1}{2}) \geq [2\,\phi(\theta) - 1]^{t-3}[2\,\phi(\theta') - 1]^2 \quad (2.37)$$

The lower bound for $P_{As}(\frac{1}{2})$ given in (2.37) above have been computed for given t, n and η. These values were then used for finding the asymptotic approximation to the least value of η for specified value of t, n and P* such that the lower bound $P_{As}(\frac{1}{2}) \geq P^*$. These are presented in table 5.

The experimentor may be interested in knowing the expected size of the selected set s, if indeed there is a superior treatment among the t treatments. Expected subset size provides a measure of efficiency of a subset selection procedure. In the following section we shall obtain an asymptotic expression for the expected subset size under slippage configuration for the selection rule R_2.

2.7 Asymptotic approximation to the expected size of a selected subset:

We shall derive the expression only under slippage configuration belonging to $C(\pi)$ defined in (2.9). Clearly the expected subset under the slippage configuration would be same for any choice of symmetrical pairs in the SP design and, therefore, without loss of generally we assume that T_t is the outlier and SP design uses the pairs (T_1, T_2), (T_2, T_3).......(T_{t-1}, T_t), (T_t, T_1). In that case the expression for expected subset size given by

$$E_{t,n(s)}(\pi, \sigma) = \sum_{r=1}^{t} P_r \{a_s - a_r < \sigma\}_{s \neq r} \qquad (2.38)$$

$$= 2 \lim_{n\to\infty} P_r\{u_i \leq \Delta_1 (i=1,2,\ldots t-4), u_{t-3} \leq \Delta_2,$$

$$u_{t-2} \leq \Delta_3, U_{t-1} \leq \Delta_4\} + 2 \lim_{n\to\infty} P_r\{v_i \leq \Delta_5, (i=1,2,\ldots t-5),$$

$$v_{t-4} \leq \Delta_6, v_{t-3} \leq \Delta_7, V_{t-2} \leq \Delta_8, V_{t-1} \leq \Delta_9\} + (t-5) \times$$

$$\lim_{n\to\infty} P_r\{w_i \leq \Delta_5 (i=1,2,\ldots t-6, w_i \leq \Delta_8 (i= t-5, t-4),$$

$$w_j \leq \Delta_7 (j= -3, t-2), w_{t-1} \leq \Delta_9\} + \lim_{n\to\infty} \{x_i \leq \Delta_{10} (i=1,2,\ldots t-3),$$

$$x_i \leq \Delta_{11} \ (i=t-2, t-1\} \qquad (2.39)$$

where $u_i = \dfrac{a_1 - a_{i+2} + \alpha_1}{\sqrt{n}\ \sigma_2}$, $(i=1,2,\ldots\ldots t-4)$

$$u_{t-3} = \frac{a_1 - a_2 + \alpha_1}{\sqrt{n}\ \sigma_1}$$

$$u_{t-2} = \frac{a_1 - a_{t-1}}{\sqrt{n}\ \sigma_3}$$

$$u_{t-1} = \frac{a_1 - a_t + 3\,\alpha_1}{\sqrt{n}\;\sigma_4}$$

$$v_i = \frac{a_2 - a_{i+3}}{\sqrt{n}\;\sigma_6}, \quad i = 1, 2, \ldots\ldots t-5$$

$$v_{t-4} = \frac{a_2 - a_1 - \alpha_1}{\sqrt{n}\;\sigma_1}$$

$$v_{t-3} = \frac{a_2 - a_3}{\sqrt{n}\;\sigma_5}$$

$$v_{t-2} = \frac{a_2 - a_{t-1} - \alpha_1}{\sqrt{n}\;\sigma_2}$$

$$v_{t-1} = \frac{a_2 - a_t + \alpha_2}{\sqrt{n}\;\sigma_3}$$

$$w_i = \frac{a_3 - a_{i+4}}{\sqrt{n}\;\sigma_6}, \quad i = 1, 2, \ldots\ldots t-6$$

$$w_{t-5} = \frac{a_3 - a_1 - \alpha_1}{\sqrt{n}\;\sigma_2}$$

$$w_{t-4} = \frac{a_3 - a_{t-1} - \alpha_1}{\sqrt{n}\;\sigma_2}$$

$$w_{t-3} = \frac{a_3 - a_2}{\sqrt{n}\;\sigma_5}$$

$$w_{t-2} = \frac{a_3 - a_4}{\sqrt{n}\;\sigma_5}$$

$$w_{t-1} = \frac{a_3 - a_t + \alpha_2}{\sqrt{n}\;\sigma_3}$$

$$x_i = \frac{a_t - a_{i+1} - \alpha_2}{\sqrt{n}\ \sigma_3} \quad , \quad i=1,2,\ldots\ldots t-3$$

$$x_{t-2} = \frac{a_t - a_1 - 3\alpha_1}{\sqrt{n}\ \sigma_4}$$

$$x_{t-1} = \frac{a_t - a_{t-1} - 3\,\alpha_1}{\sqrt{n}\ \sigma_4}$$

$$\Delta_1 = \frac{\vartheta + 1/2 + \alpha_1}{\sqrt{n}\ \sigma_2}$$

$$\Delta_2 = \frac{\vartheta + 1/2 + \alpha_1}{\sqrt{n}\ \sigma_1}$$

$$\Delta_3 = \frac{\vartheta + 1/2}{\sqrt{n}\ \sigma_3}$$

$$\Delta_4 = \frac{\vartheta + 1/2 + 3\alpha_1}{\sqrt{n}\ \sigma_4}$$

$$\Delta_5 = \frac{\vartheta + 1/2}{\sqrt{n}\ \sigma_6}$$

$$\Delta_6 = \frac{\vartheta + 1/2 - \alpha_1}{\sqrt{n}\ \sigma_1}$$

$$\Delta_7 = \frac{\vartheta + 1/2}{\sqrt{n}\ \sigma_5}$$

$$\Delta_8 = \frac{\vartheta + 1/2 - \alpha_1}{\sqrt{n}\ \sigma_2}$$

$$\Delta_9 = \frac{\vartheta + 1/2 + \alpha_2}{\sqrt{n}\ \sigma_3}$$

$$\Delta_{10} = \frac{\mathcal{J} + 1/2 - \alpha_2}{\sqrt{n}\ \sigma_3}$$

$$\Delta_{11} = \frac{\mathcal{J} + 1/2 - 3\alpha_1}{\sqrt{n}\ \sigma_4}$$ with

$$\alpha_1 = n\,[\pi - 1/2]$$

$$\alpha_2 = n\,[\,2\pi - 1]$$

$$\sigma_1^2 = n\,[\pi(1-\pi) + 5/4]$$

$$\sigma_2^2 = n\,[\pi(1-\pi) + 3/4]$$

$$\sigma_3^2 = n\,[2\pi(1-\pi) + 1/2]$$

$$\sigma_4^2 = n\,[5\pi(1-\pi) + 1/4]$$

$$\sigma_5^2 = 3/2\,n$$

$$\sigma_6^2 = n$$

Using the same argument as for deriving the expression (2.28), a lower bound for (2.39) can be obtained as

$$E_{t,n(s)}(\pi,\mathcal{J}) \geq 2[2\,\phi\,(\Delta_1)-1]^{t-4}[2\phi(\Delta_2)-1][2\phi(\Delta_3)-1]$$
$$[2\phi(\Delta_4)-1] + [2\phi(\Delta_5)-1]^{(t-6)}[2\phi(\Delta_7)-1][2\phi(\Delta_8)-1][2\phi(\Delta_9)-1]$$
$$\{2[2\phi(\Delta_5)-1][2\phi(\Delta_6)-1] + (t-5)[2\phi(\Delta_7)-1][2\phi(\Delta_8)-1]\} +$$
$$[2\phi(\Delta_{10})-1]^{(t-3)}[2\phi(\Delta_{11})-1]^2 \qquad (2.40)$$

2.8 An example

We illustrate the application of the decision rule R_2 on following data. The data are adapted from an experiment to determine which concentration of monosodium glutamate produce the prefered flavour of dehydrated apple slices. The experiment was conducted at the Verginia Agricultural Experimental Station by L.L. Davis and the data were reported in Bradely (1954). Apple slices containing four different concentrations tabled as A, B, C and D were presented in SP to seven judges. The preference table is given below:

	A	B	C	D	Total
A	-	4	-	0	4
B	3	-	5	-	8
C	-	2	-	3	5
D	7	-	4	-	11
Total	10	6	9	3	28

We have $a_{max} = 11$. To ensure that at least a pre-assigned probability $P^* = .75$, the best concentration is in the selected subset, we enter table 4 for $t=4$, $n=7$, $P^*=.75$, find $\vartheta=5$ and hence retain concentration D and B in the selected subset. For the same data if selection of the best concentration using the selection rule R_1 is done, we find that D can be selected as the best concentration with atleast a probability 0.95, using table 1, when Π is set at 0.85.

CHAPTER III

SELECTION OF BEST TREATMENT USING FULL PAIRED COMPARISON DESIGN IN PRESENCE OF TIES

3.1 Introduction

For wider applicability, models for paired comparisons should allow for expression of no preference between the treatments being compared and this requires analysing tied observations. In this chapter, selection procedures have been developed under indifference zone as well as subset selection formulation for Goal 1 as defined in section 5 of chapter I, based on the scores making allowance for ties, obtained by using FP design with equal number 'n' of replications. No judge effect, no replication effect and no order effect is assumed in the analysis.

3.2 Mathematical model:

For the general formulation of method of paired comparison with ties permitted, we can assume, that the preference probabilities

$$(\pi_{i.ij},\ \pi_{j.ij},\ \pi_{0.ij},\ i,j=1,2,\ldots\ldots t) \qquad (3.1)$$

defined in (1.10), (1.11) and (1.12) constitute the parameters of the model and $C^{t}\ (\pi_{i.ij}) = \{0 \leq \pi_{i.ij} \leq 1,\ 0 \leq \pi_{0.ij} \leq 1,$

$$\pi_{0.ij} + \pi_{i.ij} + \pi_{j.ij} = 1,\ i,j = 1,2,\ldots\ldots t\} \qquad (3.2)$$

the complete parametric space.

Under the above model, if treatment $T_i = T_j$, then clearly

$$\pi_{i.ij} = \pi_{j.ij} = (1-\pi_{0.ij})/2 \qquad (3.3)$$

and if $T_i > T_j$, then

$$\pi_{i.ij} > (1-\pi_{0.ij})/2 \qquad (3.4)$$

where $\pi_{0.ij}$ is the tie probability defined in (1.12).

In case all treatments are assumed equally good, the observed worths Y_i of the treatments would be identically and independently distributed and from the defination (1.12), tie probability $\pi_{0.ij}$ would not depend on the pair T_i, T_j. In that case let

$$\pi_{0.ij} = \theta \ , \ 0 \leq \theta \leq 1$$

and using (3.3), we conclude that if all treatments are equally good, then the parametric configuration of preference probability must belong to $C^t(\theta) = \{\pi_{i.ij} = \pi_{j.ij} = (1-\theta)/2,$

$$\pi_{0.ij}=\theta, \ 0\leq\theta\leq 1\} \qquad (3.5)$$

Define $K_{ij} = \pi_{i.ij} + (1/2)\ \pi_{0.ij}$ (3.6)

and $K_{ji} = \pi_{j.ij} + (1/2)\ \pi_{0.ij}$

Then clearly

$$0 \leq K_{ij} \leq 1$$

$$K_{ij} + K_{ji} = 1, \ i,j = 1,2,........t.$$

In terms of parametric function K_{ij} we have, that, if treatment $T_i = T_j$, then $K_{ij} = K_{ji} = \frac{1}{2}$ and if $T_i > T_j$, then $K_{ij} > \frac{1}{2}$, using conditions (3.3) and (3.4) respectively. This suggests that parameters K_{ij}, $i > j$, $i,j, = 1,2,.......t$ which are functions of those in (3.1), would be sufficient for providing mutual comparisons among the treatments. Because of this we shall define true feasible ranking of the treatments in terms of these parameters K_{ij}. In fact, let $K_i = \sum_j K_{ij}$ $i = 1,2,......t$ be the K-row-sum corresponding to the treatment T_i, then we would rank the treatments according to descending values of K_i's. If there are n replications, the corresponding $K_i = n \sum_j k_{ij}$, $i=1,2,....t$.

It is interesting to observe in the above discussion that the parameters K_{ij} play exactly in the same role in the model with ties permitted as the preference probabilities π_{ij}, when ties are not permitted (see section 1.2). Consequently, all results on ranking by row-sum as discussed by Kendall (1955), Buhlmann and Huber (1963) and Huber (1963) would be directly applicable to tie case when parameter π_{ij} are replaced by K_{ij}.

From the preceeding discussion it follows that the problem of selection of best treatment now reduces to the selection of the treatment with the maximum K-row sum. The development of selection procedures for the above goal would as usual need the distribution of scores of various treatments based on preference and the data obtained by using a given design, which is FP design in this chapter. These distributions are discussed in the following section.

3.3 DISTRIBUTION THEORY

Consider a balanced paired comparison experiment consisting of 'n' replication of all $\frac{t(t-1)}{2}$ comparisons between the treatments T_i (i=1,2,........t) referred to as FP design with n replications. Let x_{ijr} as defined below be the random score associated with the rth replication of comparison between T_i and T_j

$$x_{ijr} \begin{cases} = 1 & \text{if } T_i > T_j \\ = 0 & \text{if } T_j > T_i \\ = \tfrac{1}{2} & \text{if } T_i = T_j \\ i, j = 1, 2, \ldots\ldots t, & i \neq j \\ r = 1, 2, \ldots\ldots\ldots n. & \end{cases} \qquad (3.7)$$

we assume that all nt(t-1)/2 comparisons are independent. From the definitions of preference probabilities we have

$$P_r\ (x_{ijr} = 1) = \pi_{i.ij}$$
$$P_r\ (x_{ijr} = 0) = \pi_{j.ij} \qquad (3.8)$$
$$P_r\ (x_{ijr} = \tfrac{1}{2}) = \pi_{0.ij}$$

The score a_i for the treatment T_i is defined in the same manner as in (1.8) i.e.

$$a_i = \sum_{r=1}^{n} a_{ir} = \sum_{r=1}^{n} \sum_{j} x_{ijr}$$

clearly $a_i \in (0, \tfrac{1}{2}, 2/2, 3/2, \ldots\ldots, \frac{2n-1}{2}, n)$

$$\sum_{i=1}^{t} a_{ir} = t(t-1)/2$$

$$\sum_{i=1}^{t} a_i = nt(t-1)/2$$

In the next section we will obtain the exact distribution of the scores.

Exact distribution of scores

Let A_{rs} (r>s) be the number of times T_r is prefered to T_s and $A_{0.rs}$ be the number of ties in 'n' comparisons, then

$$n = A_{rs} + A_{sr} + A_{0.rs} \qquad (3.9)$$

In view of our assumption, (A_{rs}, A_{sr}) have independent trinomial distributions.

$$P(A_{rs}, A_{sr}, \pi_{r.rs}, \pi_{s.rs})$$
$$= \frac{n!}{A_{rs}!\ A_{sr}!\ A_{0.rs}!}\ \pi_{r.rs}^{A_{rs}}\ \pi_{s.rs}^{A_{sr}}\ \pi_{0.rs.}^{A_{0.rs}} \qquad (3.10)$$

where $A_{rs}, A_{sr} \geq 0$ subject to (3.9). Thus joint distribution of $\{A_{rs}\} = \{A_{rs}, r > s, r, s = 1, 2, \ldots\ldots.t\}$ given by $f(\{A_{rs}\}) =$

$$\prod_{r>s}^{t} P\ (A_{rs}, A_{sr}, \pi_{r,rs}, \pi_{s.rs}) \qquad (3.11)$$

Clearly score a_i can be rewritten in the form

$$a_i = \sum_{\substack{j=1 \\ j\neq i}}^{t} [A_{ij} + \tfrac{1}{2} A_{0.ij}]$$
$$= \tfrac{1}{2} [n(t-1) + \sum_{\substack{j=1 \\ i\neq j}}^{t} (A_{ij} - A_{ji})], \quad i=1,2,\ldots\ldots t. \qquad (3.12)$$

where $0 \leq A_{ij} \leq n, \; A_{ij} + A_{ji} \leq n.$

The joint distribution of the score vector $\underline{a} = (a_1, a_2, \ldots a_t)$ using (3.11) is the following

$$f[\underline{a}, C^t(\pi_{i.ij})] = \sum_{Rm} f(\{A_{rs}\}) \qquad (3.13)$$

where $\sum_{Rm}$ is the summation over A_{ij}'s subject to restriction in (3.12).

When all treatments are equally good i.e. when parametric configuration belongs to $C^t(\theta)$ defined in (3.5), then (3.13) reduces to

$$f[\underline{a}, C^t(\theta)] = \sum_{Rm} \left(\frac{1-\theta}{2}\right)^{\sum_{r>s}^{t}(n-A_{0.rs})} \theta^{\sum_{r>s}^{t} A_{0.rs}} \prod_{r>s}^{t} \frac{n!}{A_{rs}!A_{sr}!A_{0.rs}!}$$

Let $\sum_{r>s}^{t} A_{0.rs} = x$, the total number of ties, then

$$f[\underline{a} \; C^t(\theta)] = \left(\frac{1-\theta}{2}\right)^{\frac{nt(t-1)}{2}} \sum_{x=0}^{nt(t-1)/2} \left(\frac{2\theta}{1-\theta}\right)^x g_x(\underline{a}, n) \qquad (3.14)$$

where $g_x(\underline{a}, n) = \sum_{Rm^1} \prod_{r>s}^{t} \frac{n!}{A_{rs}!\,A_{sr}!\,A_{0.rs}!}$

and $\sum_{Rm^1}$ is the summation over A_{rs}, A_{sr} subject to restriction in Rm together with $\sum_{r>s}^{t} A_{0.rs} = x$. The function $g_x(\underline{a}, n)$ gives the number of ways in which the outcome that the score vector is $\underline{a}$ and these scores include exactly x tied observations be realized among nt(t-1)/2 comparisons. Further more, for any fixed x, it is a symmetrical function of scores $a_1, a_2, \ldots a_t$. From $g_x(\underline{a}, n)$ we can obtain the partition function $G_x(\underline{a}, n)$ giving the number of permissible partitions of nt(t-1)/2 into 't' scores $a_1, a_2, \ldots$ $\ldots a_t$, irrespective of order and such that these scores involve exactly 'x' tied observations. In view of the symmetry of $g_x(\underline{a}, n)$, we have

$$G_x(\underline{a}, n) = (t!/\prod_k m_k!)\ g_x(\underline{a}, n) \tag{3.15}$$

where m_k is the number of scores all of magnitude a_k. It is easy to compute $G_x(\underline{a}, n)$ by using generating function than $g_x(\underline{a}, n)$.

It is clear that the equation of the exact probability function (3.13) is very cumbersome for large t and n. Since this would be involved in the construction of tables for implementing the selection rules for selecting the best treatment to be discussed subsequently, We now develop the asymptotic distribution theory. based on multivariate central limit theorem.

<u>Asymptotic distribution for difference of scores</u>

Using (3.8) we have

$$\begin{aligned} E(a_i) &= n\,E\Big(\sum_{j\neq i} x_{ijr}\Big) = n \sum_j k_{ij} \\ V(a_i) &= n\,V(a_{ir}) \\ &= n \sum_j [\pi_{i.ij}(1-\pi_{i.ij}-\pi_{0.ij}) \\ &\qquad + (1/4)\,\pi_{0.ij}(1-\pi_{0.ij})] \end{aligned} \tag{3.16}$$

$$\text{Cov}\,(a_i,\ a_j) = -\,n[\pi_{i.ij}(1-\pi_{i.ij} - \pi_{0.ij}) + 1/4\,\pi_{0.ij}(1-\pi_{0.ij})] \qquad (3.17)$$

Distribution of the vectors of differences among scores, defined by $\underline{d} = (d_1,\ d_2, \ldots\ldots d_{t-1})$, where $d_i = (a_i - a_t)$, $i=1,2,\ldots\ldots t-1$, is of special interest. The variate d_i has mean

$$\alpha_i = n[\sum_{k=1}^{t} (\pi_{i.ik} + \tfrac{1}{2}\pi_{0.ik}) - \sum_{k=1}^{t-1}(\pi_{t.tk} + \tfrac{1}{2}\pi_{0.tk})] \qquad (3.18)$$

Its variance is given by

$$\sigma d_i d_i = n\,[\ \text{Var}\ a_{ir} + \text{var}\ \ a_{tr} - 2\ \text{Cov}\ (a_{ir},\ a_{tr})] \qquad (3.19)$$

The covariance of d_i and d_j can be obtained from the relation

$$\sigma d_i d_j = \text{var}\ a_t - \text{cov}\ (a_i,\ a_t) - \text{cov}\ (a_t,\ a_j) + \text{cov}\ (a_i,\ a_j) \quad (3.20)$$

Now $d_i = \sum_{r=1}^{n} (a_{ir} - a_{tr}) = \sum_r d_{ir}$ and d_{ir} have, for any given r, means, variance and covariance α_i/n, $\sigma d_i d_i/n$, and $\sigma d_i d_j/n$ respectively. Thus as $n \to \infty$, the vector $n^{-\frac{1}{2}}(\underline{d} - \underline{\alpha})$ is multivariate normal $N(0,\Sigma)$ where Σ is the matrix $(n^{-1}\,\sigma d_i d_j$ and $\underline{\alpha} = (\alpha_1,\ \alpha_2, \ldots\ldots \alpha_{t-1})$

Now we are in a position to develop the selection procedures for selecting the best treatment out of t treatments in the next section.

3.4 SELECTION OF THE BEST TREATMENT

As discussed in the section 3.2, we would select the treatment with largest k-row sum as the best. Also from (3.16), it follows that the scores a_i are the unbiased estimators of k_i. Thus it is reasonable to base selection procedures on these scores we define the following selection rule R_3.

Rule R_3 Obtain $(a_1, a_2, \ldots\ldots a_t)$, the scores of 't' treatments $T_1, T_2, \ldots\ldots T_t$ respectively, as defined in (3.12), using FP design with n replications per pair and declare the treatment with the score $\max\limits_{1 \leq i \leq t} a_i$ as the best treatment. If 'm' scores tie for the first place, then one of the corresponding 'm' treatments is chosen at random for the best treatment. Same rule was defined by Trawinski and David (1963) for their problem.

Without loss of generality, it can be assumed that T_t is the best treatment. Then the parametric configuration would belong to the set

$$C^{t*}(\pi_{i.ij}) = \{ \pi_{t.tj} > \frac{1-\pi_{0.tj}}{2}, \; j = 1, 2, \ldots . t-1$$

$$0 \leq \pi_{i.ij} \leq 1, \; 0 \leq \pi_{0.ij} \leq 1$$

$$\pi_{i.ij} + \pi_{j.ij} + \pi_{0.ij} = 1, \; i, \; j = 1, 2, \ldots\ldots t \}$$

For a given preassigned value of $\pi > \frac{1}{2}$ call

$$C_\pi^{t*}(\pi_{i.ij}) = \{ \pi_{t.tj} + \pi_{0.tj}/2 \geq \pi, \quad j = 1, 2, \ldots\ldots t-1$$

$$0 \leq \pi_{i.ij} \leq 1, \; 0 \leq \pi_{0.ij} \leq 1$$

$$\pi_{0.ij} + \pi_{i.ij} + \pi_{j.ij} = 1, \; i, j = 1, 2, \ldots\ldots t, \quad i \neq j \}$$

which is a subset of $C^{t*}(\pi_{i.ij})$ as a preference zone.

In case T_t is an outlier and other treatments are of equal worth then we can assume that $\pi_{0.tj} = \theta'$ for all $j = 1, 2, \ldots . t-1$ and $\pi_{0.ij} = \theta$, $\underset{i \neq j}{i, j = 1, 2, \ldots\ldots t-1}$. Further more, since T_t is considered to be the superior treatment, in comparison to others, the probability θ' of declaring a tie for the outlier T_t is to be lower than θ, the probability of declaring a tie among other treatments that are equal. Thus slippage configuration must belong to

$$C_{\pi}^{t*}(\theta, \theta') = \{ \pi_{t.tj} = \pi - \theta'/2$$

$$\pi_{0.tj} = \theta', \; j=1,2,\ldots\ldots t-1$$

$$\pi_{0.ij} = \theta \qquad (3.21)$$

$$\pi_{i.ij} = \frac{1-\theta}{2}, \; i,j=1,2,\ldots\ldots t-1, \; i \neq j$$

$$0 \leq \theta' \leq \theta \leq 1 \}$$

Let P_1 denote the probability of correct selection that the treatment T_t is infact declared as the best treatment by rule R_3. Following the general theory of indifference zone approach for implementing the selection procedure R_3, the number 'n' of replication per paired comparison in FP design is chosen such that

$$P_1 \geq P^* \qquad (3.22)$$

for given P^* and π whenever the parametric configuration is in $C_{\pi}^{t*}(\pi_{i.ij})$

Unfortunately the expression for P_1 in $C_{\pi}^{t*}(\pi_{i.ij})$ is hard to write and it is equally hard to search for least favourable configuration in $C_{\pi}^{t*}(\pi_{i.ij})$. However, following Trawinski and David (1963), we will obtain the expression for $P_1(\pi, \theta, \theta')$ in $C_{\pi}^{t*}(\theta, \theta')$ and choose least n such that

$$P_1(\pi, \theta, \theta') \geq P^* \qquad (3.23)$$

for given P^*, π, θ, θ'. As such for implementing the selection rule R_3, parameters θ, θ' should be prescribed in advance as is π. This choice of $\pi_{i.ij}$ in $C_{\pi}^{t*}(\theta, \theta')$ does not necessarily correspond to a least favourable configuration. However, the configuration $C_{\pi}^{t*}(\theta, \theta')$ is important in itself since it presents the situation in which there is a superior treatment and would be a reasonable basis for deciding about the number of replications required for detecting an outlier. In the following we will obtain the expression for $P_1(\pi, \theta, \theta')$.

<u>Exact evaluation of the probability of correct selection of the best treatment</u>

From (3.13), the joint distribution of the scores when parametric configuration is in $C_\pi^{t*}(\theta, \theta')$ can be written as

$$P[\underline{a}, C_\pi^{t*}(\theta, \theta')] = \sum_{Rm} [(1-\theta)/2]^{\sum_{r>s}^{t-1}(A_{rs}+A_{sr})} \theta^{\sum_{r>s}^{t-1} A_{o.rs}} \times$$

$$\theta'^{\sum_{j=1}^{t-1} A_{0.tj}} (\pi-\theta'/2)^{\sum_{j=1}^{t-1} A_{tj}} (1-\pi-\theta'/2)^{\sum_{j=1}^{t-1} A_{jt}} \times$$

$$\prod_{r>s}^{t} \frac{n!}{A_{rs}!\, A_{sr}!\, A_{o.rs}!}$$

By using the following relations

i) $A_{rs} + A_{sr} + A_{0.rs} = n$

ii) $A_{tj} + A_{jt} + A_{0.tj} = n$

iii) $a_t = \sum_{j=1}^{t-1} (A_{tj} + \frac{1}{2} A_{0.tj})$

and putting $\sum_{r>s}^{t} A_{0.rs} = x$, $\sum_{j=1}^{t-1} A_{o.tj} = y$

then last expression can be written as

$$P[\underline{a}, C_\pi^{t*}(\theta, \theta')] = \left(\frac{1-\theta}{2}\right)^{\frac{n(t-1)}{2}} (1-\pi-\theta'/2)^{n(t-1)} \times$$

$$\sum_{y=0}^{n(t-1)} \theta'^{y} (1-\pi-\theta'/2)^{-(a_t+y/2)} (\pi-\theta'/2)^{(a_t-y/2)} \times$$

$$\sum_{x=y}^{\frac{n(t-1)}{2}+y} \left(\frac{2\theta}{1-\theta}\right)^{(x-y)} g_{x,y}(\underline{a}, n) \qquad (3.24)$$

$$\text{where } g_{x,y}(\underline{a}, n) = \sum_{R_1m} \prod_{r>s}^{t} \frac{n!}{A_{rs}!\, A_{sr}!\, A_{0.rs}!}$$

R_1m is the restriction on A_{rs}, A_{sr} and $A_{0.rs}$ which is same as in Rm together with $\sum_{r>s}^{t} A_{0.rs} = x$, and $\sum_{j=1}^{t-1} A_{0.tj} = y$. If we assume that there are in all x ties and the treatment T_t has got exactly y ties, then $g_{x,y}(\underline{a},n)$ is the number of ways the outcome $(a_1, a_2 \ldots \ldots a_t)$ can be realized.

Let $G_{x,y}(\underline{a},n)$ be a partition function giving the number of permissible partitions of ½ nt(t-1) into t scores $a_1, a_2, \ldots\ldots a_t$ irrespective of order, such that there are, in all, x ties out of which T_t has got exactly y ties. $G_{x,y}(\underline{a},n)$ is related to $g_{x,y}(\underline{a},n)$ by the relation (2.26) as in chapter 2. Using this relationship and following arguments similar to the one which leads to (2.25), we can obtain the required probability of correct selection as

$$P_1(\pi,\theta,\theta') = \frac{1}{t}\left(\frac{1-\theta}{2}\right)^{\frac{n(t-1)}{2}} \sum_{2a_t=[n(t-1)]+1}^{2n(t-1)} \sum_{y=0}^{n(t-1)} (1-\pi-\theta'/2)^{n(t-1)-a_t-y/2} \times$$

$$\theta'^{y} (\pi-\theta'/2)^{a_t-y/2} \sum_{x=y}^{\frac{n(t-1)}{2}+y} \left(\frac{2\theta}{1-\theta}\right)^{(x-y)} \sum_{\substack{a_1, a_2, \ldots\ldots a_{t-1} \leq a_t \\ \sum_{i=1}^{t-1} a_i = n\binom{t}{2} - a_t}} G_{x,y}(\underline{a};n) \qquad (3.25)$$

where $[n(t-1)]$ is the greatest integer less than n(t-1). The $G_{x,y}(\underline{a},n)$ has been tabulated for t=3, n=1, 2 in table 6.

<u>Example:</u> If t=3, n=1 (3.25) reduces to

$$P_1(\pi,\theta,\theta') = 1/3 \left(\frac{1-\theta}{2}\right) \sum_{2a_3=2}^{4} \sum_{y=0}^{2} (1-\pi-\theta'/2)^{(2-a_3-y/2)}\, \theta'^{y} \times$$

$$(\pi-\theta'/2)^{a_3-y/2} \sum_{x=y}^{1+y} \left(\frac{2\theta}{1-\theta}\right)^{(x-y)} \sum_{\substack{a_1,\ a_2 \leq a_3 \\ a_1+a_2=3-a_3}} G_{x,y}(a_1,a_2,a_3;\ 1) \tag{3.26}$$

Using table 6, for t=3, n=1 (3.26) simplifies to

$$P_1(\pi,\theta,\theta') = \left(\pi-\frac{\theta'}{2}\right)^2 + 1/3\left(\pi-\frac{\theta'}{2}\right)(1-\pi-\theta'/2)(1-\theta)$$

$$+ \frac{\theta\theta'^2}{3} + (\pi-\theta'/2)(3+\theta)\,\theta'/2$$

when no ties are permissible $\theta' = \theta = 0$ and then

$P_1(\pi,\theta,\theta') = \pi^2 + 1/3\,\pi(1-\pi)$, which is same as that given by Tranwinski and David (1963).

We are unable to compute $P_1(\pi,\theta,\theta')$ for larger t and n for lack of knowledge of $G_{x,y}(\underline{a},n)$.

However, in the following we shall develop the asymptotic expression of P_1 for large n and t, which seems to compare favourably even for smaller n.

Asymptotic approximation

Following Tranwinski and David (1963), as $n\to\infty$ the probability of correct selection $P_1(\pi,\theta,\theta')$ is asymptotically given by

$$P_{1A}(\pi,\theta,\theta') = \lim_{n\to\infty} P_r\left(a_t \geq \max_{1\leq i\leq t-1} a_i\right)$$

$$= \lim_{n\to\infty} P_r\{d_i \leq 0,\ i=1,2,\ldots..t-1\} \tag{3.27}$$

It is clear that the (t-1) differences $d_i = a_i - a_t$ in $C_\pi^{t*}(\theta,\theta')$ are equicorrelated, identically distributed variates. By (3.18) - (3.20), their means, variances and covariances are

$$\alpha = -nt(\pi-1/2)$$

$$\sigma_d^2 = n\ [(t+2)\pi(1-\pi)+t(1-\theta)/4+(\theta-\theta')/2-(1+\tfrac{t\theta}{2})/2] \qquad (3.28)$$

$$\rho\sigma_d^2 = n[(t+1)\pi(1-\pi)-\theta'(t+1)/4-(1-\theta)/4]$$

Using their arguments we get

$$P_{1A}(\pi,\ \theta,\ \theta') = \int_{-\infty}^{+\infty} [\phi(U_t)]^{t-1}\phi(u_t)\ du_t \qquad (3.29)$$

where $U_t = [\rho/(1-\rho)]^{1/2} \times u_t - \alpha/[(1-\rho)^{1/2}\sigma_d]$.

In table 7 we have presented the values of P_{1A} given by (3.29) for some given set of values of t, n, π, θ and θ'. For numerical comparison of number of replication needed for attaining the same probability of correct selection for fixed π, and for some given set of values of t, θ and θ', we have prepared table 8.

Comments on table 7

For an experiment involving t treatments, the table gives the smallest number of replications n satisfying (3.23) for various specified values of π, θ, θ' and P*. In the construction of the table exact expression (3.25) was used for various combinations of (t,n) upto (3, 2) and asymptotic approximation (3.29) elsewhere It is clear from this table and table 8 that as θ or θ' increases n decreases. And also for fixed P*, π, θ and θ', 'n' decreases as t increases.

Charts V and VI correspond to charts 1A and 1B of Trawinski and David (1963) and are exactly in character i.e. the curves for increasing values of t cross the curves corresponding to preceding values of t. Thus, using their arguments it follows that single outlier configuration in $C_\pi^{t*}(\theta,\ \theta')$ is not the least favourable configuration.

3.5 SELECTION OF A SUBSET CONTAINING THE BEST TREATMENT

We shall adopt the following selection rule R_4 based on scores

a_i defined in (3.12) for selecting a subset s which contains the treatment corresponding to highest k-row sum.

Rule R_4: Retain treatment T_i ($i=1, 2, \ldots t$) in the subset s if its score $a_i \geq \max_{1 \leq i \leq t} a_i - \mathcal{J}$, where $\mathcal{J} \geq 0$ is the selection constant.

Note that the same rule was used by us in the previous chapter for SP design. This was also used by Trawinski and David (1963) for FP design with no ties permitted.

The constant $\mathcal{J}$ for a given n and t is the smallest positive number chosen so that for a given P** Inf $P_{CS} \geq P^{**}$ (3.30)
where Inf P_{CS} is the infimum of the probability of correct selection in $C^t(\pi_{i.ij})$.

Assuming, without loss of generality, that T_t is the best treatment and using Rule R_4, we have for any configuration in $C^t(\pi_{i.ij})$ and a chosen value of $\mathcal{J}$, the probability of correct selection

$$P_{CS} = P\left(a_t \geq \max_{1 \leq i \leq t-1} a_i - \mathcal{J}\right)$$

$$= P\left(a_t \geq a_i - \mathcal{J},\ i=1, 2, \ldots t-1\right) \qquad (3.31)$$

Now we will show that P_{CS} will be infimum at a parametric configuration belonging to $C^t(\theta)$ given in (3.5).

<u>Least favourable configuration for the probability of correct selection</u>

Without loss of generality we will assume for convenience that

$$s_1 \leq s_2 \leq \ldots\ldots\ldots \leq s_{t-1} \leq s_t \qquad (3.32)$$

We will show that P_{CS} as defined in (3.31) is minimum for $C^t(\theta)$, i.e. when all s_i are equal.

Under the assumption that linear model is valid, P_{CS} clearly increases with S_t for fixed S_i, $i=1,2,\ldots\ldots t-1$, so that S_t must be taken as small as is compatible with (3.31) viz. $S_t = S_{t-1}$. Using this fact next we show that the probability that $a_{t-1} \geq a_t + \vartheta$ is greatest when all S_i are equal. For this first note that in case $S_t = S_{t-1}$, we have

$$\pi_{t.tj} = \pi_{t-1.t-1.j} = \pi_j \quad \text{(say)}$$

$$\pi_{0.tj} = \pi_{0.t-1,j} = \theta_j \quad \text{(say)}, \; j=1,2,\ldots .t-2$$

Let $\quad \pi_{0.t,t-1} = \theta$

For some simplification we can write

$$P_r(a_{t-1} - a_t > \vartheta) = \sum_R \left(\frac{1-\theta}{2}\right)^{(n-A_{0.t,t-1})} \theta^{A_{0.t,t-1}} \times$$

$$\frac{n!}{A_{t,t-1}!\, A_{0.t,t-1}!\,(n-A_{t,t-1}-A_{0.t,t-1})!} \prod_{j=1}^{t-2}\Big\{\frac{n!}{A_{tj}!\,A_{0.tj}!\,(n-A_{tj}-A_{0.tj})!}$$

$$\frac{n!}{A_{t-1,j}!\, A_{0.t-1,j}!\, (n-A_{t-1,j}-A_{0.tj})!} \times$$

$$\pi_j^{(A_{tj} + A_{t-1,j})} (1-\theta_j - \pi_j)^{(2n-A_{tj}-A_{t-1,j}-A_{0.tj}-A_{0.t-1,j})} \times$$

$$\theta_j^{(A_{0.tj} + A_{0.t-1,j})}\Big\} \tag{3.33}$$

where $\sum_R$ extends overall permissible values of A_s' which give

$$a_{t-1} - a_t > \vartheta$$

Here a_t and a_{t-1} are as defined in (3.12).

In particular for t=3

$$P_r\{a_2 - a_3 > \vartheta\} = \sum_R \left(\frac{1-\theta}{2}\right)^{(n-A_{0.32})} \theta^{A_{0.32}} \times$$

$$\frac{n!}{A_{21}!\, A_{0.21}!\, (n-A_{21}-A_{0.21})!} \times$$

$$\frac{n!}{A_{32}!\ A_{0.32}!\ (n-A_{32}-A_{0.32})!}\ \frac{n!}{(n-A_{31}-A_{0.31})!\ A_{31}!\ A_{0.31}!} \times$$

$$\pi_1^{(A_{31}+A_{21})}\ (1-\theta_1-\pi_1)^{(2n-A_{31}-A_{21}-A_{0.21}-A_{0.31})} \times$$

$$\theta_1^{(A_{0.21}+A_{0.31})} \qquad (3.34)$$

set $A_{0.21}+A_{0.31}=x$, then $a_2-a_3>\vartheta$ is equivalent to

$$A_{21}-A_{31}-A_{0.13}>\vartheta+A_{32}-A_{23}-x/2$$

clearly (3.34) can be rewritten as

$$P_r\{a_2-a_3>\vartheta\}=\sum_{x=0}^{2n}\theta_1^{x}\sum_{0\leq A_{23}+A_{32}\leq n}\binom{n}{A_{32}}\left(\frac{1-\theta}{2}\right)^{A_{32}}\left(\frac{1+\theta}{2}\right)^{(n-A_{32})}\times$$

$$\sum_{R'}\frac{n!}{A_{31}!\ A_{0.31}!\ (n-A_{31}-A_{0.31})!}\times$$

$$\frac{n!\qquad \pi_1^{(A_{31}+A_{21})}\ (1-\theta_1-\pi_1)^{(2n-A_{31}-A_{21}-x)}}{A_{21}!\ (x-A_{0.31})!\ (n-A_{21}-x+A_{0.31})!}$$

where $\sum_{R'}$ is sum over all possible values A_{21}, A_{31}, $A_{0.13}$ such that $A_{21}-A_{31}-A_{0.13}>\vartheta+A_{32}-A_{23}-x/2$, for fixed values A_{32}, A_{23} and x.

Now consider the terms in $\sum_{R'}$. Set $A_{21}-A_{31}-A_{0.31}=C$ where $C>\vartheta+A_{32}-A_{23}-x/2$. A pair of values of A_{21}, A_{31} and $A_{0.31}$ satisfying this equation can be written as below

$$A_{0.13}=j$$
$$A_{31}=i$$
$$A_{21}=C+i+j \qquad \text{and}$$
$$A_{0.13}=x-j$$
$$A_{31}=n-i-x-c$$
$$A_{21}=n-j-i$$

with $(x - j) \geq 0$, $n - i - x - c \geq 0$ and $n - j - i \geq 0$. The terms in $\sum_{R'}$ can be paired off resulting in a typical term of the type below.

$$\binom{n}{i}\binom{n-i}{j}\binom{n}{c+i+j}\binom{n-c-i-j}{x-j}\Big[\pi_1^{(2i+j+c)} \times$$

$$(1-\theta_1-\pi_1)^{(2n-2i-j-x-c)} + (1-\theta_1-\pi_1)^{(2i+j+c)} \times$$

$$\pi_1^{(2n-2i-j-x-c)}\Big]$$

which can be easily seen to be a maximum for $\pi_1 = \frac{1-\theta_1}{2}$.

If $A_{0.31} = n-2i-\frac{x}{2}-c$

$A_{31} = i$

$A_{21} = n - \frac{x}{2} - i$ such that $n-2i-x/2-c \geq 0$, $n-x/2-i \geq 0$, only one term arises. This is proportional to

$$\pi_1^{n-x/2}\,(1-\theta_1-\pi_1)^{n-x/2}$$ which also is a maximum for $\pi_1 = \frac{1-\theta_1}{2}$. Note that such type of terms occurs only when x is even. Thus for t=3 $P_r\{a_2 - a_3 > \vartheta\}$ is greatest when

$\pi_1 = \frac{1-\theta_1}{2}$.

For t>3 refer to equation (3.33) and to start with fix A_{tk}, $A_{0.tk}$, $A_{t-1,k}$, $A_{0.t-1,k}$ for k=2, 3, ...t-2. Then

$$P_r\{a_{t-1} - a_t > \vartheta\} =$$

$$\sum \prod_{k=2}^{t-2} \Big[\frac{n!}{A_{tk}!\,(n-A_{tk}-A_{0.tk})!\,A_{0.tk}!} \times$$

$$\frac{n!}{A_{t-1,k}!\,(n-A_{t-1,k}-A_{0.t-1,k})!\,A_{0.t-1,k}!} \times$$

$$\pi_k^{(A_{tk}+A_{t-1,k})}\ \theta_k^{(A_{0.tk}+A_{0.t-1,k})} \quad x$$

$$(1-\theta_k-\pi_k)^{(2n-A_{tk}-A_{t-1,k}-A_{0.tk}-A_{0.t-1,k})}\Big] \quad x$$

$$\sum^{f} \left(\frac{1-\theta}{2}\right)^{(n-A_{0.t,t-1})} \frac{\theta^{A_{0.t,t-1}}\, n!}{A_{t,t-1}!\ A_{0.t,t-1}!\ (n-A_{t,t-1}-A_{0.t,t-1})!} \quad x$$

$$\frac{n!}{A_{t1}!\ A_{0.t1}!\ (n-A_{t1}-A_{0.t1})!} \ x \ \frac{n!}{A_{t-1,1}!\ A_{0.t-1,1}!\ (n-A_{t-1,1}-A_{0.t-1,1})!}$$

$$\pi_1^{(A_{t1}+A_{t-1,1})}\ \theta_1^{(A_{0,t1}+A_{0.t-1,1})} \quad x \quad (1-\theta_1-\pi_1)^{(2n-A_{t1}-A_{t-1,1}-A_{0.t1}-A_{0.t-1,1})} \qquad (3.35)$$

where the summation $\sum$ now extends over all permissible values of $A_{t,t-1}$, $A_{0.t,t-1}$, $A_{t\,1}$, $A_{0.t\,1}$, $A_{t-1,1}$, $A_{0.t-1,1}$ which give $a_{t-1}-a_t > \vartheta$, while $\sum^{f}$ is summation over all fixed A's. As for the case of t=3, this sum $\sum$ is maximum for $\pi_1 = \frac{1-\theta_1}{2}$, a result which holds for all permissible choices of A_{tk}, $A_{t-1,k}$, $A_{0.tk}$ and $A_{0.t-1,k}$. The argument can be repeated with A_{t2}, $A_{t-1,2}$, $A_{0.t2}$, $A_{0.t-1,2}$, π_2 replacing A_{t1}, $A_{t-1,1}$, $A_{0.t1}$, $A_{0.t-1,1}$, π_1 respectively to show that to maximise (3.35) we require also $\pi_2 = \frac{1-\theta_2}{2}$ and and in fact $\pi_3 = \frac{1-\theta_3}{2}$, ... $\pi_{t-2} = \frac{1-\theta_{t-2}}{2}$.

The chance that one or more of the first (t-2) treatments has a score exceeding $a_t+\vartheta$ is improved by making the merits of these treatments as large as possible, that is, by making $S_1 = S_2 = \quad = S_{t-1} = S_t$.

Also in case all treatments are equally good, the observed worth Y_i of the treatments would be identically and independently distributed and from the defination (1.2), tie probability $\pi_{0.ij}$ would not depend on the pair T_i, T_j. In that case $\theta_i \to \theta$ for all i. Thus the LFC for probabilities would become

$$\pi_1 = \pi_2 = \ldots\ldots = \pi_t = \frac{1-\theta}{2}$$

which is a configuration in $c^t(\theta)$ defined by (3.5). This complets proof.

Hence for fixed t, n and ϑ, the configuration $c^t(\theta)$ leads to the infimum of P_{cs} for any $c^t(\pi_{i.ij})$ satisfying a linear model. The probability requirement (3.30) is equivalent to

$$P_{cs}\{c^t(\theta)\} \geq P^{**} \tag{3.36}$$

because of above discussion. Now to determine ϑ so that (3.36) is satisfied, we require an exact formula for $P_{cs}\{c^t(\theta)\}$ which is done below.

Exact evaluation of ϑ

A convenient expression of $P_{cs}\{c^t(\theta)\}$ may be obtained in terms of the partition function $G_x(\underline{a};n)$ defined in (3.15). Let x be the total of ties, then

$$P_{cs}\{c^t(\theta)\} = P(a_t \geq a_{max} - \vartheta / c^t(\theta))$$

$$= \sum_{x=0}^{nt(t-1)/2} P\{a_t \geq a_{max} - \vartheta / x,\ c^t(\theta)\}\ P(X=x)$$

$$= \sum_{x=0}^{nt(t-1)/2} \sum_{\underline{a}} P(a_t \geq a_{max} - \vartheta / x,\ \underline{a},\ c^t(\theta)) \quad \star$$

$$P(\underline{a},\ x,\ c^t(\theta))$$

$$= \left(\frac{1-\theta}{2}\right)^{nt(t-1)/2} \sum_{x=0}^{nt(t-1)/2} \left(\frac{2\theta}{1-\theta}\right)^x \quad \times$$

$$[\sum p\{a_t \geq a_{max} - \vartheta/x,\ \underline{a},\ c^t(\theta)\}\ G_x(\underline{a},\ n)]$$

$$= \left(\frac{1-\theta}{2}\right)^{nt(t-1)/2} \sum_{x=0}^{nt(t-1)/2} \left(\frac{2\theta}{1-\theta}\right)^x \sum Q(\underline{a},\ x,\ n,\vartheta)\ \text{(say)}$$

where $P(\underline{a},\ x,\ c^t(\theta))$ is the joint probability function of partition of scores and total no. of ties, where the last summation extends over all the distinct partions $\underline{a}$ of ½ nt(t-1) such that there are x ties in all. Following the argument similar to that of Trawinski and David (1963), we can obtain the quantity $Q(\underline{a},\ x,\ n,\vartheta)$ as $Q(\underline{a},\ x,\ n,\vartheta) = M_x(\underline{a},\vartheta)\ G_x(\underline{a},\ n)/t$, the multiple $M_x(\underline{a},\vartheta)$ being the number of a's in $\underline{a}$, which exceeds or equal $a_{max}-\vartheta$, for a given x. For a given n, t and ϑ, we have

$$P_{cs}\{c^t(\theta)\} = t^{-1}\left(\frac{1-\theta}{2}\right)^{\frac{nt(t-1)}{2}} \sum_{x=0}^{nt(t-1)/2} \left(\frac{2\theta}{1-\theta}\right)^x \times$$

$$\sum M_x(\underline{a},\vartheta)\ G_x(\underline{a},\ n) \qquad (3.37)$$

We illustrate the above procedure for t=3, n=1. In this case there are 7 permissible partitions of which a typical one is o, 1, 2 with x=0 and frequency $G_0(0,1,2;1) = 6$. Also we see that for

$\vartheta = 0$	M=1	corresponding to	$a_t=2$
$\vartheta = ½$	M=1	" "	$a_t=2$
$\vartheta = 1$	M=2	" "	$a_t=2,1$
$\vartheta = 3/2$	M=2	" "	$a_t=2,1$
$\vartheta = 2$	M=3	" "	$a_t=2,1,0$

Thus the contribution to $P_{cs}\{c^t(\theta)\}$ from this partition is, for a given ϑ, $1/3\left(\frac{1-\theta}{2}\right)^3 M_0(\underline{a},\vartheta)\times 6$. If the corresponding contribution for all 7 partitions are added up, the resulting values of $P_{cs}\{c^t(\theta)\}$ are

ϑ	$P_{CS}\ C^t(\theta)$
0	$1/4\,[2-\theta+3\theta^3]$
½	$1/4\,[2+\theta+\theta^3]$
1	$3/4\,(1+\theta^2)-\frac{1}{2}\theta^3$
3/2	$1/4\,[3(1+\theta-\theta^2)+\theta^3]$
2	1

We are unable to compute $P_{CS}\{C^t(\theta)\}$ for larger t and n for lack of availability of $G_X(\underline{a}, n)$. However, in the following we shall develop the asymptotic expression of $P_{CS}\{C^t(\theta)\}$ for large n and t.

Asymptotic approximation to ϑ

The asymptotic probability of correct selection under rule R_4, may be written, with a continuity correction on ϑ as

$$P_{CS_A} = \lim_{n\to\infty} P_r\{a_{max} - a_t < \vartheta + \tfrac{1}{2}\}$$

$$= \lim_{n\to\infty} P_r\{d_i < \vartheta + \tfrac{1}{2},\ i=1,2,\ldots\ldots t-1\}$$

when the preference probabilities are in $C^t(\theta)$, the equations (3.28) are simply $\alpha = 0,\ \sigma_d^2 = nt(1-\theta)/2,\ \rho\sigma_d^2 = nt(1-\theta)/4$ and $\rho \doteq \frac{1}{2}$

$$P_{CS_A}\{C^t(\theta)\} = \lim_{n\to\infty} P_r\{\frac{d_i}{\sigma_d} < (\vartheta+\tfrac{1}{2})/\sigma_d,\ i=1,2,\ldots\ldots t-1)\}$$

$$= \lim_{n\to\infty} P_r\ (v_i < \Delta\ ,\ i=1,2,\ldots\ldots t-1)$$

where $v_i = d_i/\sigma_d$, $\Delta = (\vartheta+\frac{1}{2})/\sigma_d$

Using the same arguments which leads to the equation (3.29), we get

$$P_{CS_A}\{C^t(\theta)\} \doteq \int_{-\infty}^{\infty} [\Phi(U't)^{t-1}\,\phi\,(ut)\ dut \qquad (3.38)$$

where $U_t' = [\rho/(1-\rho)]^{\frac{1}{2}} \times u_t + (\vartheta+\frac{1}{2})/[(1-\rho)^{\frac{1}{2}}\,\sigma_d]$.

Some values of ϑ to implement selection rule R_4 is presented in table 9.

Next we will obtain an asymptotic approximation to the expected size of the selected subset S.

3.6 Asymptotic approximation to the expected size of a selected subset

We shall derive the expression only under slippage configuration belonging to $C_{\pi}^{t*}(\theta, \theta')$ defined in (3.21). The expression for the expected size of the selected subset

$$E_{t,n}(\pi, \vartheta) = \sum_{i=1}^{t} P_r\{ \max_{i \neq j} (a_j - a_i) \leq \vartheta \} \tag{3.39}$$

Let $\underline{d_i}$, the vector random variable with components $d_{ij}=a_j-a_i$, $i \neq j$, $i=1,2,\ldots\ldots t$, have expectation α_i. The asymptotic distribution of $(\underline{d_i} - \underline{\alpha_i})n^{-\frac{1}{2}}$ is $IN(0, \sum_i)$. The form of $\sum_i$ is implied by the discussion below. For $i \neq j \neq k$, $i,j,k, =1,2,\ldots\ldots t-1$, we have

$$\begin{aligned}
\text{Var}\ (a_i - a_j) &= \sigma^2 \\
&= 2n\,[\pi(1-\pi)+(t-1)(1-\theta)/4 - \theta'/4] \\
\text{Cov}\ (a_i-a_j,\ a_i-a_k) &= n[\pi(1-\pi)+(t-1)(1-\theta)/4-\theta'/4] \\
&= 1/2\ \sigma^2 \\
\text{Var}\ (a_i-a_t) &= \sigma_t^2 \\
&= n[(t+2)\pi(1-\pi)+t(1-\theta)/4+(\theta-\theta')/2-(\tfrac{1+t\theta'}{2})/2] \\
\text{Cov}\ (a_i-a_j,\ a_i-a_t) &= n[\pi(1-\pi)+(t-1)(1-\theta)/4 - \theta'/4] \\
&= 1/2\ \sigma^2 \\
\text{Cov}\ (a_i-a_t,\ a_j-a_t) &= n[(t+1)\pi(1-\pi)-(t+1)\theta'/4 - (1-\theta)/4] \\
&= C_t .
\end{aligned}$$

Following Trawinski (1969), we can obtain the asymptotic expected size of the selected subset

$$E_{t;1(n)}(\pi,\vartheta)=\int_{-\infty}^{\infty}[(t-1)\{\phi(w_1)\}^{(t-2)}\phi(w_2)+\{\phi(w)\}^{(t-1)}]d\,\phi\,(z). \quad (3.40)$$

where $w_1 = z + 2^{1/2}\Delta$

$w_2 = 2^{1/2}(\rho/\rho_t^{1/2}) + \Delta_{t-1}/\rho_t^{1/2}$

$w = \{\rho_t/(1-\rho_t)\}^{1/2} z + \Delta_t/(1-\rho_t)^{1/2}$

$\Delta = (\vartheta+1/2)/\sigma$, $\Delta_{t-1} = (\vartheta+1/2-\alpha)/\sigma_t$

$\Delta_t = (\vartheta+1/2+\alpha)/\sigma_t$, $\alpha = nt(\pi-1/2)$

$\rho = 1/2\sigma/\sigma_t$, $\rho_t = c_t/\sigma_t^2$

The accuracy of this approximation to $E_{t,n}(\pi,\vartheta)$ have been illustrated by Trawinski (1969), when there are no ties and when $\pi=1$.

3.7 An Example

We illustrate the application of the decision rules, in this chapter, on the following data. The data is from an experiment on taste testing conducted at IARI, New Delhi, on four varieties of drum stick fruites (Moringa-Oleifera), using FP design with ties. Three judges were selected for taste acquity. Each judge was presented with six pairs. The pairs were randomised and allocated to the judges in a random order with a view to cancel out the order effects as far as possible. The experiment was repeated twice. The data for preferences by the judges pooled over repetitions are given below.

Pair	$A_{i.ij}$	$A_{0.ij}$	$A_{j.ij}$	Total
(1, 2)	4	0	2	6
(1, 3)	4	1	1	6
(1, 4)	2	0	4	6
(2, 3)	5	0	1	6
(2, 4)	1	2	3	6
(3, 4)	0	0	6	6

Variety	Score	No. of ties
1	10½	1
2	9	2
3	2½	1
4	14	2

Using the above data for the selection of the best variety, we find by selection rule R_3, variety 4 can be treated as the best variety with atleast a probability 0.99 setting π =.75. (Using table VII, for θ'=.1, θ=.15, t=4, n=6). We have a_{max}= 14. To ensure that with at least a preassigned probability P*=.90, the best variety is in the selected subset, we enter table IX for t=4, n=6, P*=.9, θ=.15, find η= 5.5 and hence retain varieties 1,2 and 4 in the selected subset.

CHAPTER IV

SELECTION OF BEST TREATMENT USING SYMMETRICAL PAIRED COMPARISON DESIGN IN PRESENCE OF TIES

4.1 Introduction

In this chapter, selection procedures have been developed under indifference zone as well as subset selection formulation for Goal 1 as defined in section 5 of chapter I, based on the scores making allowence for ties, obtained by using SP design with equal number 'n' of replications. No judge effect, no replication effect and no order effect is assumed in the analysis.

The methods developed in this chapter are the generalisation of those in chapter II, and can be applied to the type of problems cited in that chapter. The major difference here is that the methods in the present chapter also incorporate tie effect. In view of this, in this chapter also, it will be assumed that 't' treatments to be compared have a single outlier, while the other treatments differ only slightly. The Goal is thus the selection of outlier. Because of the special nature of the problem considered we shall assume a simplified preference probability structure in line with those assumed in chapter II.

4.2 A mathematical model for preference probabilities

For the purpose of defining preference probabilities as in section (2.2) it is convenient to assume here that the 't' treatment T_1, T_2, ...T_t are such that $T_1 \leq T_2 \leq T_3 \leq \ldots\ldots \leq T_t$ and T_t is the outlier. Further in the presence of ties, the preference probabilities have the following structure.

$$\pi_{t.tj} = p+(t-j)\,\epsilon', \; j=1,2,\ldots\ldots t-1$$

$$\pi_{0.tj} = j\,\theta' \qquad (4.1)$$

$$\pi_{t.tj} + \pi_{j.tj} + \pi_{0.tj} = 1$$

$$\pi_{i.ij} = \tfrac{1}{2} + (i-j)\,\epsilon, \; i>j$$

$$\pi_{0.ij} = (t-1+i-j)\,\theta, \quad i,j = 1,2,\ldots\ldots t-1$$

$$\pi_{i.ij} + \pi_{j.ij} + \pi_{0.ij} = 1$$

where p, ϵ, ϵ', θ, θ' are the parameters of the model. θ and θ' are the tie parameters. Parameters ϵ' and ϵ are the same as defined in section (2.2). The parameter θ' gives the tie probability when outlier is compared with the worst treatment i.e. $\pi_{0.t1} = \theta'$. Similarly parameters θ gives the tie probability when the worst treatment is compared with the best treatment (apart from the outlier) i.e. $\pi_{0.1,t-1} = \theta$.

As discussed in section (3.2) the parameter(K_{ij}, i>j, i,j= 1,2,.......t) defined in (3.36) would be sufficient for providing mutual comparisons among the treatments and a feasible ranking of the treatments can be provided in terms of K-row sum. Since we are using SP design, the K-row sum per replication corresponding to treatment T_i under this design would modify to

$$K_i = K_{il} + K_{im}$$

when the treatment T_i is paired with T_l and T_m in the SP design used. The treatment corresponding to the maximum value of K_i would be termed as the best.

Consequently the parameters K_{ij} play exactly the same role in the model with ties permitted as π_{ij} in chapter II when ties were not permitted. (See proof of Theorm III Section (2.3)). In fact, Theorm III of Section (2.3) would hold for K-row sums in case K_{ij} satisfy condition

$$\min_{1 \leq j \leq t-1} K_{tj} > \max_{1 \leq i,j \leq t-1} K_{ij} \tag{4.2}$$

which had to be satisfied by π_{ij}'s earlier in chapter II.

For the model to be meaningful p, ϵ, ϵ', θ, θ' must satisfy the following restrictions.

$$\begin{array}{ll} \text{i)} & p \geq \tfrac{1}{2} \\ \text{ii)} & 0 \leq \epsilon \leq \epsilon' \\ \text{iii)} & 0 \leq \theta' \leq \theta \\ \text{iv)} & 0 \leq \epsilon < \frac{p-\frac{1}{2}}{(t-2)} + \frac{(t-1)\,\theta' - \theta}{2(t-2)} + \epsilon'/(t-2) \\ \text{v)} & \epsilon' \leq (1-p-\theta'/2)/(t-1) \end{array} \tag{4.3}$$

The restrictions (i), (ii) and (iii) in (4.3) are obvious. Restriction (iv) arises because we need to satisfy (4.2) in order that model (4.1) always detects the outlier. The last restriction arises from the fact that $K_{ti} \leq 1$. Through out this chapter we assume that preference probabilities satisfy the model (4.1) with restriction (4.3).

The development of selection procedures for the selection of the treatment with the highest K-row sum, need the distribution of scores of different treatments based on preferences data obtained by SP design. These distributions are discussed in the following section.

4.3 DISTRIBUTION THEORY

The results to be obtained here follow considerations similar to those in Section (2.4) or in Section (3.3) and therefore, some details are omitted. Consider an SP design consisting of 'n' replications of each of the pair $(T_1, T_2), (T_2, T_3) \ldots\ldots (T_{t-1}, T_t)$, (T_t, T_1) involved in one such design. The characteristic variable x_{ijr} can now be defined by

$$x_{ijr} = \begin{cases} 1 & \text{if } T_i > T_j \\ 0 & \text{if } T_j > T_i \\ \tfrac{1}{2} & \text{if } T_i = T_j, \quad i = 1, 2, \ldots\ldots t \\ & \quad j = \left.\begin{matrix} i-1 \text{ or} \\ i+1 \end{matrix}\right\} \bmod t \\ & \quad r = 1, 2, \ldots\ldots n. \end{cases} \tag{4.4}$$

and its distribution is same as (3.8). As usual we assume that there is no replication effect and that all nt comparisons are independent. The score a_i for the treatment T_i is defined in the same manner as in (2.5) i.e.

$$a_i = \sum_{r=1}^{n} a_{ir} = \sum_{r=1}^{n} \sum_{\substack{j=i-1 \\ \text{or} \\ i+1}} x_{ijr} \quad \bmod t \tag{4.5}$$

Clearly $a_i \in (0, \tfrac{1}{2}, 2/2, 3/2, \ldots\ldots \frac{2n-1}{2}, n)$

$$\sum_{i=1}^{t} a_{ir} = t, \qquad \sum_{i=1}^{t} a_i = n$$

If $j = \left.\begin{matrix} i-1 \text{ or} \\ i+1 \end{matrix}\right\} \bmod t$, a_{ir} and a_{jr} are correlated as are a_i and a_j. Other correlations among them are zero.

<u>Exact distribution of scores</u>

Defining A_{rs} and $A_{0.rs}$ as in section (3.3), the joint distribution of $(A_{rs}) = (A_{rs}, r>s, \quad r=1, 2, \ldots\ldots t$, $s = \left.\begin{matrix} r-1 \text{ or} \\ r+1 \end{matrix}\right\} \bmod t$)

Can be written as

$$f(\{A_{rs}\}) = \prod_{\substack{r>s \\ s=r-1 \text{ or } \\ r+1}}^{t} {}_{\text{mod } t} P(A_{rs}, A_{sr}, \pi_{r.rs}, \pi_{s.rs}) \tag{4.6}$$

where $P(A_{rs}, A_{sr}, \pi_{r.rs}, \pi_{s.rs})$ is given in (3.10). Now the scores for the t treatments may be expressed as

$$a_t = n + \tfrac{1}{2}(A_{t1} - A_{1t}) + \tfrac{1}{2}(A_{t,t-1} - A_{t-1,t})$$

$$a_i = n + \tfrac{1}{2}(A_{i,i+1} - A_{i+1,i}) + \tfrac{1}{2}(A_{i,i-1} - A_{i-1,i}), \quad i=2,3,\ldots\ldots t-1$$

$$a_1 = n + \tfrac{1}{2}(A_{12} - A_{21}) + \tfrac{1}{2}(A_{1t} - A_{t1}) \tag{4.7}$$

The joint distribution of scores $\underline{a} = (a_1, a_2, \ldots\ldots a_t)$ using (4.6) is $f[\underline{a}, C_1^t(\pi_{i.ij})] = \sum_{Rs} f(\{A_{rs}\})$ (4.8)

where $\sum_{Rs}$ is the summation over the restriction (4.7) and the parametric configuration belongs to the complete parameter space defind in (3.2).

In particular, if the parametric configuration belong to the set of outlier configuration given in (3.21), the distribution of scores given in (4.8) simplify to

$$f[\underline{a}, C_{1\pi}^t(\theta, \theta')] = \left(\frac{1-\theta}{2}\right)^{(t-2)n} \left(\pi - \frac{\theta'}{2}\right)^{a_t} \times$$

$$\left(1 - \frac{\theta'}{2} - \pi\right)^{(2n-a_t)} \sum_{y'=0}^{2n} \left(\frac{\theta'}{\sqrt{(\pi-\theta'/2)(1-\pi-\theta'/2)}}\right)^{y'} \times$$

$$\sum_{x'=y'}^{(t-2)n+y'} \left(\frac{2\theta}{1-\theta}\right)^{(x'-y')} \sum_{Rs}' \prod_{\substack{r=1 \\ s=r-1 \text{ or } \\ r+1}} {}_{\text{mod } t} \frac{n!}{A_{rs}!\, A_{sr}!\, A_{0.rs}!} \tag{4.9}$$

where $y' = A_{0.t1} + A_{0.t,t-1}$

$$x' = \sum_{\substack{r=1 \\ s=r-1 \text{ or } r+1 \ (\text{mod } t)}}^{t} A_{0.rs}$$

and $\sum_{Rs}'$ is the summation under the restriction Rs for fixed x' and y'.

As stated earlier, we are assuming the model (4.1) for the preference probabilities in this chapter. Under this model, the complete parameter space denoted by C_1^* $(p, \epsilon, \epsilon', \theta, \theta')$ could be defined only in terms of the parameters, $p, \epsilon, \epsilon', \theta, \theta'$ restricted by (4.3), while preference zone would become

$C_{1\pi}^*$ $(p, \epsilon, \epsilon', \theta, \theta') = \{p > \frac{1}{2},\ 0 \leq \epsilon \leq \epsilon',\ 0 \leq \theta' \leq \theta,$

$0 \leq \epsilon \leq \frac{p-\frac{1}{2}}{(t-2)} + \frac{(t-1)\ \theta' - \theta}{2\ (t-2)} + \frac{\epsilon'}{(t-2)},\ 0 \leq \epsilon' \leq \frac{1-p-\theta'/2}{(t-1)}, p+\epsilon \geq \pi - \theta'/2$

Also the slippage configuration defined in (3.21) with reference to $C_{1\pi}^*$ $(p, \epsilon, \epsilon', \theta, \theta')$ can be rewritten as

$$C_1^* \ (\pi, \theta, \theta') = \{ p = \pi - \frac{\theta'}{2},\ \epsilon = 0,\ 0 \leq \theta' \leq \theta \ \epsilon' = 0\} \tag{4.10}$$

Also if all treatments are equally good, the parametric configuration belong to $C_1^t(\theta) = \{ \theta' = \theta,\ p = \frac{1-\theta}{2},\ \epsilon = 0,\ \epsilon' = 0,\ 0 \leq \theta\}$

under equality configuration the joint distribution of the scores simplifies to

$$f\ [\underline{a},\ C_1^t(\theta)\ ] = \left(\frac{1-\theta}{2}\right)^{nt} \sum_{x'=0}^{nt} \left(\frac{2\ \theta}{1-\theta}\right)^{x'} g_{x'}^*(\underline{a}, n)$$

$$\text{where } g_{x'}^*(\underline{a}, n) = \sum_{R's} \prod_{\substack{r=1 \\ s=r-1 \text{ or } r+1 \ (\text{mod } t)}}^{t} \frac{n!}{A_{rs}!\ A_{sr}!\ A_{0.rs}!}$$

and $\sum_{R's}$ is the summation over A_{rs}, A_{sr} subject to restriction in Rs togather with $\sum_{\substack{r=1 \\ s=r-1 \text{ or } r+1 \pmod t}}^{t} A_{0.rs} = x'$.

The function $g^*_{x'}(\underline{a},n)$ gives the number of ways outcome $\underline{a}$ can be realized, given that there are x' ties among nt comparisons. From $g^*_{x'}(\underline{a}, n)$, we can obtain the partition function $G^*_{x'}(\underline{a},n)$ giving the number of permissible partitions of nt into 't' scores $a_1, a_2, \ldots\ldots a_t$ irrespective of order given that there are x' ties and the relation between the two is same as obtained previously in (3.15) chapter III i.e.

$$G^*_{x'}(\underline{a},n) = (t!/\prod_k m_k!)\, g^*_{x'}(\underline{a},n) \tag{4.12}$$

In the following we now derive the asymptotic distribution similar to those in Section (3.3).

<u>Asymptotic distribution theory</u>

Using (3.8), we have

$$E(a_i) = n\, E \sum_{\substack{j=i-1 \text{ or } i+1 \pmod t}} x_{ijr}$$

$$= n \sum_{\substack{j=i-1 \text{ or } i+1 \pmod t}} K_{ij} \qquad = nk_i \tag{4.13}$$

$$\text{Var}(a_i) = n\, \text{Var}(a_{ir})$$

$$= n \sum_{\substack{j=i-1 \text{ or } i+1 \pmod t}} [\pi_{i.ij}(1-\pi_{i.ij}-\pi_{0.ij}) + \tfrac{1}{4}\pi_{0.ij}(1-\pi_{0.ij})]$$

$$\text{Cov}(a_i, a_j) = -n[\pi_{i.ij}(1-\pi_{i.ij}-\pi_{0.ij}) + \tfrac{1}{4}\pi_{0.ij}(1-\pi_{0.ij}),] \text{ if } j=i-1 \text{ or } i+1 \pmod t \tag{4.14}$$

$$= 0 \text{ if } j \neq i$$
$$\neq \begin{matrix} i-1 \text{ or} \\ i+1 \end{matrix} \bigg\} \bmod t$$

The distribution of the vector of difference among scores, defined by $\underline{d} = (d_1, d_2, \ldots \ldots d_{t-1})$, where $d_i = (a_i - a_t), i=1,2,\ldots.t-1$ is of special interest. The variance covariance matrix of $\underline{d}$ would have a similar structure as for the model without ties discussed in section (2.4) A table corresponding to table (2.1) can be easily prepared for the present case.

In case there is a single outlier T_t and parametric configuration satisfy (3.21), it may be noted that two categories of variances and five categories of covariances become equal within each category. In fact in $C^t_{1\pi}(\theta, \theta')$, the means of d_i's are

$$\alpha_i = \begin{cases} \alpha = -2n(\pi - 1/2), & i=2,\ldots..t-2 \\ \alpha' = -3n(\pi - 1/2), & i=1, t-1 \end{cases} \tag{4.15}$$

and their variances are

$$\sigma'_{ii} = \begin{cases} \sigma^2_d = 2n[\pi(1-\pi)-(\theta'+\theta)/4+1/4], & i=2,\ldots.t-2 \\ \sigma^2_{d'} = n[5\pi(1-\pi)-(5\theta'+\theta)/4+1/4], & i=1,t-1 \end{cases} \tag{4.16}$$

As in section (2.4), the vector $\sqrt{n}\,[(d_1/n-\alpha_1)/\sqrt{\sigma_{11}'}, \ldots\ldots (d_{t-1}/n-\alpha_{t-1})/\sqrt{\sigma'_{t-1,t-1}}]$ converges in law to the multivariate random variable with distribution IN $(0, \rho')$. Under outlier configuration $C^t_{1\pi}(\theta, \theta')$, ρ' takes the same form that of ρ in section (2.4) with elements ρ_i replaced by new elements ρ'_i.

We now discuss the selection procedures for the selection of the best treatment, with the help of the distribution of scores, when the data is obtained using SP design discussed earlier. This is done in the next section.

4.4 SELECTION OF THE BEST TREATMENT

Selection rule for best treatment

From (4.13), it follows that the scores a_i are the unbiased estimator of $n\ k_i$. Therefore, we base our selection procedures on these scores and define Rule R_5 in the familiar manner.

Rule R_5 : Obtain $(a_1, a_2, \ldots\ldots a_t)$, the scores of 't' treatments $T_1, T_2, \ldots\ldots T_t$ respectively by using SP design in the presence of ties with n replication per pair and declare the treatment with score $\max_{1 \leq i \leq t} a_i$ as the best treatment. If 'm' scores tie for the first place, then one of the corresponding 'm' treatments is chosen at random for the best treatment.

Assuming that T_t is the best treatment and P_2 is the probability of its selection as the best treatment by rule R_5, as usual we choose the number of replications 'n' so as to satisfy the probability requirement

$$P_2 \geq P^* \tag{4.17}$$

for all configuration in $C_1^* (p, \epsilon, \epsilon', \theta, \theta')$

As in chapter II, we have been able to conjucture by numerical verification that the infimum $P_2 = P_{2\pi}$ (say), in the preference zone $C_{1\pi}^*(p, \epsilon, \epsilon', \theta, \theta')$ occurs at the configuration given by (4.10). This numerical verification was done for 4 treatments in a single replication design. In this case, the possible distinct sets of pairs under SP design are as in (2.21). Assuming $T_1 \leq T_2 \leq T_3 \leq T_4$, the corresponding expressions for the probability of correct selection, when n=1 are, respectively

$$P_{1t} = [\pi_{12}\pi_{23}\pi_{34}\pi_{41} + \pi_{21}\pi_{32}\pi_{43}\pi_{14} + \pi_{0.12}\pi_{0.23}\pi_{0.34}\pi_{0.41}]/4 + \pi_{43}\pi_{41}(1-\pi_{21}\pi_{23}/2) + \pi_{43}\pi_{0.14}[\pi_{21}(\pi_{32}+\pi_{0.32}/2) + \pi_{0.12}(1-\pi_{23}/2) + \pi_{12}/2] + \pi_{41}\pi_{0.43}[\pi_{12}(1-\pi_{32}/2) + \pi_{21}(\pi_{32} + \pi_{0.32})/2 + \pi_{0.12}(1+\pi_{0.32})/2]$$

$$P_{2t} = [\pi_{12}\pi_{31}\pi_{43}\pi_{24} + \pi_{21}\pi_{13}\pi_{34}\pi_{42} + \pi_{0.12}\pi_{0.31}\pi_{0.43}\pi_{0.24}]/4 + \pi_{43}\pi_{42}\{1-\pi_{12}\pi_{13}/2\} + \pi_{43}\pi_{0.24}[\pi_{12}(\pi_{31}+\pi_{0.13}/2) + \pi_{0.13}(1-\pi_{13}/2) + \pi_{21}/2] + \pi_{42}\pi_{0.43}[\pi_{21}(1-\pi_{31}/2) + \pi_{12}(\pi_{31}+\pi_{0.13})/2 + \pi_{0.12}(1+\pi_{0.13})/2]$$

$$P_{3t} = [\pi_{23}\pi_{31}\pi_{14}\pi_{42} + \pi_{32}\pi_{13}\pi_{41}\pi_{24} + \pi_{0.23}\pi_{0.31}\pi_{0.14}\pi_{0.42}]/4 + \pi_{41}\pi_{42}(1-\pi_{32}\pi_{31}/2) + \pi_{41}\pi_{0.24}[\pi_{32}(\pi_{13} + \pi_{0.13}/2) + \pi_{0.32}(1-\pi_{31}/2) + \pi_{23}/2] + \pi_{42}\pi_{0.14}[\pi_{23}(1-\pi_{13}/2) + \pi_{32}(\pi_{13}+\pi_{0.13})/2 + \pi_{0.32}(1+\pi_{0.13})/2]$$

Each of the probability of correct selection as given by P_{1t}, P_{2t} and P_{3t} have been computed for different values of p, ϵ, ϵ', θ and θ' in C^*_1 (p, ϵ, ϵ', θ, θ'), assuming that the preference probabilities satisfy model (4.2). (Tr...) It has been observed that each of P_{1t}, P_{2t}, P_{3t} attains the same infimum and at the same configurations as given by (4.10).

In general $(t-1)!/2$ SP designs involving t treatments have same infimum for probability of correct selection in the preference zone $C^*_{1\pi}$ (p, ϵ, ϵ', θ, θ') and this occurs at the same configuration as given in (4.10).

Probability of correct selection under least favourable configuration

The joint distribution of scores under conjuctured LFC defined in (4.10) was obtained in expression (4.9). Using this and following the arguments which lead to (2.25) starting from (2.4), we can obtain the required probability of correct selection as

$$P_{2\pi} = t^{-1} \left(\frac{1-\theta}{2}\right)^{(t-2)n} \sum_{2a_t=2n}^{4n} (\pi - \theta'/2)^{a_t} (1-\theta'/2-\pi)^{(2n-a_t)} \times$$

$$\sum_{y'=0}^{2n} \left(\frac{\theta'}{\sqrt{(\pi-\theta'/2)(1-\pi-\theta'/2)}}\right)^{y'} \sum_{x'=y'}^{n(t-2)+y'} \left(\frac{2\theta}{1-\theta}\right)^{(x'-y')} \times$$

$$\Sigma\, G^{*}_{x', y'}(\underline{a},n) \qquad (4.18)$$

where the last summation extends over $\sum_{i=1}^{t-1} a_i = nt-a_t$, $a_1, a_2, \ldots$ $\ldots a_{t-1} \leq a_t$ and $G^{*}_{x',y'}(\underline{a}, n)$ is the number of permissible partition of nt into 't' scores $a_1, a_2, \ldots\ldots a_t$ irrespective of order, given that there are x' ties in all and exactly y' ties for treatment with the top score. The $G^{*}_{x',y'}(\underline{a}, n)$ has been tabulated for t=3, 4 and n=1 in table 10.

Example: If t=3, n=1, then (4.18) recudes to

$$P_{2\pi} = 3^{-1} \left(\frac{1-\theta}{2}\right) \sum_{2a_t=2}^{4} (\pi-\theta'/2)^{a_t} (1-\pi-\theta'/2)^{(2-a_t)} \sum_{y'=0}^{2} \times \left(\frac{\theta'}{\sqrt{(\pi-\theta'/2)(1-\pi-\theta'/2)}}\right)^{y'} \sum_{x'=y'}^{1+y'} \left(\frac{2\theta}{1-\theta}\right)^{x'-y'} \Sigma G^{*}_{x',y'}(\underline{a},n) \quad (4.19)$$

using table 10, for t=3, n=1 (4.19) simplifies to

$$P_{2\pi} = 3^{-1} (\pi-\theta'/2)(1-\theta)(1-\pi-\theta'/2) - \theta\theta'^2/3 + 3\left(\frac{1-\theta}{2}\right)(\pi-\theta'/2)\,\theta'$$
$$+2\theta\,(\pi-\theta'/2)^{3/2}(1-\pi-\theta'/2)^{1/2} + (1-\theta)(\pi-\theta'/2)^2 + \theta\,(\pi-\theta'/2)^2$$

when there are no ties i.e. when $\theta=0$, $\theta'=0$, $P_{2\pi}$ reduces to $1/3\ \pi \times (1-\pi) + \pi^2$, which is the same as symmetrical pairs without ties, when t=3, n=1.

We are unable to compute $P_{2\pi}$ for $t \geq 5$ and $n \geq 2$ for lack of knowledge about $G^{*}_{x',y'}(\underline{a};n)$. However, in the following we shall develop the asymptotic expression of $P_{2\pi}$ for large n and t.

Asymptotic approximation to the infimum of probability of correct selection

Using the agruments which leads to the bound (2.32) in chapter 2, we can obtain a similar bound for the asymptotic expression $P_{2\pi A}$ for the probability of correct selection under rule R_5 as given below:

$$P_{2\pi A} \geq [2\phi(\Delta)-1]^{t-3}[2\phi(\Delta')-1]^2 \qquad (4.20)$$

with α, α', σd and $\sigma d'$ as obtained in (4.15) and (4.16).

The lower bound for $P_{2\pi A}$ given in (4.20) have been computed for given t, n, π, θ, and θ'. Table 11 gives an upper bound to the smallest number of replications 'n', which ensures that the highest score in an experiment of size (t, n) will corresponds to the best treatment with at least a pre-assigned probability p*, for specified values of π, θ and θ'.

Comparisons of FP and SP designs with ties

Table 7 and Table 11 are used to construct table 12, which indicates minimum reduction in the size of the experiment for SP design in comparison to FP design (both with ties) for large t, large π, small θ and θ' for various values of P*. The computations utilise bound (4.20) in case of SP design, while the equation (3.28) for FP design for fixing P* value. In fact the computations have shown that SP design requires much smaller experiment size if either t is large or π is close to 1, while θ and θ' are small.

Table 13 gives the number of replications needed for attaining the same probability of correct selection for fixed π and for some given set of values of t, θ and θ'. It is evident from this table, that for a fixed t, as θ or θ' increase, n decreases.

In the next section we proceed to develop a subset selection rule for the selection of the best treatment.

4.5 SELECTION OF A SUBSET CONTAINING THE BEST TREATMENT

We shall adopt the following seledtion rule R_6 based on scores a_i defined in (4.5), for selecting a subset s which contains the treatment corresponding to highest k-row sum as defined in (3.6).

Rule R_6: Retain treatment T_i (i=1, 2,.....t) in the subset s if its score $a_i \geq \max_{1 \leq i \leq t} a_i - \vartheta$, where $\vartheta \geq 0$ is the selection constant.

Note that the same rule was used by us in the previous chapters. As usual the constant ϑ is the smallest positive number so that for a given p**.

$$\text{Inf } P_{ts} \geq p^{**} \tag{4.21}$$

where infimum is taken over all the parametric configuration in C_1^* (p, ϵ, ϵ', θ, θ') given by (4.3).

Proceeding in the same manner as for rule R_2 in chapter 2, we conjucture that the infimum of probability of correct selection under rule R_6 in parametric space C_1^* (p, ϵ, ϵ', θ, θ') is attained at the configuration C_1^t (θ), i.e. at the configuration implying that all treatments are equally good. The detailed expression for P_{ts} were written for 3 and 4 treatments under all possible sets of symmetrical pairs given by (2.21). Here we give a typical expression for the case when t=4, n=1 and SP design including pairs given by (ii) of (2.21) is used. This expression for $\vartheta=\frac{1}{2}$ is

$$P_{ts} = [\pi_{21}\pi_{13}\pi_{34}\pi_{42} + \pi_{12}\pi_{31}\pi_{43}\pi_{24} + \pi_{0.12}\pi_{0.13}\pi_{0.34}\pi_{0.42}]/4 + \pi_{43}\pi_{42}(1-\pi_{12}\pi_{13}/2) + \pi_{43}\pi_{0.42}[1-(\pi_{12}\pi_{0.13} + \pi_{21}+\pi_{13}\pi_{0.12})/2] + \pi_{42}\pi_{0.43}[1+\pi_{12}\pi_{13}+ \pi_{21}-\pi_{21}\pi_{31}+\pi_{0.12}\pi_{0.13}]/2 .$$

Numerical computations were done for all expressions of the above type for various values of p, ϵ, ϵ', θ, θ' in C_1^* $(p, \epsilon, \epsilon', \theta, \theta')$ assuming model (4.2) holds. These computations suggested that all $(t-1)!/2$ SP designs involving t treatments have same infimum for P_{ts} in the preference zone $C_{1\pi}^*$ $(p, \epsilon, \epsilon', \theta, \theta')$ which is attained at the configuration C_1^t (θ). Thus, the requirement (4.22) is equivalent to choosing least ϑ such that

$$P_{ts}\ (\theta) \geq P^{**} \tag{4.22}$$

where $P_{ts}(\theta)$ is the value of P_{ts} at the least favourable configuration C_1^t (θ). The expression for $P_{ts}(\theta)$ can be simplified as follows.

<u>Exact evaluation of ϑ</u>

Proceeding as in R_4 which lead to expression (3.36), we can obtain a convenient expression of $P_{ts}(\theta)$ in terms of the partition function $G^*_{x'}$ $(\underline{a}, n)$ defined in (4.12). Let x' be the total of ties, then we can write

$$P_{ts}(\theta) = \left(\frac{1-\theta}{2}\right)^{nt} \sum_{x'=o}^{nt} \left(\frac{2\ \theta}{1-\theta}\right)^{x'} \sum Q'(\underline{a}, x', n, \vartheta) \quad \text{(say)}$$

where the last summation extends over all distinct partitions of $\underline{a}$ of nt such that there are x' ties in all.

Further more Q' $(\underline{a}, x', n, \vartheta) = M'_{x'}$ $(\underline{a}, \vartheta)$ $G^*_{x'}(\underline{a}, n)/t$ where the multiple $M'_{x'}$ is the number of a's in $\underline{a}$ which exceeds or equal to $a_{max} - \vartheta$ for a given x'. We have finally, for a given n, t, ϑ and x'.

$$P_{ts}(\theta) = t^{-1} \left(\frac{1-\theta}{2}\right)^{nt} \sum_{x'=0}^{nt} \left(\frac{2\theta}{1-\theta}\right)^{x'} \sum M'_x (\underline{a}, \vartheta)\, G^*_{x'}(\underline{a}, n) \quad (2.23)$$

For the sake of illustration let t=4, n=1. In this case, there are 13 permissible partitions of which a typical one is (2, 1, ½, ½) with x'=1 and frequency $G^*_1(\underline{a}, n)=8$. Also we see that for

$\vartheta = 0$	M'=1	Subset includes treatments with score 2
$\vartheta = ½$	M'=1	" 2
$\vartheta = 1$	M'=2	" 2, 1
$\vartheta = 3/2$	M'=4	" 2, 1, ½, ½

Thusthe contribution to $P_{ts}(\theta)$ from this partition for a given ϑ is $1/4 \left(\frac{1-\theta}{2}\right)^4 \left(\frac{2\theta}{1-\theta}\right) M'(\underline{a}, \vartheta) \times 8$. If the corresponding contributions for all 13 partitions are added up, the resulting values of $P_{ts}(\theta)$ are

ϑ	$P_{ts}(\theta)$
0	$(3-2\theta+6\theta^2-10\theta+11\theta^4)/8$
½	$(3+4\theta-2\theta^2+4\theta^3-\theta^4)/8$
1	$(3-\theta+6\theta^2-5\theta^3+\theta^4)/4$
3/2	$(3+3\theta-7\theta^2+9\theta^3-4\theta^4)/4$
2	1

In the following we shall develop the asymptotic expression of $P_{ts}(\theta)$ for large n and t.

4.6 Asymptotic approximation to ϑ

When the preference probabilities are in $C_1^t(\theta)$, the equations (4.16) are simply

$$\sigma_d^2 = n\,(1-\theta-\theta^2/2)$$

$$\sigma_d'^2 = n\{3(1-\theta)-5\theta^2/2\}/2$$

Using the same arguments which leads to the equation (2.37), a lower bound for $P_{tsA}(\theta)$, the asymptotic probability of correct selection is given by

$$P_{tsA}(\theta) \geq [2\phi\,(\Delta)-1]^{t-3}\,[2\phi(\Delta')-1]^2 \tag{4.24}$$

Using (4.24), the lower bounds for $P_{tsA}(\theta)$ have been computed for given t, n, η and θ and it is presented in table 14.

At the end we note that, following the arguments similar to those in Section (2.7), we can obtain an asymptotic expression for the expected subset size, under slippage configuration. This expression is exactly the same as (2.40) with σ_i^2, i=1, 2, 3, 4, 5, 6, now given as below.

$$\sigma_1^2 = n\,[\pi(1-\pi) - (\theta' + 5\theta)/4 + 5/4]$$

$$\sigma_2^2 = n\,[\pi(1-\pi) - (\theta' + 3\theta)/4 + 3/4]$$

$$\sigma_3^2 = n\,[2\pi(1-\pi) - (\theta' + \theta)/2 + 1/2]$$

$$\sigma_4^2 = n\,[5\pi\,(1-\pi) - (5\theta' + \theta)\,/4 + 1/4]$$

$$\sigma_5^2 = 3n\,(1-\theta)/2$$

$$\sigma_6^2 = n\,(1-\theta)$$

CHAPTER V

SELECTION PROCEDURES FOR SOME OTHER GOALS USING FP DESIGN

5.1 Introduction

In all the three previous chapters we were concerned with Goal 1 as defined in section 5 of chapter I. In this chapter we consider two remaining Goals, viz., Goal 2 and 3. The selection procedures will be developed assuming that there are no ties, no order effects, and the data are obtained by using FP design with equal no. of replications for each pair. Specifically the following three problems have been discussed.

Problem 1 Goal 2 of selection of k-best of t treatments using FP design

In earlier chapters we have dealt at some length, with the goal of selecting the single best treatment from a set of t treatments using data from a paired comparison design. There are situations, however, where we may wish to choose the two best treatments or the three best treatments without ordering them. As an example, consider wheat breeding experiments in Agriculture. Usually the high yielding varieties of wheat are not having very good bread making qualities such as taste, texture, loaf volume, dough formation time, etc. Also bakeries would not like to depend on a single good variety, since its short supply at times may lead to short fall in production or total stoppage in production. Thus bakeries would be interested in selecting a few varieties of wheat having good bread making qualities. In section (5.2) we consider the problem of selecting a fixed number k of best treatments out of a group of t treatments using FP design under indifference zone formulation.

Problem 2 Goal 3 of selection of treatments better than a standard using FP design

Continuing with the example of selection of wheat varieties with respect to their bread making qualities, the bakeries would always be in look out for those varieties which are at least as good as the one being currently used by them. Even the best one among a set of new available varieties will not be worthy of consideration unless its bread making qualities are atleast the same if not better than that of the variety in use already. Thus the problem here would be that of selection of varieties that are better than a standard (currently in use). Let there be (t+1) treatments in all, of which one is labelled as the standard. The problem of selection of the best treatment amongst those of 't' non standard treatments that are better than the standard has been dealt in Section (5.4). In section (5.5), we have considered the problem of selection of all treatments better than a standard.

Problem 3 Selection of best treatment assuming Thurstone-Mosteller model and using FP design

In situations where TM model as defined in (1.3) holds, one may develope large sample selection procedure for Goal 1 under both the indifference zone as well as subset selection formulation. These selection procedures will be developed in Section (5.6) when data are taken by using FP design with equal number 'n' (large) of replications per paired comparisons.

5.2 Selection procedure for problem 1

Basic formulations and assumptions regarding the preference probabilities and scoring system for the t treatments would be exactly the same as used by Trawinski and David (1963). Thus the probability distribution of the score vector (1.48) and other results such as (1.49) can be directly used in the discussion of selection procedures for the selection of k-best treatments, which is the problem to be discussed here. We define the following selection rule for the present problem.

<u>Selection Rule R_7</u>

Obtain $(a_1, a_2, \ldots\ldots a_t)$, the scores of 't' treatments T_1, $T_2, \ldots\ldots T_t$ respectively, as defined by (1.8), using FP design with n replications per pair and declare the k treatments with the highest scores as the best treatments. If m ($\leq$ t-1) of them are tied for the last (k-1) places, (k-1) of them are declared at random as the best treatments, where $1 = 0,1,2,\ldots\ldots k-1$ are the number of best scores out of k, where there are no ties.

Without loss of generality we shall assume that T_{t-k+1}, $T_{t-k+2}, \ldots\ldots T_{t-1}, T_t$ are the k best treatments. Then the true worth S_i corresponding to each of these k treatments would be larger than true worth of each of the remaining treatments $T_1, T_2 \ldots T_{t-k}$. Thus the complete parametric space can be defined by

$$C^k(\pi_{ij}) = \left\{ \begin{array}{ll} \pi_{lj} > 1/2, & l = t-k+1, \ldots\ldots t \\ & j = 1,2,\ldots\ldots t-k \\ 0 \leq \pi_{ij} \leq 1, & \pi_{ij} + \pi_{ji} = 1, \; i,j = 1,2,\ldots\ldots t \end{array} \right\}$$

In case $T_{t-k+1}, \ldots\ldots T_{t-1}, T_t$ are equal and the remaining (t-k) inferior treatments are also equal, we have the set of slippage configurations defined by

$$C_\pi^k = \left\{ \begin{array}{l} \pi_{lj} = \pi > 1/2, \ l = t-k+1, \ \ldots\ldots t, \ j=1,2,..t-k \\ \pi_{lm} = 1/2, \ l,m=t-k+1,\ldots.t, \ \ l \neq m \\ \pi_{ij} = 1/2 \ , \ i,j=1,2,\ldots.t-k, \ \ i \neq j \end{array} \right\} \quad (5.1)$$

Let $P(\pi)$ denote the probability of correct selection under slippage configuration, that the treatments $T_{t-k+1}, \ldots\ldots T_{t-1}, T_t$ are in fact declared as the best treatments by selection rule R_7. Following the arguments of Trawinski and David (1963), we choose the parameter n, the number of replication per pair in FP design to be the least n such that

$$P(\pi) \geq P^* \quad (5.2)$$

for given π and P^*. The choice of π_{ij} in C_π^k does not necessarily correspond to a least favourable configuration. However, the configuration C_π^k is important in itself, since it presents the situation in which there are k superior treatments and would be a reasonable basis for deciding about the number of replications required for detecting the k outliers. In the following we will obtain the expression for $P(\pi)$.

<u>Exact evaluation of the probability of correct selection under slippage configuration</u>

From (1.48), the joint distribution of the scores when the parametric configuration is in C_π^k can be written as

$$f(\underline{a}) = 2^{-n[\binom{t}{2}-k(t-k)]} \ \pi^{\Sigma a_i} \ (1-\pi)^{nk[t-\frac{k+1}{2}]-\Sigma a_i} \ g(\underline{a};n), \quad i=t-k+1,\ldots t \quad (5.3)$$

where $g(\underline{a};n)$ is given in (1.48B). Of the $G(\underline{a};n)/g(\underline{a};n)$ distinct permutations of the scores $a_1, a_2, \ldots\ldots a_t$, a proportion $\binom{m}{k-1}/\binom{t}{k}$ must have top scores in the last k places associated with $T_{t-k+1}, \ldots\ldots\ldots T_t$. Here $\binom{m}{k-1}$ is the number of ways the last k places can be filled in by the k top scores in a given partition.

The corresponding contribution to the probability of correct selection $P(\pi)$ is $2^{-n[\binom{t}{2}-k(t-k)]}\pi^{\Sigma a_i}(1-\pi)^{nk(t-\frac{k+1}{2})-\Sigma a_i}\ \dfrac{g(\underline{a};n)}{\binom{m}{k-1}}\times$

$$\frac{G(\underline{a};n)}{g(\underline{a};n)}\times\frac{\binom{m}{k-1}}{\binom{t}{k}} \qquad (5.4)$$

$P(\pi)$ is then given by summing the above over all a_i, $i=t-k+1\ldots.t$ which can be k top scores and over all permissible values of other scores and may be expressed as $P(\pi) = 2^{-n[\binom{t}{2}-k(t-k)]}\sum^{nk(t-\frac{k+1}{2})}_{\sum_{i=t-k+1}^{t} a_i = c}\times$

$$\pi^{\Sigma a_i}\ (1-\pi)^{[nk(t-\frac{k+1}{2})-\Sigma a_i]}\ \frac{\Sigma\, G(\underline{a};n)}{\binom{t}{k}} \qquad (5.5)$$

where the last summation extends over $\sum_{i=1}^{t-k} a_i = n\binom{t}{2} - \sum_{i=t-k+1}^{t} a_i$ and C is the smallest integer greater than or equal $nk(t-1)/2$.

Example If $t=3$, $n=1$, $k=2$, then from (5.5)

$$\begin{aligned}P(\pi) &= 2^{-1}\sum_{a_2+a_3=2}^{3}\pi^{(a_2+a_3)}(1-\pi)^{(3-a_2-a_3)}\ \Sigma\, G(\underline{a};1)/3\\ &= 2^{-1}[\pi^2(1-\pi)\,G(1,1,1;1)/3 + \pi^3G(0,1,2;1)/3]\\ &= [\pi^2(1-\pi)\,2/3 + 6\pi^3/3]/2\\ &= \pi^3 + \pi^2(1-\pi)/3\end{aligned}$$

If $k=1$, then

$$\begin{aligned}P(\pi) &= 2^{-1}\sum_{a_3=1}^{2}\pi^{a_3}(1-\pi)^{(2-a_3)}\ \Sigma G(a;1)/3\\ &= [\pi(1-\pi)\,G(1,1,1;1)/3+\pi^2G(0,1,2;1)/3]/2\\ &= [\pi(1-\pi)2/3+ 6\pi^2/3]/2\\ &= \pi^2+ \pi(1-\pi)/3,\end{aligned}$$

which is the same as the expression of Trawinski and David (1963) for a single outlier.

In the following we will give the distribution of the vector of difference among the scores.

5.3 Asymptotic distribution theory

Consider the vector of differences $\underline{d}_l = (d_1, d_2,\ldots..d_{t-k}\ldots$ $\ldots d_{l-1}, d_{l+1},\ldots..d_t)$, where $d_i = a_i - a_l$, $i=1,2,\ldots..t-k, l-1,$ $l+1,\ldots..t$, $l= t-k+1,\ldots\ldots..,t$. The variate d_i has mean

$$\alpha_i = n\Big[\sum_{\substack{j=1\\ j\neq i}}^{t} \pi_{ij} - \sum_{\substack{j=1\\ j\neq 1}}^{t} \pi_{1j}\Big]$$

and variance

$$\sigma_{ii} = n\Big[\sum_{\substack{j=1\\ j\neq i\neq 1}}^{t} (\pi_{ij}\pi_{ji}+\pi_{1j}\pi_{j1}) + 4\,\pi_{i1}\pi_{1i}\Big]$$

$$\sigma_{ij} = n\Big[\sum_{\substack{j=1\\ j\neq i\neq 1}}^{t} \pi_{1j}\pi_{j1}+(\pi_{i1}\pi_{1i}+\pi_{j1}\pi_{1j}-\pi_{ij}\pi_{ji})\Big] \qquad (5.6)$$

under the slippage configuration (5.1)

$$\alpha_i = \begin{cases} -nt\,(\pi - 1/2), & i=1,2,\ldots t-k,\ i\neq 1 \\ 0, & i=t-k+1,\ldots t \end{cases}$$

and the structure of the variance covariance matrix of $\underline{d}_1$ under (5.1) is

$$n\Sigma = \begin{array}{c|cccccccc} d_i{}'s & 1 & 2 & 3\ldots & t-k & t-k+1\ldots & 1-1 & 1+1\ldots & t \\ \hline 1 & A & C & C\cdots & C & D\cdots & D & D & D \\ 2 & C & A & C\cdots & C & D\cdots & D & D & D \\ 3 & C & C & A\cdots & C & D\cdots & D & D & D \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ t-k & C & C & C\cdots & A & D\cdots & D & D & D \\ t-k+1 & D & D & D\cdots & D & B\cdots & D & D & D \\ 1-1 & D & D & D\cdots & D & D\cdots & B & D & D \\ 1+1 & D & D & D\cdots & D & D\cdots & D & B & D \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ t & D & D & D\cdots & D & D\cdots & D & D & B \end{array}$$

where

$$\begin{aligned} A &= n[(t+2)\pi(1-\pi) + (t-2)/4\,] \\ B &= n[2\pi(1-\pi)(t-k) + k/2] \\ C &= n[(t-k+2)\pi(1-\pi) + (k-2)/4] \\ D &= n[(t-k)\pi(1-\pi) + k/4] \end{aligned} \qquad (5.7)$$

It is easy to verify that in general d_i's can be put in to two types with respect to mean and variance sturcture. However, d_i's fall in three categories with respect to covariance sturucture. The three categories of covariance are

i) Cov (d_i, d_j), $i \neq j$, $i,j = 1,2,\ldots\ldots t-k$

ii) Cov (d_i, d_j), $i \neq j$, $i,j = t-k+1,\ldots\ldots l-1, l+1\ldots\ldots t$

iii) Cov (d_i, d_j), $i=1,2,\ldots\ldots t-k$, $j = t-k+1\ldots\ldots l-1\ldots\ldots t$.

Under (5.1), within the categories (i), (ii) and (iii), the covariances are equal. Also the expression for covariance in category (ii) is equal to the expression for covariance in category (iii). Let $R = (r_{ij})$ be the correlation matrix corresponding to variance covariance matrix $n\sum$ for any configuration in C_π^k, then clearly, by multivariate central limit theorem, the vector

$$\left[\frac{d_1-\alpha_1}{\sqrt{\sigma_{11}}}, \ldots\ldots \frac{d_{t-1}-\alpha_{t-1}}{\sqrt{\sigma_{t-1,t-1}}}, \frac{d_{l+1}-\alpha_{l+1}}{\sqrt{\sigma_{l+1,l+1}}}, \frac{d_t-\alpha_t}{\sqrt{\sigma_{tt}}} \right]$$

converges in law to the multivariate random variable with distribution N (O,R). Note that R takes the following form because of the remark about the equality of covariances made above.

$$R = \left\{ \begin{array}{llll} 1 & r\ldots\ldots r & r_1\ldots\ldots\ldots r_1 & \\ r & 1\ldots\ldots r & r_1\ldots\ldots\ldots r_1 & \\ \ldots\ldots & \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots & \\ r & r \quad 1 \quad r & r_1\ldots\ldots\ldots r_1 & \\ r & r \quad r \quad 1 & r_1 \ldots\ldots r_1 & \\ r_1 & r_1\ldots\ldots r_1 & 1 \ldots\ldots r_1 & \\ \ldots\ldots & \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots & \\ \ldots\ldots & \ldots\ldots\ldots\ldots & \ldots\ldots 1 \quad r_1 & \\ r_1 & r_1\ldots\ldots r_1\ldots r_1\ldots r_1 & \quad 1 & \end{array} \right\} \qquad (5.8)$$

With the help of this distribution, we will obtain the asymptotic expression for the probability of correct selection.

5.4 Asymptotic approximation to the probability of correct selection

The probability of correct selection under the parametric configuration in (5.1) asymptotically, as $n \to \infty$ is given by

$$P_A(\pi) = \lim_{n\to\infty} P\,(\min(a_t, \ldots, a_{t-k+1}) > \max(a_1 \ldots\ldots a_{t-k})$$

$$= \lim_{n\to\infty} \sum_{l=t-k+1}^{t} P\,[\,\min\,(a_t, \ldots, a_{t-k+1}) = a_l, \quad a_l > a_i,\ i=1,2,\ldots,t-k\,]$$

$$= \lim_{n\to\infty} \sum_{l=t-k+1}^{t} P\,[a_l > a_i,\ i=1,2,\ldots t-k \quad a_l < a_j,\ j\neq l, j=t-k+1,\ldots\ldots t\,]$$

$$= \lim_{n\to\infty} \sum_{l=t-k+1}^{t} P\,[\,d_i < 0,\ i=1,2,\ldots t-k, d_j > 0, j=t-k+1,\ldots t, j\neq l]$$

$$= \lim_{n\to\infty} kP\,[\,d_i<0,\ i=1,2,\ldots t-k,\ d_j > 0,\ j=t-k+1,\ldots t,\ j\neq l]$$

$$> k \lim_{n\to\infty} P\,[\,d_i<0,\ i=1,2,\ldots t-k]\ P[d_j>0, j=t-k+1,\ldots t,\ j\neq l]$$

using Sidak's inequality we have

$$P_A(\pi) > k\,[\,P_1 \times P_2]$$

where $P_1 = \lim_{n\to\infty} P[\,d_i<0,\ i=1,2,\ldots\ldots t-k]$

$$P_2 = \lim_{n\to\infty} P[\,d_j>0,\ j=t-k+1,\ldots\ldots t]$$

Now consider P_1

$$P_1 = \lim_{n\to\infty} P[\,v_i < \Delta\ ,\ i=1,2,\ldots\ldots t-k]$$

where $v_i = (d_i - \alpha_0)/\sigma$ and $\Delta = -\alpha_0/\sigma$ where $\alpha_0 = -nt\,(\pi - \frac{1}{2})$

$$\sigma^2 = n\,[\ (t+2)\,\pi\,(1-\pi) + 1/4\ (t-2)\,]$$

From the limiting multivariate normality of v_i, we have

$$P_1 = (2\pi)^{-(1/2)(t-k)} |R_1|^{-1/2} \int_{-\infty}^{\Delta}\!\!\ldots\!\int_{-\infty}^{\Delta} \exp\{-(1/2)\ V' R_1^{-1} V\}\ dv_1 \ldots dv_{t-k} \quad (5.9)$$

R_1 being the correlation matrix of v_i, with elements 1 along the principal diagonal and r else where and

$$r\sigma^2 = n\,[\,(t-k+2)\,\pi\,(1-\pi) + (1/4)(k-2)] \quad \text{for all } i=1,2,\ldots t-k.$$

As v_i are equi-correlated, it is possible to simplify (5.9) following arguments similar to those of Trawinski and David (1963) as $P_1 = \int_{-\infty}^{\infty} [\phi(Z)]^{t-k} \phi(x)\, dx$ where

$$Z = [\, r(1-r)]^{1/2} x - \alpha_0 / [(1-r)^{1/2} \sigma\,]$$

Now consider P_2

$$P_2 = \lim_{n \to \infty} P\, [v_j > 0,\ j = t-k+1, \ldots\ldots t\,]$$

where $v_j = d_j / \sigma'$ and

$$\sigma'^2 = n[\, 2\, \pi(1-\pi)(t-k) + k/2]$$

From the limiting multivariate normality of v_j, we have

$$P_2 = (2\pi)^{-(1/2)(k-1)} |R_2|^{-1/2} \int_0^{\infty} \ldots \int_0^{\infty} \exp\, [-\tfrac{1}{2}\, V' R_2^{-1} V]\, dv_{t-k+2} \ldots dv_t \quad (5.10)$$

R_2 being the correlation matrix of v_j with elements 1 along principal diagonal and ½ else where. Again as v_j are equi-correlated, it is possible to simplify (5.10) using the arguments as stated above for obtaining P_1.

$$P_2 = \lim_{n \to \infty} P\, [\, u_j > u_{t-k+1},\ j = t-k+2, \ldots\ldots t\,]$$

$$= \int_{-\infty}^{\infty} [\, P\{\, u_j > u_{t-k+1} / u_{t-k+1}\}\,]^{k-1} \phi\, (u_{t-k+1})\, du_{t-k+1}$$

This leads to

$$P_2 = \int_{-\infty}^{\infty} [1-\phi\, (x)]^{k-1} \phi\, (x)\, dx \quad (5.11)$$

Thus the asymptotic probability of correct selection

$$P_A(\pi) \geq k \int_{-\infty}^{\infty} [\phi(Z)]^{t-k} \phi\, (x)\, dx \int_{-\infty}^{\infty} [1-\phi(x)]^{k-1} \phi(x)\, dx \quad (5.12)$$

For an experiment involving t treatments, the above lower bound is set at P* and the resulting equation is solved for n, the smallest number of replications which ensure that the highest k scores in an experiment of t treatments with n replications will correspond to the best k treatments with at least a pre-assigned probability P*. The integrals involved on the R.H.S. of (5.12) have been tabulated by Gupta (1956, 1963) and would be helpful for computing n.

5.5 Selection procedures for problem 2

Let $T_0, T_1, \ldots T_t$ be (t+1) treatments with T_0 as standard or control treatment. In the present section we develop selection procedures for the following two goals.

(A) Selection of a best treatment amongst $(T_1, T_2, \ldots\ldots T_t)$ that is better than standard T_0, if such treatment exists, otherwise no treatment is selected.

(B) Selection of all treatments if any amongst $(T_1, T_2 \ldots\ldots T_t)$, that are better than control T_0.

Selection procedures will be defined on the basis of data obtained by FP design with equal number n of replications per pair. Goal A will be discussed under indifference zone formulation while Goal B under subset selection formulation. First we shall consider the selection procedure for Goal A. Let π_{ij}, i, j=0, 1, ...t, i≠j, as usual represent the preference probability of T_i over T_j, assuming no ties are permissible. Assume general linear model (1.2) for preference probabilities. Without loss of generality we can assume that T_t be the best treatment amongst $(T_1, T_2 \ldots T_t)$.

Let

$$\begin{aligned} C(\pi_{ij}) &= \{0 \le \pi_{ij} \le 1,\ \pi_{ij} + \pi_{ji} = 1, i, j = 0, 1, \ldots\ldots t\} \\ C_0(\pi_{ij}) &= \{\pi_{0j} > 1/2 + \alpha_0,\ j = 1, 2, \ldots\ldots t\} \qquad (5.13) \\ C_t(\pi_{ij}) &= \{\pi_{tj} > 1/2 + \alpha_1,\ j = 0, 1, 2, \ldots\ldots t-1\} \end{aligned}$$

where α_0, α_1, are specified constant in the interval $(0, \frac{1}{2})$. The parametric configurations in $C_0(\pi_{ij})$ imply T_o is an outlier treatment while that in $C_t(\pi_{ij})$ imply T_t as the outlier treatment.

Let $a_{[1]} \leq a_{[2]} \leq \ldots\ldots \leq a_{[t]}$ be the ordered score for the 't' treatments $T_1 \ldots\ldots T_t$ and a_0 be the score for the standard treatment T_0 obtained by using FP design with n replications per pair. We define the following selection rule for Goal A as stated above.

Selection rule R_8

Choose T_0 as the best treatment if $a_{[t]} < a_0 + C$. $C > 0$ otherwise choose the treatment that produce the score $a_{[t]}$ as the best treatment. In the latter case, if m scores tie for the last place, choose any one of the corresponding treatments as the best with probability 1/m.

The number of replication 'n' and the constant 'C' are needed in order to implement the selection Rule R_8. For specified probabilities P_1^*, P_2^*, such that $2^{-t} < P_1^* < 1$ and $(1-2^{-t})t^{-1} < P_2^* < 1$, we choose 'n' and 'C' so that the following probability requirements are satisfied by Rule R_8.

$$P_r\{ T_0 \text{ is selected}\} \geq P_1^* \text{ whenever } \pi_{ij} \in C_0(\pi_{ij}) \tag{5.14}$$

$$P_r\{ T_t \text{ is selected}\} \geq P_2^* \text{ whenever } \pi_{ij} \in C_t(\pi_{ij}) \tag{5.15}$$

In case T_0 is an outlier and other treatments are of equal worth, the parametric configuration for preference probability must belong to

$$C_0(\alpha_0) = \{ \pi_{0j} = \tfrac{1}{2} + \alpha_0, \; j = 1, 2, \ldots\ldots t \qquad \pi_{ij} = \tfrac{1}{2} \;\; i \neq j = 1, 2, \ldots\ldots t \} \tag{5.16}$$

which is a configuration in $C_0(\pi_{ij})$.

Similarly if T_t is an outlier and other treatments are of equal worth, the slippage configuration must belong to

$$C_t(\alpha_1) = \{ \pi_{tj} = \tfrac{1}{2} + \alpha_1, \; j = 0, 1, \ldots\ldots t-1 \qquad \pi_{ij} = \tfrac{1}{2} \;\; i \neq j, = 0, 1, \ldots\ldots\ldots t-1 \} \tag{5.17}$$

$$l_i\ (\underline{a}) + C > l_1\ (\underline{a}),\ l_2(\underline{a}),\ldots\ldots l_{i-1}(\underline{a}),\ l_{i+1}(\underline{a})\ldots l_k(\underline{a}) \qquad (5.19)$$

where $\alpha_{1i} = 1$ if (5.19) hold good

= 0 otherwise.

Note that the requirement (5.19) is equivalent to the requirement that a score vector $\underline{a}$ with score $a_0 = l_i$ satisfies

$$a_0 + C > \max\ (a_1,\ a_2 \ldots\ldots a_t).$$

Contribution to the probability of correct selection from score vector having l_i in the first place and which are permutations of partition vector $\underline{a}^*$ is

$$2^{-\frac{1}{2}nt(t-1)} \frac{G(\underline{a};n)}{g(\underline{a};n)} \times \frac{m_i}{t+1}\ \alpha_{1i} \times g(\underline{a};n) \times (\tfrac{1}{2} + \alpha_0)^{l_i(\underline{a})} \times (\tfrac{1}{2} - \alpha_0)^{nt - l_i(\underline{a})}$$

Hence contribution to the probability from all $\underline{a}^*$ is given by

$$\frac{2^{-\frac{1}{2}nt(t-1)}}{(t+1)} G(\underline{a},n) \sum_{i=1}^{k} m_i\ \alpha_{1i}\ (\tfrac{1}{2} + \alpha_0)^{l_i(\underline{a})} (\tfrac{1}{2} - \alpha_0)^{nt - l_i(\underline{a})} \qquad (5.20)$$

Thus the probability of correct selection $P_{1(cs)}$ is obtained by adding (5.20) over all permissible partition $\underline{a}^*$ of total score $n\binom{t+1}{2}$

$$P_{1(cs)} = 2^{-\frac{1}{2}nt(t-1)} \sum_{\underline{a}^*} \frac{G\ (\underline{a};n)}{(t+1)} \times \sum_{i=1}^{k} m_i\ \alpha_{1i}\ (\tfrac{1}{2} + \alpha_0)^{l_i(\underline{a})}\ (\tfrac{1}{2} - \alpha_0)^{nt - l_i(\underline{a})} \qquad (5.21)$$

Now we shall obtain the expression for 2nd probability in (5.18) under the configuration $C_t(\alpha_1)$.

Let $P_{2(cs)} = \sum_{m=1}^{t} \frac{1}{m}\ P_r\{a_t \geq \max\ (a_0 + c, a_1, \ldots\ldots, a_{t-1})$, and $a_t = a_{[t]} = a_{[t-m+1]}\}$. Keeping in mind that there are (t+1) treatments instead of t, we can use (1.48) to write the joint distribution of the scores under slippage configuration $C_t(\alpha_1)$ as

$$f(\underline{a}) = 2^{-\frac{1}{2}nt(t-1)}(\tfrac{1}{2}+\alpha_1)^{a_t}(\tfrac{1}{2}-\alpha_1)^{nt-a_t}\; g(\underline{a},n).$$

Let $M_1(\underline{a},c)$ is the number of a_i's in $\underline{a}$, (other than max a_i), such that $a_i+C \leq \max a_i$ and m' be the number of scores tied for the last place in a partition. Then of the $\frac{G(\underline{a},n)}{g(\underline{a},n)}$ distinct partition of the scores $(a_0, a_1, \ldots\ldots a_t)$, a proportion $\frac{m'}{(t+1)}$ x $\frac{M_1(\underline{a},C)}{t}$ must have highest score in the last place satisfying the requirement

$$a_t \geq \max(a_0+c, a_1, \ldots\ldots a_{t-1}) \quad \text{and}$$

$$a_t = a_{[t]} = a_{[t-m+1]} \tag{5.21}$$

for some $C > 0$. Since $C > 0$, in order that a score vector $\underline{a}$ satisfy (5.21), a_0 must not tie with a_t. Thus m'=m. The corresponding contribution to the probability of correct selection $P_{2(cs)}$ is

$$2^{-\frac{1}{2}nt(t-1)}(\tfrac{1}{2}+\alpha_1)^{a_t}(\tfrac{1}{2}-\alpha_1)^{nt-a_t} \times$$

$$\frac{M_1(\underline{a},C)}{(t+1)} \quad \frac{G(\underline{a},n)}{t} \tag{5.22}$$

For a given C, t, n and α_1, $P_{2(cs)}$ is then given by summing (5.22) over all a_t, which can be maximum score and over all permissible values of other scores and may be written as

$$P_{2(cs)} = 2^{-\frac{1}{2}nt(t-1)} \sum_{a_t=(\frac{1}{2}nt)}^{nt} (\tfrac{1}{2}+\alpha_1)^{a_t}(\tfrac{1}{2}-\alpha_1)^{nt-a_t} \times \sum \frac{M_1(\underline{a},C)}{(t+1)t}\; G(\underline{a};n) \tag{5.23}$$

where the last summation extends over $\sum_{i=0}^{t-1} a_i = n\binom{t+1}{2} - a_t$.

For implementing the selection Rule R_8, we shall, as usual, use the probability of correct selection under the slippage configuration as obtained in (5.20) and (5.23) for fixing the selection constant n, c for given α_0, α_1, P_0^* and P_1^*. In fact we shall find n, c such that

$$P_{1(cs)} \geq P_0^* \qquad P_{2(cs)} \geq P_1^* \tag{5.24}$$

Since it would be economical to use smaller number of replications per paired comparisons, we would choose that pair (n, c) which satisfy (5.24) and for which n is least.

In the following, we shall develop the asymptotic expression for $P_{1(cs)}$ and $P_{2(cs)}$ for large n and t. Basic arguments are same as that used by Trawinski and David (1963) for deriving their asymptotic expressions and therefore details would be omitted in the following.

Asymptotic probability of correct selection for rule R_8

$$P_{1A}(cs) = \lim_{n\to\infty} Pr\{a_i < a_0 + \tfrac{1}{2} + c,\ i=1,2,\ldots\ldots t\}$$

$$= \lim_{n\to\infty} P_r\{d_i < C + \tfrac{1}{2},\ i=1,2,\ldots\ldots\ldots t\}$$

where $d_i = a_i - a_0$. Under $C_0(\alpha_0)$, d_i's have got mean, variance and covariances as

$$\bar{d} = -n\ (t+1)\,\alpha_0$$
$$\sigma_d^2 = n[\tfrac{1}{2}\ (t+1) - (t+3)\,\alpha_0^2] \tag{5.25}$$
$$\rho\sigma_d^2 = n[\ 1/4\ (t+1) - (t+2)\,\alpha_0^2]$$

respectively.

$$P_{1A}(cs) = \lim_{n\to\infty} P_r\{v_i < \Delta\ ,\ i=1,2,\ldots\ldots t\} \tag{5.26}$$

where $v_i = (d_i - \bar{d})/\sigma_d$ and $\Delta = (c + \tfrac{1}{2} - \bar{d})/\sigma_d$

As v_i are equi-correlated, it is possible to simplify (5.26) as below.

$$P_{1A}(cs) = \int_{-\infty}^{\infty} [\phi(Z)]^{t} \phi(x)\, dx \tag{5.27}$$

where $Z = [\rho/(1-\rho)]^{1/2}x + (c + 1/2 - \bar{d})/(1-\rho)^{1/2}\sigma_d$

Next the expression for $P_{2A}(cs)$ is obtained

$$P_{2A}(cs) = \lim_{n\to\infty} P_r\{a_t \geq \max(a_0+c, a_1, \ldots a_{t-1})\}$$
$$= \lim_{n\to\infty} P_r\{a_t \geq a_0 + c,\ a_t \geq a_i,\ i=1,2,\ldots t-1\}$$
$$= \lim_{n\to\infty} P_r\{d_i' \leq 1/2,\ i=1,2,\ldots t-1,\ d_0' \leq 1/2 - c\}$$

where $d_i' = a_i - a_t$, $i=0,1,2,\ldots t-1$

Under $C_1(\alpha_1)$, d_i's have got mean, variance and covariance as given by (5.25) with α_0 replaced by α_1.

Thus

$$P_{2A}(cs) = \lim_{n\to\infty} P_r\{v_i < \Delta_i,\ i=0,1,2,\ldots t-1\}$$

where $v_i = (d_i' - \bar{d}')/\sigma_{d'}$, $i=0,1,2,\ldots t-1$

$\Delta_i = (1/2 - \bar{d}')/\sigma_{d'} = \Delta$, $i=1,2,\ldots t-1$

$\Delta_0 = (1/2 - c - \bar{d}')/\sigma_{d'}$

The last expression simplifies to

$$P_{2A}(cs) = \int_{-\infty}^{\infty} [\phi(Z)]^{t-1}\phi(Z')\,\phi(x)\,dx \tag{5.28}$$

where $Z = [\rho'/(1-\rho')]^{1/2}x + (1/2-\bar{d}')/[(1-\rho')^{1/2}\sigma_{d'}]$

$Z' = [\rho'/(1-\rho')]^{1/2}x + (1/2-c-\bar{d})/[(1-\rho')^{1/2}\sigma_{d'}]$

A lower bound of (5.28) can be obtained by using Sidak's inequality as

$$P_{2A}(cs) \geq \phi(\Delta_0) \int_{-\infty}^{\infty} [\phi(Z)]^{t-1} \phi(x)\, dx \tag{5.29}$$

The integral on the RHS of (5.28) does not seem to have been tabulated in the literature of ranking and selection procedures. Instead we can utilise the lower bound given in (5.29), in order to get conservative values of (n, c) satisfying the probability requirement (5.24). The integral involved in (5.27) and (5.29) have been tabulated extensively as stated earlier in this chapter.

Now we will define the selection rule R_9 for Goal B as defined at the beginning of the present section. The problem here is to select a subset of t treatments $T_1, T_2, \ldots\ldots T_t$, that includes all treatments that are better than standard T_0.

Without loss of generality we can assume that

$T_i > T_0$ if $i \in I$ and $T_j < T_0$ if $j \in I^c$, where I is a subset of $\{1, 2, \ldots\ldots t\}$.

Let

$$C_I(\pi_{ij}) = \begin{Bmatrix} \pi_{i0} > \tfrac{1}{2}, & i \in I \\ \pi_{k0} \leq \tfrac{1}{2}, & k \in I^c \end{Bmatrix} \tag{5.30}$$

The parametric configuration $C_I(\pi_{ij})$ imply that $T_1, T_2, \ldots\ldots, T_i > T_0 \geq T_{i+1}, \ldots\ldots T_t$.

Let $a_0, a_1, \ldots\ldots a_t$ be the score for the (t+1) treatments $T_0, T_1, \ldots\ldots T_t$ obtained by using FP design with n replication per pair. We define the following selection rule R_9.

<u>Selection rule R_9</u>

Retain in the selected set of treatments only those treatment T_i for which

$$a_i > a_0 - d$$

wher $d > 0$ is the selection constant. The selection constant 'd' is a function of 't' and 'n', the number of treatments and the number of replications per paired comparisons. For specified probability P*, we choose 'n' and 'd' such that

$$P_r\{T_i \text{ is selected, } i \in I\} \geq P^* \tag{5.31}$$

for all configuration in $C_I(\pi_{ij})$.

In case all treatments are of equal worth, the parametric configuration for preference probability must belong to

$$C_E(\tfrac{1}{2}) = \{ \pi_{ij} = \tfrac{1}{2} \quad i \neq j = 0,1,\ldots\ldots t \}$$

which is a configuration in $C_I(\pi_{ij})$.

In the following, we shall obtain the probability of correct selection for the rule R_9 under $C_E(\tfrac{1}{2})$.

Probability of correct selection for rule R_9

The probability of correct selection for selection rule R_9 is given by

$$P_{(cs)} = P\{ a_i \geq a_0 - d,\ i \in I/\ t,\ n,\ C_E(\tfrac{1}{2}) \}$$

$$\geq P\{ a_i \geq a_0 - d,\ i=1,2,\ldots t/t,n,\ C_E(\tfrac{1}{2}) \}$$

$$= P\{ \min (a_1, a_2, \ldots\ldots a_t) \geq a_0 - d/t,\ n,\ C_E(\tfrac{1}{2}) \}$$

$$= \sum_{\underline{a}} P\{ \min (a_1, a_2, \ldots\ldots a_t) \geq a_0 - d/\underline{a},\ C_E(\tfrac{1}{2}) \} \times P(\underline{a},\ C_E(\tfrac{1}{2}))$$

$$= 2^{-\frac{1}{2}nt(t+1)} \sum [P\{ \min (a_1, a_2, \ldots\ldots a_t) \geq a_0 - d/\underline{a},\ C_E(\tfrac{1}{2}) \} \times G(\underline{a}; n)]$$

where $P(\underline{a}, C_E(\tfrac{1}{2}))$ is the partition probability function and where the summation extend overall unordered distinct partition $\underline{a}$ of $\tfrac{1}{2}nt(t+1)$. (Using (1.48) in $C_E(\tfrac{1}{2})$. $P\{\min (a_1, a_2, \ldots\ldots a_t) \geq a_0 - d/\underline{a},\ C_E(\tfrac{1}{2})\} \times G(\underline{a}; n)$ is the frequency of score vectors corresponding to the partition $\underline{a}$ which satisfies the condition

$$a_i \geq a_0 - d, \quad i=1,2,\ldots\ldots t.$$

This is given by

$$\frac{M(\underline{a}, d)}{(t+1)} G(\underline{a}; n)$$

where $M(\underline{a}, d)$ is the number of a_i's in $\underline{a}$ which are less than or equ to $\min_{i=0,1,2,\ldots\ldots t} a_i + d$

For given n, t and d we have

$$P_{(cs)} = (t+1)^{-1}\ 2^{-\frac{1}{2}nt(t+1)} \sum M(\underline{a}, d)\ \ G(\underline{a}; n) \qquad (5.32)$$

As an illustration of the above procedure , consider the case when t=3, n=1. In this case, there are four permissible partitions of which a typical one (written in ascending order) is [0123] with frequency G(0123; 1) = 24. Also we see for

d=0	M=1
d=1	M=2
d=2	M=3
d=3	M=4

Thus the contribution to $P_{(cs)}$ from this partition is, for a given d is $4^{-1}\ 2^{-6}\ M(d) \times 24 = \frac{3M(d)}{2^5}$

If the corresponding contributions from all four partitions are added up, the resulting values are

d	0	1	2	3
$P_{(cs)}$	0.4062	0.6875	0.9062	1

In the following, we shall obtain an asymptotic expression for $P_{(cs)}$. As in the previous section the basic arguments are the same as that used by Trawinski and David (1963) for developing their asymptotic expression.

<u>Asymptotic approximation to d</u>

Here we have

$$P_{(cs)} > \lim_{n\to\infty} P_r \{ a_0 - a_i < d + 1/2\}$$

$$= \lim_{n\to\infty} P_r \{d_i < d + 1/2,\ i=1,2,\ldots.t\}$$

Under C_E (1/2) $E(d_i) = 0$

$$\mathrm{Var}(d_i) = \sigma^2_d = n(t+1)/2$$

$\text{Cov}\ (d_i, d_j) = \rho\sigma_d^2 = n(t+1)/4$ with $\rho = 1/2$.

Then using standard arguments

$$P_{(cs)} = \int_{-\infty}^{\infty} [\phi(u_t+w)]^t \ \phi(u_t)\ du_t \qquad (5.33)$$

where $w = 2\ (d+\frac{1}{2})/\ [n(t+1)]^{1/2}$

The integral involved in (5.33) have been tabulated by Trawinski and David (1963) and would be helpful for computing d.

5.6 Selection procedures for problem 3.

We assume that the observed merits of t treatments $T_1, T_2 \ldots \ldots T_t$ satisfy the condition of TM model and hence the prefernece probabilities are given by equation (1.3).

As before let S_i denote the true response to the ith treatment T_i (i=1, 2,.....t). Then using the arguments of Sadasivan (1981), which in fact utilise the ideas used by Glemn and David (1960), the least square estimates S_i^* of S_i (i=2, 3,.....t), S_1 being taken as orgin are given by

$$S^* = \begin{pmatrix} S_2^* \\ S_3^* \\ \cdot \\ \cdot \\ \cdot \\ S_t^* \end{pmatrix} = (X'\ X)^{-1}\ D \qquad (5.34)$$

$$\text{where } (X'\ X)^{-1} = \frac{1}{t} \begin{bmatrix} 2 & 1 & 1 \ldots\ldots\ldots 1 \\ 1 & 2 & 1 \ldots\ldots\ldots 1 \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \\ 1 & 1 \ldots\ldots\ldots 1 & 2 \end{bmatrix}$$

$$D' = \left[\sum_j{}' D_{2j} \ldots\ldots \sum_j{}' D_{tj}\right]$$

$$D_{ij} = S_i^* - S_j^* = \sin^{-1}(2\,p_{ij}-1)$$

p_{ij} being proportion of preference for T_i over T_j.

The variance-covariance matrix associated with the vector S* is given by

$$\sum = (X'X)^{-1}/n$$

It is well known that under inverse sin transformation $\theta = \sin^{-1}(2P-1)$, where P is the observed proportion from binomial sample of size n, θ has approximate normal density, with variance 1/n, for large n. Thus D_{ij}'s are normal for large samples.

$$\mathrm{Var}\,(D_{ij}) = \mathrm{Var}\,[\sin^{-1}(2\,p_{ij}-1)] = 1/n$$

Further, for distinct pairs (i,j) and (k,l), D_{ij} and D_{kl} are independent. Thus, the joint distribution of D_{ij} is multivariate normal. Then by definition (equation (5.34)).

$$S^* = \begin{pmatrix} S_2^* \\ S_3^* \\ \vdots \\ S_t^* \end{pmatrix} \quad \text{has multivatiate}$$

normal distribution IN $(S, \sum)$ for large n. Clearly

$$\mathrm{Var}\,(S_i^*) = 2/nt \text{ and}$$

$$\mathrm{Corr}\,(S_i^*, S_j^*) = \tfrac{1}{2} \text{ for all } i \neq j = 2, 3, \ldots.t.$$

Since the interest lies in the relative merit of various treatments and in the selection of treatment with the maximum worth, selection procedures for the present Goal can be based on least square estimates of S_i (i=2,........t) given by (5.34), where S_1 is taken as the orgin. First we will discuss large sample selection procedures for Goal 1 under indifference zone formulation We define the following selection rule R_{10} for selecting the best treatment.

Rule R_{10}

Let $S_1^* = 0$

If max $(S_1^*, S_2^*, \ldots S_t^*) = S_i^*$, select T_i as the best treatment $i = 1, 2, \ldots t$. With reference to S_1 as orgin, we can set $S_1 = 0$ and then the true response to T_i is S_i $(i = 2, \ldots t)$. Without loss of generality assume $S_2 \leq S_3 \leq \ldots \leq S_t$. The complete parameter space is then

$$\Omega = \{ S_2 \leq S_3 \leq \ldots \leq S_t \}.$$

Let
$$\Omega_1 = \{ S_2 \leq S_3 \leq \ldots \leq S_t \leq 0 \}$$
$$\Omega_2 = \{ S_2 \leq S_3 \leq \ldots \leq S_t \geq 0 \}$$

Note that Ω_1 implies that T_1 is the best treatment while Ω_2 implies that T_t is the best treatment and clearly

$$\Omega = \Omega_1 \cup \Omega_2$$

Preference zone for T_1 will be $\Omega_1^* = \{ S_2 \leq S_3 \leq \ldots \leq S_{t-1} \leq S_t \leq -\alpha \}$ where $\alpha \geq 0$ and that for T_t will be $\Omega_2^* = \{ S_2 \leq S_3 \leq \ldots \leq S_{t-1} \leq S_t - \alpha , S_t \geq 0 \}$. In case T_1 is superior and the remaining (t-1) inferior treatments are equal, we have the set of slippage configuration defined by

$\Omega'_1 = \{ S_2 = S_3 = \ldots = S_t = -\alpha \}$. If T_t is the outlier, the corresponding slippage configuration will be defined by

$$\Omega'_2 = \{ S_2 = S_3 = \ldots = S_{t-1} = -\alpha, S_t = 0 \}.$$

Next, we will obtain the expression for the probability of correct selection.

Probability of correct selection

Let T_1 be best treatment, then the probability of its correct selection by rule R_{10}

$$P_{cs} = P_r\{ S_i^* < 0, \ i=2,\ldots\ldots t / \Omega_1^* \}$$

$$= P_r\{ v_i < h_i, \ i=2,\ldots\ldots t / \Omega_1^* \}$$

where $v_i = (S_i^* - S_i)/\sigma$ and $h_i = - S_i/\sigma$

with $\sigma = (2/nt)^{\frac{1}{2}}$. In Ω_1^*, $h_i > 0$ for all i. Since v_i's are equi correlated with correlation $\rho = \frac{1}{2}$, we have

$$P_{cs} = \int_{-\infty}^{\infty} \phi(u_0) \prod_{i=2}^{t} \phi(\sqrt{2}\, h_i - u_0)\, du_0 \qquad (5.35)$$

Using arguments of Stuart (1958). The infimum of the expression (5.35) in Ω_1^* attained as

$S_2 = S_3 = \ldots S_t \rightarrow \alpha$ that is at the slippage configuration Ω_1'.

$$\underset{\Omega_1'}{\text{Infimum}}\ P_{cs} = \int_{-\infty}^{\infty} \phi^{(t-1)}(\sqrt{2}\, h - u_0)\, \phi(u_0)\, du_0 \qquad (5.36)$$

where $h = \alpha/\sigma$

When T_t be the best treatment, then the probability of its correct selection is given by

$$P_{cs} = [P_r\{ S_t^* > 0, \ S_t^* - S_j^* > 0, \ j=2,\ldots.t-1\} / \Omega_2^*]$$

$$= P_r\{ Z_i > 0, \ i=2\ldots\ldots t \}$$

where $Z_t = S_t^*$, $Z_j = S_t^* - S_j^*$, $j=2,3,\ldots\ldots t-1$. Variance-covariance matirx of Z_j's, $j=2,3,\ldots\ldots t$, remains the same as that of S_j^*'s, $j=2,3,\ldots\ldots t$, and

$$P_{cs} = P_r\{ v_j > h_j \ , \ j=2,3,\ldots\ldots t\}$$

where $v_t = \dfrac{Z_t - S_t}{\sigma}$, $v_j = \dfrac{Z_j - (S_t - S_j)}{\sigma}$, $j=2,\ldots\ldots,t-1$

$h_t = - S_t/\sigma$ $h_j = - (S_t - S_j)/\sigma$ with $\sigma^2 = 2/nt$. In Ω_2^*, $h_j < 0$ for all j and we can get as in P_{cs}

$$P_{1cs} = P_r[\bigcap_{j=2}^{t} (v_j < - h_j)$$

$$= \int_{-\infty}^{\infty} \phi(u_0) \prod_{j=2}^{t} \phi(-\sqrt{2}\, h_j - u_0)\, du_0 \qquad (5.37)$$

since v_j's are equi-correlated with correlation $\rho = \frac{1}{2}$. The expression (5.37) attains its infimum when h_j's attain their maximum. This will occure when all S_j's, $j=2,3,....t-1 \rightarrow S_t - \alpha$ and then $S_t \rightarrow 0$. Hence the expression (5.37) attain its infimum at the slippage configuration Ω_2'.

Thus $\text{Inf } P_{CS} = \int_{-\infty}^{\infty} [1-\phi(u_0)] \, \phi^{(t-2)}(\sqrt{2}\, h-u_0) \, \varphi(u_0) \, du_0$ (5.38)

where $h = \alpha/\sigma$.

Comparing R.H.S. of (5.36) and (5.38) we note that

$\underset{\Omega_2^*}{\text{Inf}} \; P_{CS} \leq \underset{\Omega_4^*}{\text{Inf}} \; P_{CS}$, for all values of h. Hence

$$\underset{\Omega}{\text{Inf}} \; P_{CS} = \int_{-\infty}^{\infty} [1-\phi(u_0)] \, \phi^{t-2}(\sqrt{2}\, h-u_0) \, \phi(u_0) \, du_0 \qquad (5.39)$$

with least favourable configuration given by Ω_2^*.

For implementing the selection procedure given by rule R_{10}, the replication size n will be determined as the minimum integer 'n' for which R.H.S. of (5.39) is greater than or equal to given value of P*. A lower bound of (5.39) can be obtained by using Sidak's in equality as

$$\underset{\Omega}{\text{Inf}} \; P_{CS} \geq 1/2 \int_{-\infty}^{\infty} \phi^{(t-2)}(\sqrt{2}\, h-u_0) \, \phi(u_0) \, du_0 \qquad (5.40)$$

Next, we will give the selection procedures for selecting a subset which contain the best treatment.

<u>Selection of a subset containing the best treatment</u>

<u>Rule R_{11}</u> : Set $S_1^* = 0$. Retain the treatment T_i (i=1, 2,....t) in the selected subset if $S_i^* \geq \max S_i^* - C$, where $\max S_i^*$ is the maximum of $\{0, S_2^*, S_3^*, ...S_t^*\}$ and $C > 0$ is the selection constant to be

determined to satisfy P* conditon (1.18). Now we will obtain the expressions for the probability of correct selection for Rule R_{11}.

Probability of correct selection

If T_1 is the best treatment, that is the parameter S_i belong to Ω_1, then the probability of its correct selection P_{CS} by rule R_{11} is given by

$$P\{ S_1^* > \max S_i^* - C,\ i=2,\ldots.t/\Omega_1\}$$
$$= P(0 > \max S_i^* - C,\ i=2,3,\ldots..t)$$
$$= P(S_i^* < C,\ i=2,3,\ldots..t)$$
$$= P\{ v_i < h_i,\ i=2,3,\ldots..t\}$$

where $v_i = (S_i^* - S_i)/\sigma$ and $h_i = (C - S_i)/\sigma$. Since v_i's are equi-correlated with correlation $\rho = \frac{1}{2}$, we have

$$P_{CS} = \int_{-\infty}^{\infty} \varphi(u_0) \prod_{i=2}^{t} \phi(\sqrt{2}\, h_i - u_0)\, du_0$$

The infimum of P_{CS} in Ω_1, attained as $S_2 = S_3 = \ldots\ldots = S_t \to 0$.

Thus LFC in Ω_1 is $S_2 = S_3 \ldots\ldots = S_t = 0$

and

$$\inf_{\Omega_1} P_{CS} = \int_{-\infty}^{\infty} \phi^{t-1}(\sqrt{2}\, h - u_0)\, \varphi(u_0)\, du_0 \qquad (5.41)$$

where $h = C/\sigma$

Now if T_t is the best treatment, parameters lie in Ω_2 and probability P_{CS} of its being included in the selected subset by rule R_{11} is given by

$$P_{CS} = P\{ S_t^* > \max S_i^* - C\}$$
$$= P\{ S_t^* > -C,\ S_t^* > S_i^* - C,\ i=2,3,\ldots.t-1\}$$
$$= P\{ S_t^* > -C,\ S_t^* - S_i^* > -C,\ i=2,3,\ldots.t-1\}$$
$$= P\{ Z_j > -C,\ j=2,3,\ldots..t\}$$

where $Z_j = S_t^* - S_j^*$, $j=2,3,\ldots.t-1$, $Z_t = S_t^*$

Now $P_{cs} = P\{v_j > h_j, \quad j=2,3,\ldots..t\}$

where $v_j = \dfrac{Z_j - (S_t - S_j)}{\sigma}, \quad j=2,3,\ldots.t-1$

$v_t = (Z_t - S_t)/\sigma$ and

$h_j = -[C + (S_t - S_j)] \ /\sigma \quad j= 2,3,\ldots.t-1$

$h_t = -(C + S_t)/\sigma$ with $\sigma^2 = 2/nt.$

proceeding in the same manner as we obtained (5.37), we have

$$P_{cs} = \int_{-\infty}^{\infty} \varphi(u_0) \prod_{j=2}^{t} \phi(-\sqrt{2}\, h_j - u_0)\, du_0$$

in Ω_2.

The infimum of P_{cs} in Ω_2 attained at the configuration $S_2=S_3=\ldots.$ $\ldots.=S_t=0$ and $\inf_{\Omega_2} P_{cs}$ is the same as R.H.S. of (5.41). Thus Inf P_{cs} under R_{11} is given by (5.41). Thus the selection constant C can be fixed by equating R.H.S. of (5.41) to a given value P*. The integral involved in (5.41) have been tabulated by Gupta (1963) and would be helpful in choosing C.

CHAPTER VI

PAIRED COMPARISON MODELS INCORPORATING ORDER EFFECTS

6.1 Introduction

In all the four previous chpaters, we have considered the problem of ranking and selection using paired comparison designs. The present chapter falls apart from the remaining chapters in the sense that it does not consider any ranking or selection problem. In fact, we will be considering the extensions of some of the existing models for paired comparison experiments to incorporate order effects due to treatment specified characteristics. The problem of estimation and testing adequacy of these models would also be of interest. We assume that there are no ties.

In section (6.2), we shall discuss multiplicative order effect for BT model. Section (6.3) deals with additive order effect for TM model.

6.2 Multiplicative order effect for BT model

We will present a variant of the multiplicative order effect model due to Davidson and Beaver (1977). Consider treatments T_i, $i=1,2,\ldots.t$ for which BT model as defined in (1.5) holds. An important feature of BT model is that the value of $S_1=\ln\pi_1,\ldots$ $\ldots, S_t=\ln\pi_t$ can be used to represent the true worths of the t treatments on a linear scale. Thus, it is natural to assume log π_i, rather than π_i, themselves are affected additively by the order of presentation. Thus, when treatments T_i and T_j appear togather in a pair, their relative worths π_i and π_j would be subject to a multiplicative order effect. Hence, we postulate that the order of presentation of treatments T_i and T_j affects their ratings. Therefore, the resulting preference probabilities for the ordered pair (T_i, T_j) can be defined as

$$\pi_{i.ij} = \frac{\theta_i \pi_i}{\theta_i \pi_i + \theta_j' \pi_j}, \quad \pi_{j.ij} = \frac{\theta_j' \pi_j}{\theta_i \pi_i + \theta_j' \pi_j} \qquad (6.1)$$

and that for the ordered pair (T_j, T_i) are

$$\pi_{i.ji} = \frac{\theta_i' \pi_i}{\theta_i' \pi_i + \theta_j \pi_j}, \quad \pi_{j.ji} = \frac{\theta_j \pi_j}{\theta_i' \pi_i + \theta_j \pi_j} \qquad (6.2)$$

where $\theta_i \geq 0$, $\theta_i' \geq 0$, $i = 1, 2, \ldots\ldots t$.

$\pi_{i.ij}$ is the preference probability of T_i over T_j in the ordered pair (T_i, T_j) and $\pi_{i.ji}$ is the preference probability of T_i over T_j in the ordered pair (T_j, T_i).

Note that $\underline{\theta}_i$ and $\underline{\theta}_i'$ are parameters which represent the factor by which the relative worth π_i of treatment T_i is affected as a result of presenting T_i in the first place and second place respectively. When $\theta_i = \theta_j'$ and $\theta_i' = \theta_j$, there is no order effect for the pair (T_i, T_j).

If we let $\theta_i = \theta^*$ and $\theta_i' = \theta^{**}$ for all i in the above model, we get Davison and Beaver model (1.45) with $\beta = \theta^{**}/\theta^*$ and $\theta = 1$.

The basic difference between Davidson and Beaver model and ours is the following. Davidson and Beaver in their model assumed constant within pair order effects for all pairs. But we have introduced the order effect parameter as a specific treatment characteristic. The advantage of this is that we are able to introduce different within pair order effects which change from pair to pair.

Maximum likelihood estimation

We shall obtain ML estimates for the parameters of the above mode. Consider a paired comparison experiment in which n_{ij} independent responses are obtained for the ordered pair (T_i, T_j), the total number of comparisons being $N = \sum\sum n_{ij}$. Let $w_{ij}(1)$, $w_{ij}(2)$ be the frequencies of preference for treatment presented first (treatment T_i), the treatment presented second (treatment T_j). Clearly $n_{ij} = w_{ij}(1) + w_{ij}(2)$. Let $\sum_{j \neq i} w_{ij}(1) = K_i$, $\sum_{j \neq i} w_{ji}(2) = L_i$. Then the total number of preferences for the treatment T_i is given by $w_i = k_i + L_i$.

The joint likelihood function for the whole experiment is

$$L(\underline{\pi}, \underline{\theta}, \underline{\theta}') = \prod_{\substack{i,j \\ i \neq j}}^{t} \frac{(\theta_i \pi_i)^{w_{ij}(1)} (\theta_j' \pi_j)^{w_{ij}(2)}}{(\theta_i \pi_i + \theta_j' \pi_j)^{n_{ij}}}$$

we can write

$$\ln L(\underline{\pi}, \underline{\theta}, \underline{\theta}') = \sum_{i=1}^{t} [\, w_i \ln \pi_i + K_i \ln \theta_i + L_i \ln \theta_i'] - \sum_{i \neq j}\sum n_{ij} \ln (\theta_i \pi_i + \theta_j' \pi_j) \qquad (6.3)$$

Maximizing (6.3) subject to $\sum_i \pi_i = 1$, the ML estimates $(\hat{\underline{\pi}}, \hat{\underline{\theta}}, \hat{\underline{\theta}}')$ of $(\underline{\pi}, \underline{\theta}, \underline{\theta}')$ are obtained as the solution of the following system of equations

$$\begin{aligned} w_i/\hat{\pi}_i &= g_i\,(\hat{\underline{\pi}}, \hat{\underline{\theta}}, \hat{\underline{\theta}}') \\ \sum_{i=1}^{t} \hat{\pi}_i &= 1 \\ k_i/\hat{\theta}_i &= K_i\,(\hat{\underline{\pi}}, \hat{\underline{\theta}}, \hat{\underline{\theta}}') \\ L_i/\hat{\theta}_i' &= h_i\,(\hat{\underline{\pi}}, \hat{\underline{\theta}}, \hat{\underline{\theta}}'), \quad i=1,2,\ldots\ldots t \end{aligned} \qquad (6.4)$$

where

$$g_i\,(\hat{\underline{\pi}}, \hat{\underline{\theta}}, \hat{\underline{\theta}}') = \sum_{\substack{j \neq i \\ j=1}}^{t} \left[\frac{n_{ij}\, \hat{\theta}_i}{\hat{\theta}_i \hat{\pi}_i + \hat{\theta}_j' \hat{\pi}_j} + \frac{n_{ji}\, \hat{\theta}_i'}{\hat{\theta}_i' \hat{\pi}_i + \hat{\theta}_j \hat{\pi}_j} \right.$$

$$K_i(\hat{\underline{\pi}}, \hat{\underline{\theta}}, \hat{\underline{\theta}}') = \sum_{\substack{j \neq i \\ j=1}}^{t} (n_{ij}\hat{\pi}_i)/(\hat{\theta}_i\hat{\pi}_i + \hat{\theta}'_j\hat{\pi}_j)$$

$$h_i(\hat{\underline{\pi}}, \hat{\underline{\theta}}, \hat{\underline{\theta}}') = \sum_{\substack{j \neq i \\ j=1}}^{t} (n_{ji}\hat{\pi}_i)/(\hat{\theta}'_i\hat{\pi}_i + \hat{\theta}_j\hat{\pi}_j)$$

For solving this system of equations, one need consider any (t-1) of the first t equations in the system, since the first 't' equations are linearly dependent.

The following is the proposed iterative scheme for obtaining the solution to the likelihood equation (6.4).

$$\text{Let} \quad \rho_i \begin{cases} = \pi_i, & i=1,2,\ldots\ldots t \\ = \theta_{i-t}, & i= t+1,\ldots\ldots 2t \\ = \theta'_{i-2t}, & i=2t+1,\ldots\ldots 3t \end{cases}$$

$$G_i(\hat{\rho}) \quad \begin{cases} = g_i(\hat{\pi}, \hat{\theta}, \hat{\theta}'), & i=1,2,\ldots\ldots t \\ = k_{i-t}(\hat{\pi}, \hat{\theta}, \hat{\theta}') & i=t+1,\ldots\ldots 2t \\ = h_{i-2t}(\hat{\pi}, \hat{\theta}, \hat{\theta}'), & i=2t+1,\ldots\ldots 3t \end{cases}$$

$$v_i \quad \begin{cases} = w_i, & i= 1,2,\ldots\ldots t \\ = k_{i-t}, & i= t+1,\ldots\ldots 2t \\ = L_{i-2t}, & i= 2t+1,\ldots\ldots 3t \end{cases}$$

The iterations are indexed by M, M=1,2,...., since one revised value of estimate $\hat{\underline{\rho}}$ of $\underline{\rho}$ is obtained for each value of M. M iterations would be completed in 3Mt stages, starting the first iteration in stage 1, second in stage 3t+1, and so on, the Mth in stage 3(M-1)t+1. Thus each iteration passes through 3t stages, Let $\hat{\underline{\rho}}^{(n)}$ be the estimate of $\underline{\rho}$ in the nth stage. A new estimate of $\underline{\rho}$ is generated cyclically through change of one element of $\hat{\underline{\rho}}$ per stage. The (n+1)th stage value of $\hat{\underline{\rho}}^{(n+1)}$ is obtained from the nth stage value $\hat{\rho}^{(n)}$ through replacement of the $\hat{\rho}_i^{(n)}$ for which $i=(n+1) \bmod 3t$.

Then $\hat{\rho}_i^{(n+1)} = v_i/G_i(\hat{\rho}^{(n)})$

Iteration is to continued till two successive sets values are sufficiently close. As an initial estimates, one may use

$$\hat{\pi}_i^{(0)} = 1/t$$

$$\hat{\theta}_i^{(0)} = 1$$

$$\hat{\theta}_i'^{(0)} = 1, \quad i=1,2,......t$$

Tests of hypothesis

Three testing situations are of interest. They are (i) a test for presence of order effect, (ii) test for equality of order effect and (iii) a test of the appropriateness of a particular model. Large sample tests based on the likelihood ratio statistic are easy to perform when conducted in conjunction with the estimation procedure described in the previous section.

Test for the presence of order effect

The hypothesis of no order effect $H_0: \theta_i = \theta_j'$, $i,j=1,2,....t$, $i \neq j$ is to be tested against the general alternative $H_1: \theta_i \neq \theta_j'$ for atleast one pair (ij), $i,j=1,2,.....t$ $i \neq j$. Under H_0 likelihood function is independent of θ_i and θ_j' and one can use the estimation procedure of the preceding section to obtain ML estimates of π_0 of π under H_0 and corresponding likelihood $L(\hat{\pi}_0)$. The test is then conducted using the statistic.

$$S_1 = -2\ln\lambda_1 = 2\left[\ln L(\hat{\pi}, \hat{\theta}, \hat{\theta}') - \ln L(\hat{\pi}_0)\right]$$

Under H_0, the statistic S_1 has limiting chi-square distribution with 2t degree of freedom (d.f.).

Test for equality of order effect

The hypothesis of equal order effect H_0: $\theta_i = \theta$, $\theta'_j = \theta'$, $i,j=1,2,\ldots.t$ is to be tested against the general alternative H_1: $\theta_i \neq \theta$ or $\theta'_j \neq \theta'$ for all $i,j=1,2,\ldots.t$. One can use the estimation procedure of the preceding section to obtain ML estimates of $\hat{\pi}_0$, $\hat{\theta}_0$, $\hat{\theta}'_0$ under H_0 and corresponding likelihood $L(\hat{\pi}_0, \hat{\theta}_0, \hat{\theta}'_0)$. The test is then conducted using the statistic.

$$S_2 = -2 \ln \Lambda_2 = 2 \left[\ln L(\hat{\underline{\pi}}, \hat{\underline{\theta}}, \hat{\underline{\theta}}') - \ln L(\hat{\pi}_0, \hat{\theta}_0, \hat{\theta}'_0)\right]$$

Under H_0, the statistic S_2 has a limiting chi-square distribution with $2(t-1)$ d.f.

Test for the goodness of fit of the model

As a test of the goodness of the proposed model, we compare the observed numbers in each category with expected numbers obtained by using the likelihood estimates of π_i, θ_i, θ'_i, $i=1,2\ldots\ldots t$ in the expression (6.1) and (6.2) and multiplying by n_{ij} and n_{ji} respectively. Let $w'_{ij}(1)$, $w'_{ij}(2)$ be the expected frequencies of preference for the treatment presented first (treatment T_i), the treatment presented second (treatment T_j). Then

$$n_{ij} = w'_{ij}(1) + w'_{ij}(2)$$

In testing the null hypothesis that the observed and expected numbers are in agreement, we employ the chi-square statistic which will take the form

$$X_1^2 = \sum_{i \neq j} \left[\{ (w_{ij}(1) - w'_{ij}(1))^2 / w'_{ij}(1)\} + \{ (w_{ij}(2) - w'_{ij}(2))^2 / w'_{ij}(2)\}\right]$$

For sufficiently large values of expected numbers, the quantity X_1^2 is distributed approximately as a chi-square with $2\binom{t}{2}+1-3t$ degrees of freedom.

Example: Beaver and Gokhale (1975) have presented the results of weight judging experiment with wide mouth pill bottles filled with appropriate amounts of lead shot and paraffin and identical in all respects except for their weights. Each of 50 subjects responded to all 20 ordered pairs. For each pair the bottles were lifted only once in order and the subject expressed a judgement as to whether the first or the second bottle felt heavier when lifted. The frequencies of judgement are summarized in table 1.

Table 1

Summary of responses for the weights judging experiment

(i,j)	$w_{ij}(1)$	$w_{ij}(2)$	(i,j)	$w_{ij}(1)$	$w_{ij}(2)$
(1,2)	14	36	(2,4)	7	43
(2,1)	32	18	(4,2)	43	7
(1,3)	6	44	(2,5)	2	48
(3,1)	36	14	(5,2)	46	4
(1,4)	2	48	(3,4)	12	38
(4,1)	47	3	(4,3)	34	16
(1,5)	1	49	(3,5)	5	45
(5,1)	47	3	(5,3)	40	10
(2,3)	14	36	(4,5)	10	40
(3,2)	34	16	(5,4)	28	22

The ML estimates of ($\underline{\pi}$, $\underline{\theta}$, $\underline{\theta}'$) are tabulated below:

Treatment	$\hat{\pi}$	$\hat{\theta}$	$\hat{\theta}'$
1	0.0470	0.3245	0.5664
2	0.0742	0.6217	0.5321
3	0.1263	0.6649	0.9085
4	0.2409	1.0680	1.2510
5	0.5116	0.8179	1.8353

The test statistics for the three hypothesis described in section (6.4) and their associated critical levels are presented in table 2'.

Table 2'

Summary of tests of hypothesis

Test	d.f.	Statistic	Critical level
No order effect	10	$S_1 = 16.0758$	0.098
Equal order effect	8	$S_2 = 5212.8218$	0.00001
Goodness of fit	6	$x^2 = 1.9654$	0.9320

For this data order effect parameters are significant, order effects are significantly different and the fit of the model is good.

Remark: The present model is more suited to this data than Davidson and Beaver model. However, it should be mentioned that this x^2 value may have been little higher than expected due to the fact that several cell frequencies were quite small.

In the next section we will be considering a modified Thurstone Mosteller model incorporating order effect in paired comparisons.

6.3 Additive order effect for Thurstone-Mosteller model

TM model was introduced in section 1.2 and large sample selection procedures based on this model was developed in Section (5.6). In these no order effect or tie effect was assumed. This model was modified for tie effect by Glenn and David (1960) (see review section (1.4) of the thesis). Our aim here is to modify the model so as to incorporate order effect. As before, we consider 't' treatments $T_1, T_2, \ldots\ldots T_t$ having true merits $S_1, S_2, \ldots\ldots, S_t$ respectively represented on a linear scale. It is reasonable to assume that if order effect is present and treatments (T_i, T_j) are presented in this order, respective merits, S_i, S_j would change to $S_i + \theta_i$, $S_j + \theta_j'$. where θ_i, θ_j' are some real parameters. The paramerter θ_i represents amount of perturbation in the worth of treatment T_i, which was presented first while parameter θ_j' represents this perturbation in the treatment T_j which was presented second. Thus according to TM model difference $Y_i - Y_j$ between the observed merits (sensations) by the judge would follow a normal density with mean $S_i + \theta_i - S_j - \theta_j'$ and unit variance and therefore preference probabilities can be defined by

$$\begin{aligned} \pi_{i.ij} &= P(Y_i > Y_j) = \phi(S_i - S_j + \theta_i - \theta_j') \\ \pi_{j.ij} &= 1 - \pi_{i.ij} \\ \pi_{i.ji} &= P(Y_i > Y_j) = \phi(S_i - S_j + \theta_i' - \theta_j) \\ \pi_{j.ji} &= 1 - \pi_{i.ji} \quad \text{for all } i,j=1,2,\ldots.t. \end{aligned} \tag{6.5}$$

Suppose there are n observations each for the ordered pair (T_i, T_j) let the proportion of preference for T_i in the ordered pair (T_i, T_j) be

$$p_{i.ij} = n_{i.ij}/n$$

and the proportion preference for T_i in the ordered pair (T_j, T_i) be

$$P_{i.ji} = n_{i.ji}/n$$

for all $i, j = 1, 2, \ldots, t$, $i \neq j$, where $n_{i.ij}$, $n_{i.ji}$ are number of preference for T_i in the ordered pair (T_i, T_j) and (T_j, T_i) respectively and

$$n_{i.ij} + n_{j.ij} = n$$
$$n_{i.ji} + n_{j.ji} = n$$

Replacing parameters by their estimates in the first and third equations of (6.5) we have

$$P_{i.ij} = \phi(\hat{S}_i - \hat{S}_j + \hat{\theta}_i - \hat{\theta}_j')$$
$$P_{i.ji} = \phi(\hat{S}_i - \hat{S}_j + \hat{\theta}_i' - \hat{\theta}_j)$$

where $\hat{S}_i$, $\hat{S}_j$, $\hat{\theta}_i$, $\hat{\theta}_i'$, $\hat{\theta}_j$, $\hat{\theta}_j'$ are respectively the estimated values of S_i, S_j, θ_i, θ_j, θ_j', θ_i'. We may now write

$$\hat{S}_i - \hat{S}_j + \hat{\theta}_i - \hat{\theta}_j' = \phi^{-1}(P_{i.ij})$$
$$\hat{S}_i - \hat{S}_j + \hat{\theta}_i' - \hat{\theta}_j = \phi^{-1}(P_{i.ji}) \qquad (6.6)$$

Using the approximation $\phi(x) = \frac{1}{2}(1 + \sin x)$, where x represents an angle in radian measure, such that $-\pi/2 \leq x \leq \pi/2$, in equation (6.6) we have

$$\hat{S}_i - \hat{S}_j + \hat{\theta}_i - \hat{\theta}_j' = \sin^{-1}(2P_{i.ij} - 1)$$
$$\hat{S}_i - \hat{S}_j + \hat{\theta}_i' - \hat{\theta}_j = \sin^{-1}(2P_{i.ji} - 1)$$

for all $i \neq j$, $i, j = 1, 2, \ldots, t$ (6.7)

$$\text{Var}\left[\sin^{-1}(2P_{i.ij} - 1)\right] = \text{Var}\left[\sin^{-1}(2P_{i.ji} - 1)\right] = 1/n$$

We assume that $\hat{\theta}_i - \hat{\theta}_j' = -(\hat{\theta}_i' - \hat{\theta}_j)$. This assumption imply that change in the difference between the true worths of any pair of treatments say, (T_i, T_j), due to presenting T_i first is equal in magnitude, but opposite in sign to that presenting T_i second. Then $\hat{\theta}_i + \hat{\theta}_i' = \hat{\theta}_j + \hat{\theta}_j'$. Let $\hat{\theta}_i + \hat{\theta}_i' = \beta$, $i = 1, 2, \ldots, t$. Then the total

number of parameters in the model will be 2t+1, viz. (θ_1, θ_2......
...θ_t, β , S_1, S_2,......S_t). Under the assumption made above (6.7) can be written as

$$\hat{S}_i - \hat{S}_j + \hat{\theta}_i + \hat{\theta}_j - \beta = \sin^{-1}(2P_{i.ij}-1)$$

$$\hat{S}_i - \hat{S}_j - \hat{\theta}_i - \hat{\theta}_j + \beta = \sin^{-1}(2P_{i.ji}-1)$$

These equations can be solved as

$$\hat{\theta}_i + \hat{\theta}_j - \beta = \tfrac{1}{2}[\sin^{-1}(2P_{i.ij}-1) - \sin^{-1}(2P_{i.ji}-1)] = G_{ij} \qquad (6.8)$$

$$\hat{S}_i - \hat{S}_j = \tfrac{1}{2}[\sin^{-1}(2P_{i.ij}-1) + \sin^{-1}(2P_{i.ji}-1)] = H_{ij} \qquad (6.9)$$

Using these equations, we would find least square estimates of the parameters. Since our interest is in the relative merit of the treatments, we can set $S_1 = 0$ and $\theta_1 = 1$ with out loss of generality. First we will estimate $\hat{\theta}_i$ (i=2,3,....t),β , $\hat{\theta}_1$ being take as 1. Writting system (6.8) in the form

$$XB = Y$$

where Y is a column vector of size $\frac{t(t-1)}{2}$ i.e.

$$Y' = [G_{12}^{-1}, G_{13}^{-1}, \ldots G_{1t}^{-1}, G_{ij} \ldots\ldots G_{t-1,t}]$$

and B'represent 1 x t row vector of θ_i and β.

$$B' = [\theta_2, \theta_3, \ldots\ldots \theta_t, \beta]$$

Finally X is a $\binom{t}{2}$x t matrix consisting of 1's, - 1's and 0's. Typically for 5 treatments it has the following form

$$X = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 & -1 \\ 1 & 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & 1 & -1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{bmatrix}$$

Then the required vector of least squares estimates is

$$B^{*} = (X'X)^{-1} X' Y$$

X'X is of order t has the following form

$$X'X = \begin{bmatrix} (t-1) & 1 & 1 & 1 \cdots\cdots & 1 & -(t-1) \\ 1 & (t-1) & 1 & 1 \cdots\cdots & 1 & -(t-1) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 1 & 1 \cdots\cdots & (t-1) & -(t-1) \\ -(t-1) & -(t-1) & \cdots & \cdots & -(t-1) & \frac{t(t-1)}{2} \end{bmatrix}$$

and

$$(X'X)^{-1} = \frac{1}{(t-2)} \begin{bmatrix} 2 & 1 & 1 & 1 \cdots\cdots & 2 \\ 1 & 2 & 1 & 1 \cdots\cdots & 2 \\ 1 & 1 & 2 & 1 \cdots\cdots & 2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & 2 \\ 2 & 2 & 2 \cdots\cdots & 2 & \alpha \end{bmatrix}$$

where $\alpha = \frac{4k-1}{k}$ when $t = 2k+1$

$= \frac{8k-6}{2k-1}$ when $t=2k$ for all $t > 2$

Similarly using (6.9) and taking $S_1=0$ we can obtain the least square estimates

$$B_1^{*} = (X_1^{'} X_1)^{-1} X_1^{'} Y_1 \quad \text{where}$$

$$B_1^{'} = (S_2, S_3 \ldots\ldots\ldots S_t)$$

$$Y_1^{'} = [H_{12}, \ldots\ldots H_{ij} \ldots\ldots H_{t-1,t}] \quad \text{and}$$

$$(X_1^{'}X_1)^{-1} = \frac{1}{t} \begin{bmatrix} 2 & 1 \ldots\ldots\ldots 1 \\ 1 & 2 \ldots\ldots\ldots 1 \\ \ldots & \ldots\ldots\ldots\ldots \\ 1 & 1 \ldots\ldots\ldots 2 \end{bmatrix}$$

Next, we will discusses a test for the goodness of fit of the model.

<u>Test for goodness of fit</u>

As in section (6.2), we compare the observed numbers in each category with expected numbers obtained by using the least square estimates of S_i, θ_i, $i=1,2,\ldots\ldots t$ and β in the expression

$$P_{i.ij} = \tfrac{1}{2} [\, 1 + \sin (\hat{S}_i - \hat{S}_j + \hat{\theta}_i + \hat{\theta}_j - \beta)\,]$$

$$P_{i.ji} = \tfrac{1}{2} [\, 1 + \sin (\hat{S}_i - \hat{S}_j + \beta - \hat{\theta}_i - \hat{\theta}_j)]$$

for all $i \neq j$, $i,j=1,2,\ldots\ldots t$ and multiplying by n. The appropriateness of the model (6.5) can be assessed through use of the ratio U of the ML under this model to that for the most general model possible, a multinomial model with t(t-1) independent components. As in the case of Davidson (1970), the goodness of fit of the model (6.5) is tested using the statistic

$$U = -2\ln \Lambda = 2 \sum_{i \neq j} [n_{i.ij} \ln (n_{i.ji}/n^{!}_{i.ij}) + n_{i.ji} \ln (n_{i.ji}/n^{!}_{i.ji})]$$

where $n^{!}_{i.ij}$ and $n^{!}_{i.ji}$ are the expected number of preference for T_i in the ordered pair (T_i, T_j) and (T_j, T_i) respectively. Under the assumed model U has a limiting chi-square with $2\binom{t}{2} + 1-2t$ degrees of freedom.

Example: For this example, we will use the same data which we have used in section (6.2). The proportions of preferences are summarised in table 3'.

Table 3'

Proportion of preferences for the weights judging experiment

(i, j)	$P_{i.ij}$	$P_{j.ij}$
(1, 2)	0.28	0.72
(2, 1)	0.64	0.36
(1, 3)	0.12	0.88
(3, 1)	0.72	0.28
(1, 4)	0.04	0.96
(4, 1)	0.94	0.06
(1, 5)	0.02	0.98
(5, 1)	0.94	0.06
(2, 3)	0.28	0.72
(3, 2)	0.68	0.32
(2, 4)	0.14	0.86
(4, 2)	0.86	0.14
(2, 5)	0.04	0.96
(5, 2)	0.92	0.08
(3, 4)	0.24	0.76
(4, 3)	0.68	0.32
(3, 5)	0.10	0.90
(5, 3)	0.80	0.20
(4, 5)	0.20	0.80
(5, 4)	0.56	0.44

The least square estimates of $(\underline{S}, \underline{\theta})$ and β are tabulated below.

Treatment	$\hat{s}$	$\hat{\theta}$
1	0.0	1.0
2	0.2808	1.0754
3	0.6322	0.9592
4	1.0668	0.9832
5	1.3528	0.9490

$\beta = 2.1012$

The fit of the model is adequate for the data at hand (U=10.1847 for 11 d.f.).

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C H A R T S

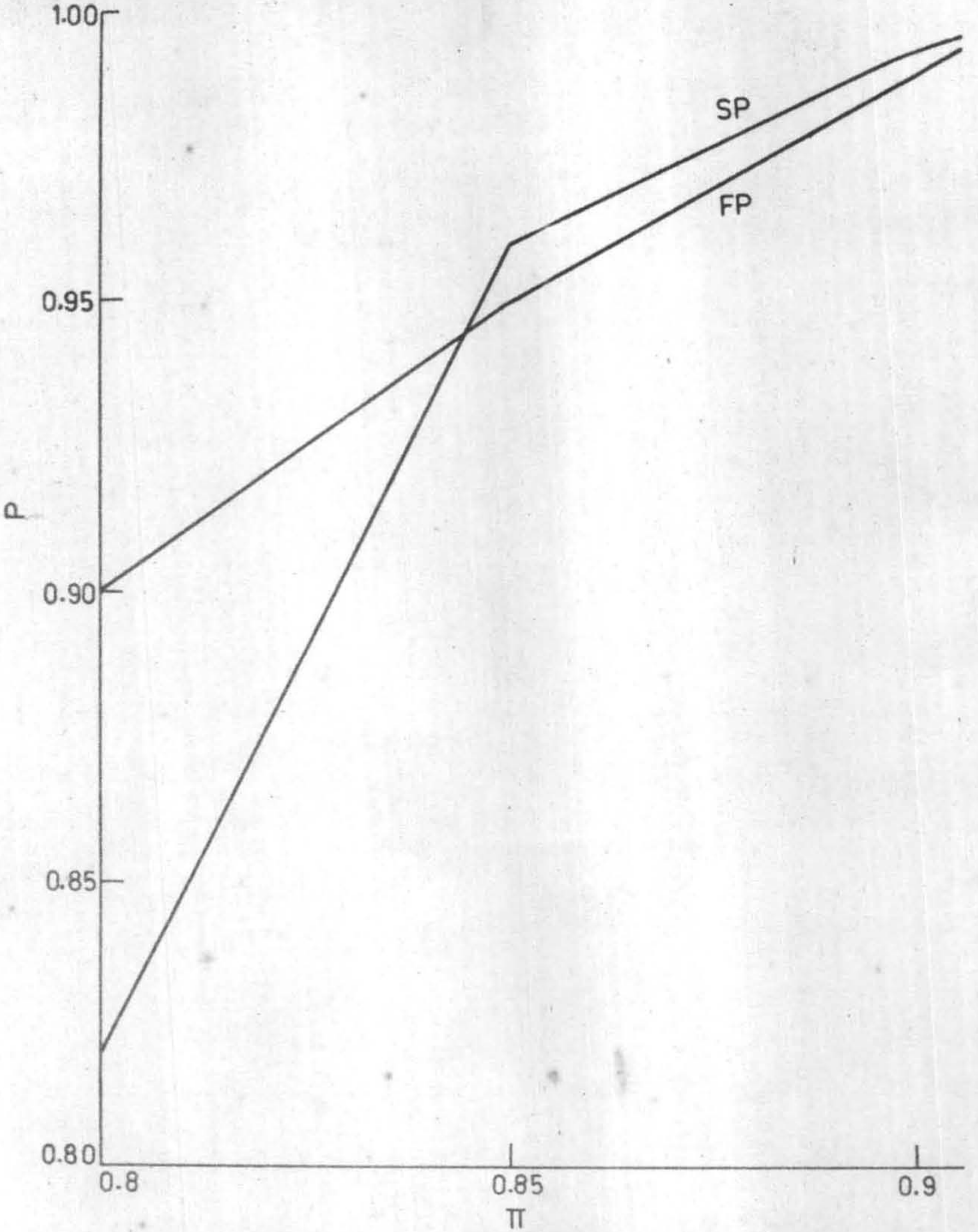

Chart 1: Probability of correct selection of the superior treatment for $t = 14$ and $N = 182$.

Present the main conclusions from these charts.

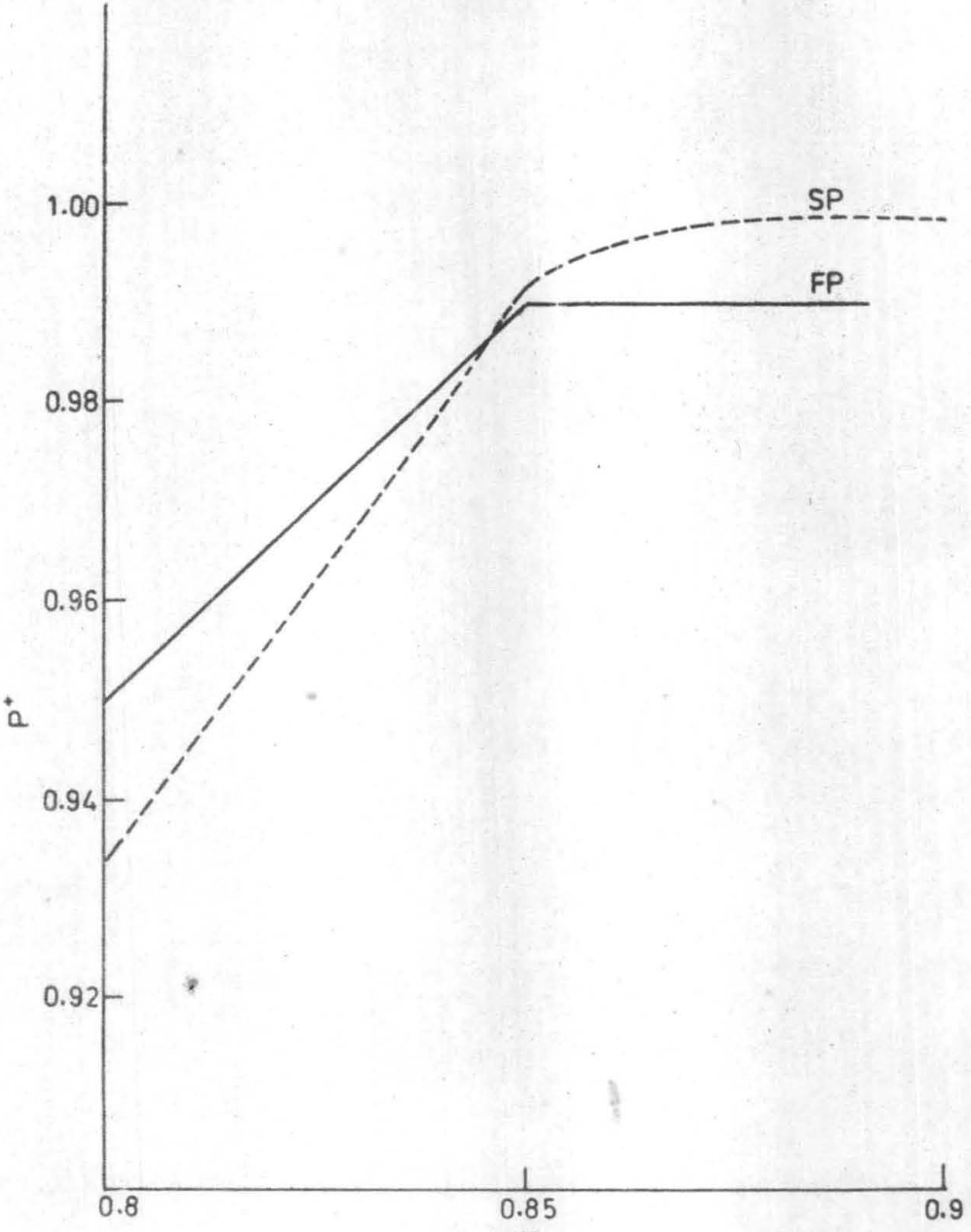

Chart 2: Probability of correct selection of the superior treatment for t = 20 and N=380

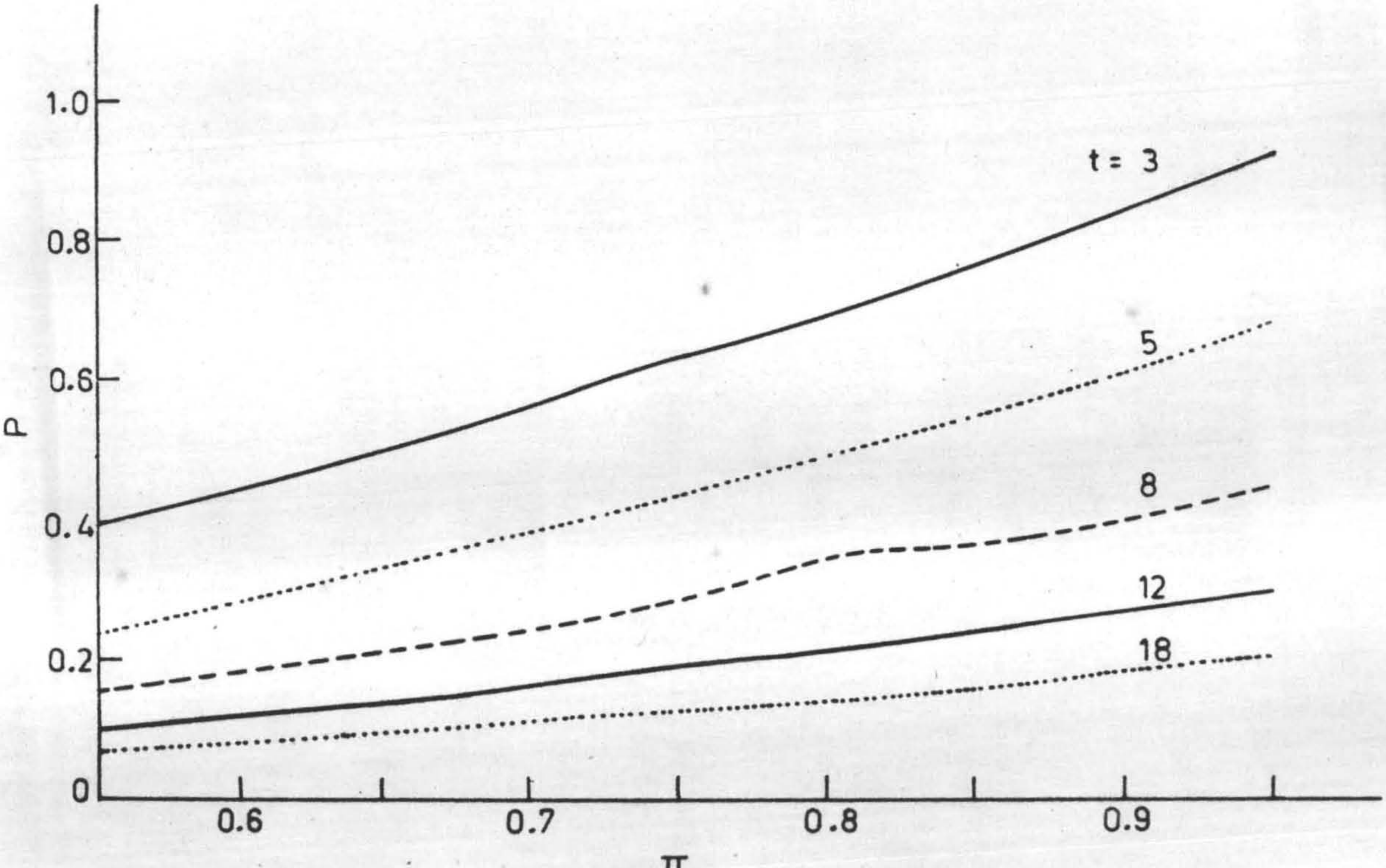

Chart 3: Probability for correct selection n = 1

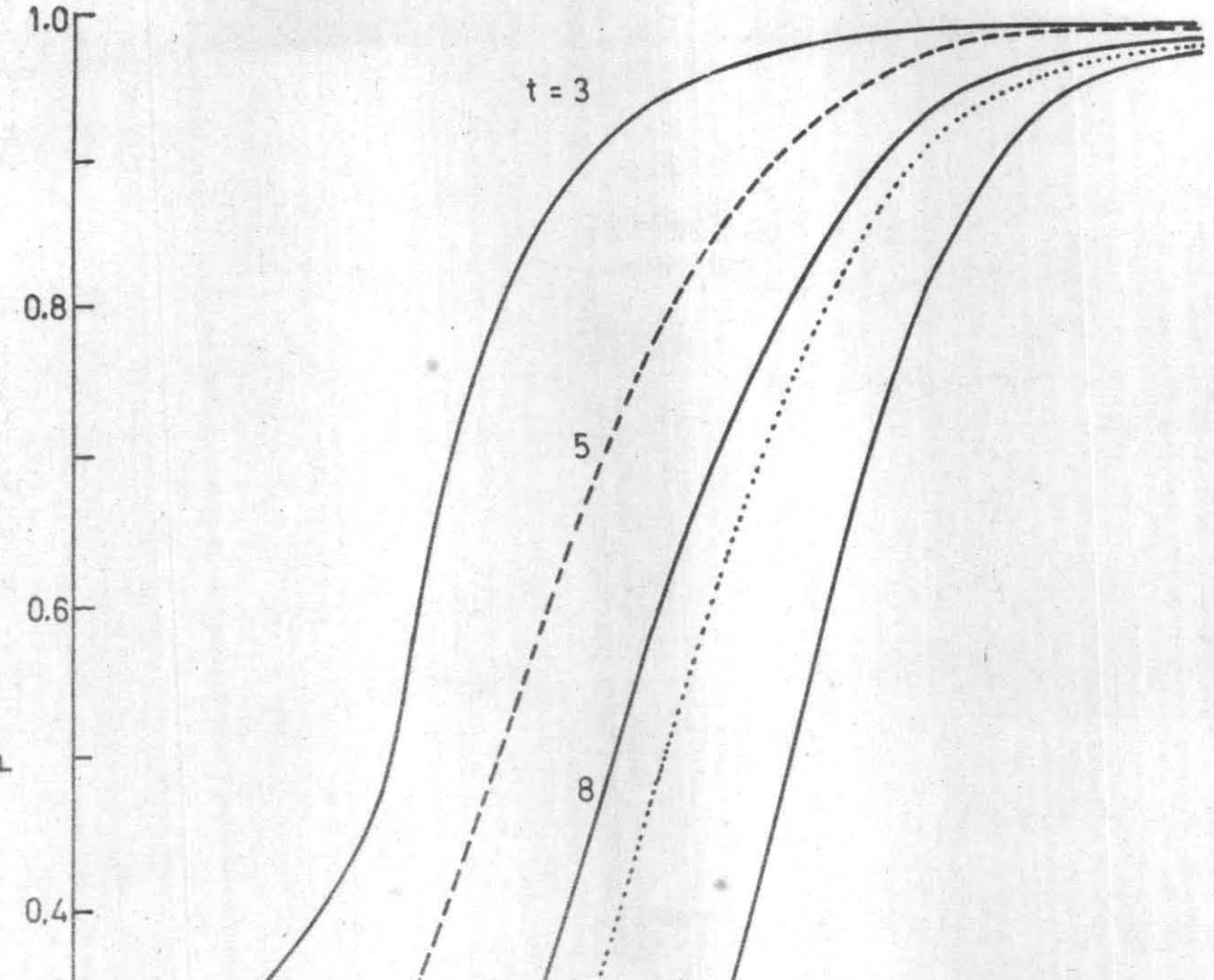

Chart 4: Probability of correct selection n = 10

omit

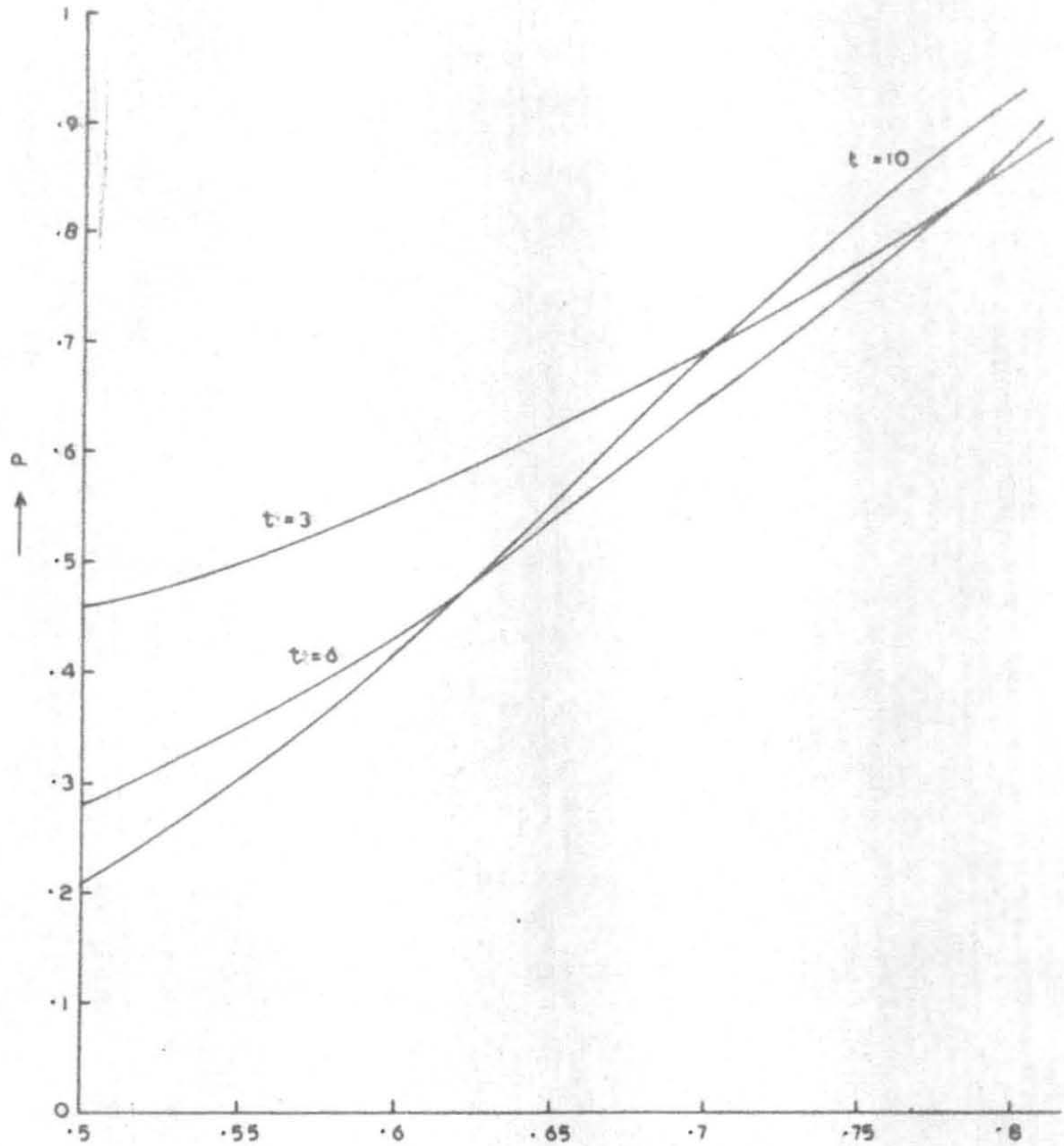

CHART-5 : PROBABILITY OF CORRECT SELECTION $n = 1$, $\theta' = \cdot 15$, $\theta = \cdot 2$

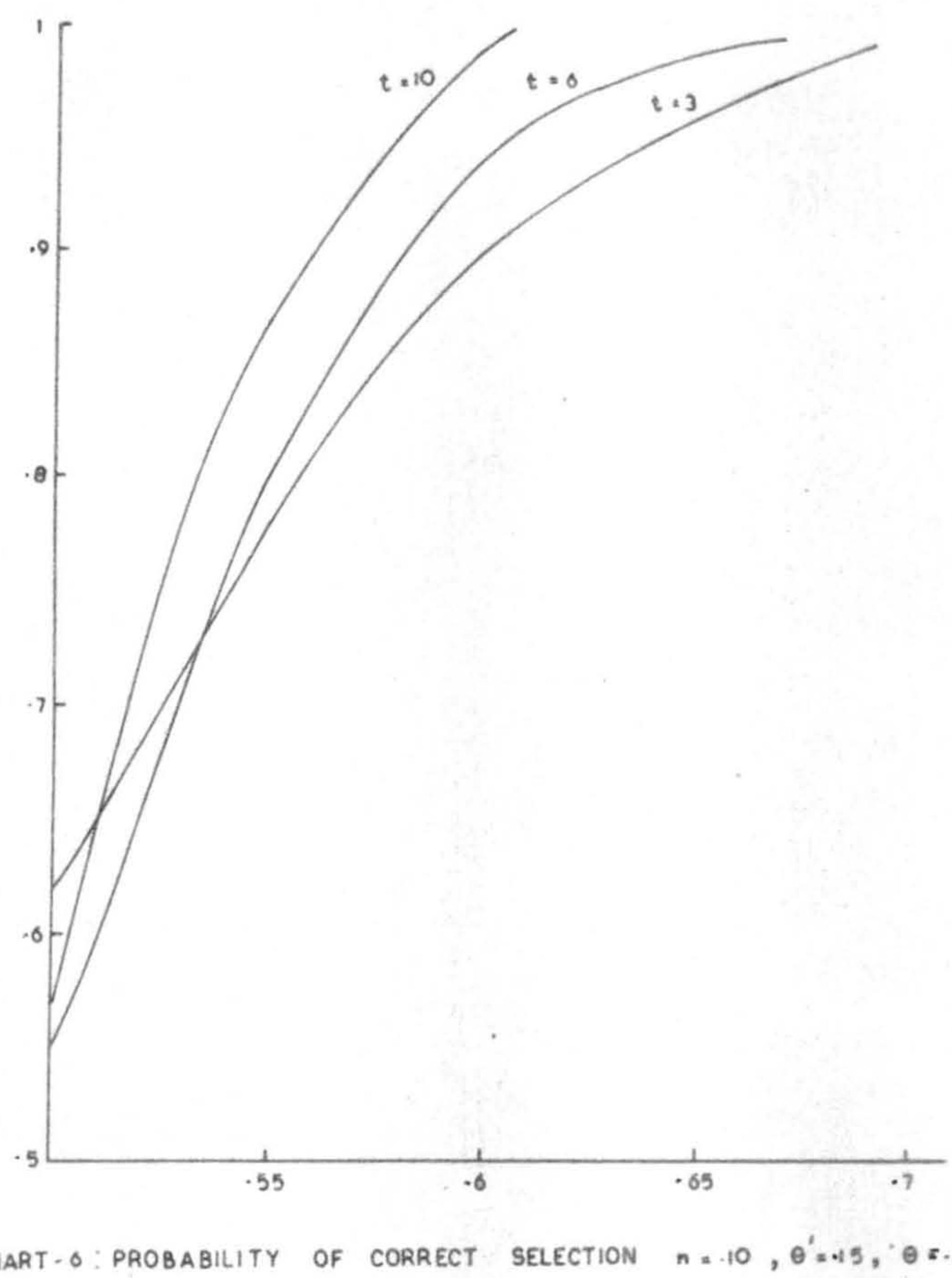

CHART-6 : PROBABILITY OF CORRECT SELECTION $n = 10$, $\theta' = \cdot 15$, $\theta = \cdot 2$

T A B L E S

Table - 1

Partitions of scores and their frequencies in symmetrical paired-comparison experiment

(a) No. of replication = 1.

t = 3

(210)	6
1^3	2
Total	8

t = 4

(2200)	2
$21^2\,0$	12
1^4	2
Total	16

t = 5

(22100)	10
$21^3\,0$	20
1^5	2
Total	32

t = 6

(222000)	2
$2^2\,1^2\,00$	30
$2\,1^4\,0$	30
1^6	2
Total	64

t=7

(2221000)	14
$2^2\,1^3\,00$	70
$2\,1^5\,0$	42
1^7	2
Total	128

t = 8

(22220000)	2
$2^3\,1^2\,000$	56
$2^2\,1^4\,00$	140
$2\,1^6\,0$	56
1^8	2
Total	256

t = 9

(222210000)	18
$2^3\,1^3\,000$	168
$2^2\,1^5\,00$	252
$21^7\,0$	72
1^9	2
Total	512

Table-1(Contd.)

t = 10

(2222200000)	2
$2^4\ 1^2$ 0000	90
$2^3\ 1^4$ 000	420
$2^2\ 1^6$ 00	420
$2\ 1^8$ 0	90
1^{10}	2
Total	1024

t = 11

(22222100000)	22
$2^4\ 1^3$ 0000	330
$2^3\ 1^5$ 000	924
$2^2\ 1^7$ 00	660
$2\ 1^9$ 0	110
1^{11}	2
Total	2048

t = 12

(222222000000)	2
$2^5\ 1^2$ 00000	132
$2^4\ 1^4$ 0000	990
$2^3\ 1^6$ 000	1848
$2^2\ 1^8$ 00	990
$2\ 1^{10}$ 0	132
1^{12}	2
Total	4096

t = 13

(2222221000000)	26
$2^5\ 1^3$ 00000	572
$2^4\ 1^5$ 0000	2574
$2^3\ 1^7$ 000	3432
$2^2\ 1^9$ 00	1430
$2\ 1^{11}$ 0	156
1^{13}	2
Total	8192

t = 14

(2222222 0000000)	2
$2^6\ 1^2$ 000000	182
$2^5\ 1^4$ 00000	2002
$2^4\ 1^6$ 0000	6006
$2^3\ 1^8$ 000	6006
$2^2\ 1^{10}$ 00	2002
$2\ 1^{12}$ 0	182
1^{14}	2
Total	16384

t = 15

(222222210000000)	30
$2^6\ 1^3$ 000000	910
$2^5\ 1^5$ 00000	6006
$2^4\ 1^7$ 0000	12870
$2^3\ 1^9$ 000	10010
$2^2\ 1^{11}$ 00	2730
$2\ 1^{13}$ 0	210
1^{15}	2
Total	32768

Table-1(Contd.)

t = 16

(2222222200000000)	2
$2^7\ 1^2 0000000$	240
$2^6\ 1^4 000000$	3640
$2^5\ 1^6 00000$	16016
$2^4\ 1^8 0000$	25740
$2^3\ 1^{10} 000$	16016
$2^2\ 1^{12} 00$	3640
$2\ 1^{14} 0$	240
1^{16}	2
Total	65536

t = 17

(22222222100000000)	34
$2^7\ 1^3\ 0000000$	1360
$2^6\ 1^5\ 000000$	12376
$2^5\ 1^7\ 00000$	38896
$2^4\ 1^9\ 0000$	48620
$2^3\ 1^{11} 000$	24752
$2^2\ 1^{13} 00$	4760
$2\ 1^{15}\ 0$	272
1^{17}	2
Total	131072

Table-1(Contd.)

(b) No. of replication = 2

t = 3

(420)	6
41^2	6
$3^2 0$	6
3 2 1	36
2^3	10
Total	64

t = 4

(4400)	2
4310	16
$42^2 0$	12
$4\ 2\ 1^2$	32
$3^2 20$	32
$32^2\ 1$	112
$3^2\ 1^2$	32
2^4	18
Total	256

t = 5

(44200)	1.0
$4^2\ 1^2\ 0$	1.0
$43^2\ 00$	1.0
43210	100
431^3	40
$42^3 0$	20
$42^2\ 1^2$	110
$3^3\ 1\ 0$	40
$3^2\ 2^2\ 0$	110
$3^2\ 2\ 1^2$	240
$3\ 2^3\ 1$	300
2^5	34
Total	1024

t = 6

(444000)	2
$4^2\ 3100$	24
$4^2\ 2^2\ 0$	30
$4^2\ 21^2 0$	72
$4^2 1^4$	12
$43^2 200$	72
$43^2\ 1^2\ 0$	120
$432^2\ 10$	384
4321^3	336
$42^4\ 0$	30
$42^3\ 1^2$	312
$3^4\ 00$	12
$3^3\ 210$	336
$3^3\ 1^3$	128
$3^2\ 2^3\ 0$	312
$3^2\ 2^2\ 1^2$	1104
$3\ 2^4\ 1$	744
2^6	66
Total	4096

Table-1(Contd.)

t = 7

(4442000)	14
$4^3\ 1^2\ 00$	14
$4^2\ 3^2\ 000$	14
$4^2\ 32100$	196
$4^2\ 3\ 1^3\ 0$	112
$4^2\ 2^3\ 00$	70
$4^2\ 2^2\ 1^2\ 0$	308
$4^2\ 2\ 1^4$	112
$4\ 3^3\ 100$	112
$4\ 3^2\ 2^2\ 00$	308
$4\ 3^2\ 21^2\ 0$	1120
$4\ 3^2\ 1^4$	224
$4\ 3\ 2^3\ 10$	1176
$432^2\ 1^3$	1680
$4\ 2^5\ 0$	42
$4\ 2^4\ 1^2$	798
$3^4\ 200$	112
$3^4\ 1^2\ 0$	224
$3^3\ 2^2\ 10$	1680
$3^3\ 2\ 1^3$	1344
$3^2\ 2^4\ 0$	798
$3^2\ 2^3\ 1$	4032
$3\ 2^5\ 1$	1764
2^7	130
Total	16384

Table-2 Least number n of repetitions required to implement the selection procedure R_1

P	t \ Π	.55	.60	.65	.70	.75	.80	.85	.90	.95
1	2	3	4	5	6	7	8	9	10	11
.75	3	149	36	16	9	3*	2*	1*	1*	1*
	4	222	54	23	13	8	3*	2*	1*	1*
	5	276	68	30	16	10	6	4	2*	2*
	6	319	79	34	19	11	8	5	4	2*
	7	356	88	38	21	13	8	6	4	2*
	8	384	95	41	23	14	9	6	4	3
	9	411	101	44	24	15	10	7	5	3
	10	435	107	46	25	16	10	7	5	4
	12	473	117	50	28	17	11	8	5	4
	14	504	125	54	30	18	12	8	6	4
	16	531	131	57	31	19	13	9	6	4
	18	555	137	60	33	20	13	9	6	5
	20	574	142	62	34	21	14	9	7	5
.90	3	244	61	26	14	8	5	3*	2*	1*
	4	341	85	36	20	12	8	5	4	2*
	5	411	100	44	24	14	9	6	5	3
	6	460	114	49	27	16	11	7	5	4
	7	500	123	54	29	18	12	8	6	4
	8	536	132	57	31	19	13	9	6	4
	9	564	139	60	33	20	13	9	6	5
	10	588	145	63	34	21	14	9	7	5

Table-2(Contd.)

	12	627	155	67	37	23	15	10	7	5
	14	663	164	71	39	24	16	11	8	5
0.90	16	689	170	74	40	25	16	11	8	6
	18	715	176	77	42	26	17	12	8	6
	20	737	182	79	43	26	17	12	8	6
	3	332	81	35	19	11	7	5	2*	2*
	4	452	112	48	26	16	10	7	5	3
	5	531	131	57	31	19	12	8	6	4
	6	588	145	63	34	21	14	9	7	5
	7	632	156	68	37	23	15	10	7	5
	8	668	165	72	39	24	16	11	8	5
.95	9	699	173	74	41	25	16	11	8	6
	10	720	178	77	42	26	17	12	8	6
	12	764	188	82	45	27	18	12	9	6
	14	803	195	86	47	29	19	13	9	6
	16	832	205	89	49	30	19	13	9	7
	18	855	211	92	50	31	20	14	9	7
	20	878	217	94	51	31	21	14	10	7

* Values based on exact theory.

Table-5(Contd.)

	19	10	11	11	11	11	11	12	12	12	12	12	12	12	12	12	12	12	12
	20	11	11	11	11	12	12	12	12	12	12	12	12	12	13	13	13	13	13
	25	12	12	12	13	13	13	13	13	13	14	14	14	14	14	14	14	14	14
	30	13	13	14	14	14	14	15	15	15	15	15	15	15	15	15	16	16	16
.90	35	14	14	15	15	15	16	16	16	16	16	16	16	16	17	17	17	17	17
	40	15	15	16	16	16	17	17	17	17	17	17	18	18	18	18	18	18	18
	45	16	16	17	17	17	18	18	18	18	18	18	19	19	19	19	19	19	19
	50	17	17	18	18	18	19	19	19	19	19	19	20	20	20	20	20	20	20
	60	18	19	19	20	20	20	21	21	21	21	21	21	22	22	22	22	22	22
	70	20	20	21	21	22	22	22	22	23	23	23	23	23	23	24	24	24	24
	80	21	22	22	23	23	23	24	24	24	24	25	25	25	25	25	25	26	26
	90	22	23	24	24	25	25	25	25	26	26	26	26	26	27	27	27	27	27
	100	24	24	25	25	26	26	26	27	27	27	27	28	28	28	28	28	29	29

* Values based on exact theory.

Table-2(Contd.)

P	t II	.55	.60	.65	.70	.75	.80	.85	.90	.95
1	2	3	4	5	6	7	8	9	10	11
	3	515	126	54	29	17	11	7	4	3
	4	694	171	74	40	24	16	11	7	5
	5	792	195	84	46	28	18	12	9	6
	6	855	211	92	50	31	20	14	10	7
	7	908	223	97	53	32	21	14	10	7
.99	8	944	232	101	55	34	22	15	11	8
	9	975	239	104	57	35	23	16	11	8
	10	1008	248	107	58	36	23	16	11	8
	12	1076	259	112	61	37	24	17	12	8
	14	1084	267	116	63	39	25	17	12	8
	16	1117	275	120	65	40	26	18	13	8
	18	1145	282	122	67	41	27	18	13	8
	20	1165	287	125	68	41	27	19	13	9

Table-3

Comparison between NA and NB

P*	t	$\bar{\Pi}$ = .55		$\bar{\Pi}$ = .75		$\bar{\Pi}$ = .95	
	NA	916.4288		32.2362		6.7678	
		NB	NB/NA	NB	NB/NA	NB	NB/NA
.99	3	515	.5629	17	.5274	3	.4433
	5	792	.8642	28	.8686	6	.8865
	7	906	.9908	32	.9927	7	1.0343
	9	975	1.0639	35	1.0857	8	1.1821
	12	1076	1.1741	37	1.1478	8	1.1821
	16	1117	1.2188	40	1.2408	8	1.1821
	20	1165	1.2712	41	1.2719	9	1.3298
	NA	697.8134		24.5462		5.1533	
.97	3	391	.5603	13	.5296	2	.3881
	5	612	.8770	22	.8963	5	.9702
	7	720	1.0316	26	1.0592	6	1.1643
	9	786	1.1264	28	1.1407	6	1.1643
	12	855	1.2252	31	1.2629	7	1.3583
	16	920	1.3184	33	1.3444	7	1.3583
	20	969	1.3886	35	1.4259	8	1.5524

Table-4 Size of Experiment N

P* / t	n̄	.90 FP	.90 SP	.95 FP	.95 SP
	12	66	60	66	48
.75	14	91	84	91	56
	16	120	96	120	64
	18	153	108	153	90
	20	190	140	190	100
	12	66	84	66	60
.90	14	91	112	91	70
	16	120	128	120	96
	18	153	144	153	108
	20	190	160	190	120
	12	132	108	66	72
.95	14	91	126	91	84
	16	120	144	120	120
	18	153	162	153	126
	20	190	200	190	140
	12	132	132	132	96
.99	14	182	168	91	112
	16	240	208	120	128
	18	306	234	153	144
	20	380	260	190	180

Table:-5 Table of ϑ for the decision rule R_2

P_s	n \ t	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
	1	1*	1*	1*	1*	1*	1*	1*	1*	1*	1*	1*	1*	1*	1*	1*	1	1	1
	2	2*	2*	2*	2*	2*	3	3	3	3	3	3	3	3	3	3	4	4	4
	3	3	3	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
	4	4	4	4	4	4	4	5	5	5	5	5	5	5	5	5	5	5	5
	5	4	4	5	5	5	5	5	5	5	5	5	5	5	5	6	6	6	6
	6	5	5	5	5	5	5	6	6	6	6	6	6	6	6	6	6	6	6
	7	5	5	5	6	6	6	6	6	6	6	6	6	6	6	7	7	7	7
	8	5	6	6	6	6	6	6	6	7	7	7	7	7	7	7	7	7	7
	9	6	6	6	6	7	7	7	7	7	7	7	7	7	7	7	7	8	8
	10	6	6	6	7	7	7	7	7	7	7	8	8	8	8	8	8	8	8
	11	6	7	7	7	7	7	7	8	8	8	8	8	8	8	8	8	8	8
	12	6	7	7	7	8	8	8	8	8	8	8	8	8	8	9	9	9	9
.75	13	7	7	7	8	8	8	8	8	8	8	9	9	9	9	9	9	9	9
	14	7	7	8	8	8	8	8	9	9	9	9	9	9	9	9	9	9	9
	15	7	8	8	8	8	8	9	9	9	9	9	9	9	9	10	10	10	10
	16	7	8	8	8	9	9	9	9	9	9	10	10	10	10	10	10	10	10
	17	8	8	8	9	9	9	9	9	10	10	10	10	10	10	10	10	10	10
	18	8	8	9	9	9	9	10	10	10	10	10	10	10	10	10	11	11	11
	19	8	9	9	9	9	10	10	10	10	10	10	10	11	11	11	11	11	11

Table-5(Contd.)

	20	8	9	9	9	10	10	10	10	10	11	11	11	11	11	11	11	11	11
	25	9	10	10	11	11	11	11	11	12	12	12	12	12	12	12	12	13	13
	30	10	11	11	12	12	12	12	13	13	13	13	13	13	13	14	14	14	14
	35	11	12	12	13	13	13	14	14	14	14	14	14	14	14	15	15	15	15
	40	12	12	13	13	14	14	14	15	15	15	15	15	15	15	16	16	16	16
	45	12	13	14	14	15	15	15	15	16	16	16	16	16	16	17	17	17	17
0.75	50	13	14	14	15	15	16	16	16	16	17	17	17	17	17	17	18	18	18
	60	14	15	16	16	17	17	17	18	18	18	18	19	19	19	19	19	19	20
	70	15	16	17	18	18	19	19	19	19	20	20	20	20	20	21	21	21	21
	80	17	18	18	19	19	20	20	21	21	21	21	22	22	22	22	22	22	23
	90	18	19	19	20	21	21	21	22	22	22	23	23	23	23	23	24	24	24
	100	18	20	20	21	22	22	23	23	23	24	24	24	24	24	25	25	25	25

* Values based on exact theory.

Table-5(Contd.)

P_b	n \ t	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
	1	2*	2*	2*	2*	2*	2*	2*	2*	2*	2*	2*	2*	2*	2*	2*	2*	2	2
	2	4*	4*	4*	4*	4*	4	4	4	4	4	4	4	4	4	4	4	4	4
	3	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
	4	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6
	5	6	6	6	6	6	7	7	7	7	7	7	7	7	7	7	7	7	7
	6	7	7	7	7	7	7	7	7	7	7	7	7	7	8	8	8	8	8
	7	7	7	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8
0.95	8	8	8	8	8	8	8	8	8	8	9	9	9	9	9	9	9	9	9
	9	8	8	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9
	10	9	9	9	9	9	9	9	9	9	10	10	10	10	10	10	10	10	10
	11	9	9	9	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10
	12	10	10	10	10	10	10	10	10	10	10	11	11	11	11	11	11	11	11
	13	10	10	10	10	10	11	11	11	11	11	11	11	11	11	11	11	11	11
	14	10	10	11	11	11	11	11	11	11	11	11	11	11	12	12	12	12	12
	15	11	11	11	11	11	11	11	12	12	12	12	12	12	12	12	12	12	12
	16	11	11	11	12	12	12	12	12	12	12	12	12	12	12	12	12	12	13
	17	11	12	12	12	12	12	12	12	12	12	13	13	13	13	13	13	13	13
	18	12	12	12	12	12	13	13	13	13	13	13	13	13	13	13	13	13	13

Table-5(Contd.)

	20	8	9	9	9	10	10	10	10	10	11	11	11	11	11	11	11	11	11
	25	9	10	10	11	11	11	11	11	12	12	12	12	12	12	12	12	13	13
	30	10	11	11	12	12	12	12	13	13	13	13	13	13	13	14	14	14	14
	35	11	12	12	13	13	13	14	14	14	14	14	14	14	14	15	15	15	15
	40	12	12	13	13	14	14	14	15	15	15	15	15	15	15	16	16	16	16
	45	12	13	14	14	15	15	15	15	16	16	16	16	16	16	17	17	17	17
0.75	50	13	14	14	15	15	16	16	16	16	17	17	17	17	17	17	18	18	18
	60	14	15	16	16	17	17	17	18	18	18	18	19	19	19	19	19	19	20
	70	15	16	17	18	18	19	19	19	19	20	20	20	20	20	21	21	21	21
	80	17	18	18	19	19	20	20	21	21	21	21	22	22	22	22	22	22	23
	90	18	19	19	20	21	21	21	22	22	22	23	23	23	23	23	24	24	24
	100	18	20	20	21	22	22	23	23	23	24	24	24	24	24	25	25	25	25

* Values based on exact theory.

Table-5(Contd.)

P_s	n | t	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
	1	2*	2*	2*	2*	2*	2*	2*	2*	2*	2*	2*	2*	2*	2*	2*	2	2	2
	2	3*	3*	3*	3*	3*	4	4	4	4	4	4	4	4	4	4	4	4	4
	3	4	4	4	4	4	5	5	5	5	5	5	5	5	5	5	5	5	5
	4	5	5	5	5	5	5	5	5	5	5	5	6	6	6	6	6	6	6
	5	5	5	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6
	6	6	6	6	6	6	6	6	7	7	7	7	7	7	7	7	7	7	7
	7	6	6	7	7	7	7	7	7	7	7	7	7	7	7	7	8	8	8
	8	7	7	7	7	7	7	7	8	8	8	8	8	8	8	8	8	8	8
0.90	9	7	7	7	8	8	8	8	8	8	8	8	8	8	8	8	9	9	9
	10	7	8	8	8	8	8	8	8	9	9	9	9	9	9	9	9	9	9
	11	8	8	8	8	9	9	9	9	9	9	9	9	9	9	9	9	9	10
	12	8	8	9	9	9	9	9	9	9	9	9	10	10	10	10	10	10	10
	13	9	9	9	9	9	9	10	10	10	10	10	10	10	10	10	10	10	10
	14	9	9	9	10	10	10	10	10	10	10	10	10	10	10	11	11	11	11
	15	9	9	10	10	10	10	10	10	10	11	11	11	11	11	11	11	11	11
	16	9	10	10	10	10	11	11	11	11	11	11	11	11	11	11	11	11	11
	17	10	10	10	10	11	11	11	11	11	11	11	11	11	12	12	12	12	12
	18	10	10	11	11	11	11	11	11	11	12	12	12	12	12	12	12	12	12

Table-5(Contd.)

P_b	n \ t	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
	1	2*	2*	2*	2*	2*	2*	2*	2*	2*	2*	2*	2*	2*	2*	2*	2*	2	2
	2	4*	4*	4*	4*	4*	4	4	4	4	4	4	4	4	4	4	4	4	4
	3	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
	4	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6
	5	6	6	6	6	6	7	7	7	7	7	7	7	7	7	7	7	7	7
	6	7	7	7	7	7	7	7	7	7	7	7	7	7	8	8	8	8	8
	7	7	7	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8
0.95	8	8	8	8	8	8	8	8	8	8	9	9	9	9	9	9	9	9	9
	9	8	8	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9
	10	9	9	9	9	9	9	9	9	9	10	10	10	10	10	10	10	10	10
	11	9	9	9	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10
	12	10	10	10	10	10	10	10	10	10	10	11	11	11	11	11	11	11	11
	13	10	10	10	10	10	11	11	11	11	11	11	11	11	11	11	11	11	11
	14	10	10	11	11	11	11	11	11	11	11	11	11	11	12	12	12	12	12
	15	11	11	11	11	11	11	11	12	12	12	12	12	12	12	12	12	12	12
	16	11	11	11	12	12	12	12	12	12	12	12	12	12	12	12	12	12	13
	17	11	12	12	12	12	12	12	12	12	12	13	13	13	13	13	13	13	13
	18	12	12	12	12	12	13	13	13	13	13	13	13	13	13	13	13	13	13

Table-5(Contd.)

	19	15	15	15	15	15	15	15	15	15	16	16	16	16	16	16	16	16	16
	20	15	15	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16
	25	17	17	17	17	18	18	18	18	18	18	18	18	18	18	18	18	18	18
	30	19	19	19	19	19	19	19	19	19	20	20	20	20	20	20	20	20	20
	35	20	20	21	21	21	21	21	21	21	21	21	21	21	21	21	21	22	22
	40	22	22	22	22	22	22	22	22	22	23	23	23	23	23	23	23	23	23
0.99	45	23	23	23	23	23	24	24	24	24	24	24	24	24	24	24	24	24	24
	50	24	24	25	25	25	25	25	25	25	25	25	25	25	26	26	26	26	26
	60	27	27	27	27	27	27	27	27	28	28	28	28	28	28	28	28	28	28
	70	29	29	29	29	29	29	30	30	30	30	30	30	30	30	30	30	30	31
	80	31	31	31	31	31	31	32	32	32	32	32	32	32	32	32	33	33	33
	90	33	33	33	33	33	33	33	34	34	34	34	34	34	34	34	34	35	35
	100	34	34	35	35	35	35	35	35	36	36	36	36	36	36	36	36	36	36

* Values based on exact theory.

Table.5(Contd.)

	19	12	12	12	13	13	13	13	13	13	13	13	13	13	13	14	14	14	14
	20	12	13	13	13	13	13	13	13	13	14	14	14	14	14	14	14	14	14
	25	14	14	14	14	15	15	15	15	15	15	15	15	15	15	15	16	16	16
	30	15	15	16	16	16	16	16	16	16	17	17	17	17	17	17	17	17	17
	35	16	17	17	17	17	17	18	18	18	18	18	18	18	18	18	18	18	19
0.95	40	17	18	18	18	18	19	19	19	19	19	19	19	19	20	20	20	20	20
	45	18	19	19	19	19	20	20	20	20	20	20	20	21	21	21	21	21	21
	50	19	20	20	20	21	21	21	21	21	21	21	22	22	22	22	22	22	22
	60	21	22	22	22	22	23	23	23	23	23	24	24	24	24	24	24	24	24
	70	23	23	24	24	24	25	25	25	25	25	25	26	26	26	26	26	26	26
	80	25	25	25	26	26	26	26	27	27	27	27	27	27	28	28	28	28	28
	90	26	27	27	27	28	28	28	28	28	29	29	29	29	29	29	30	30	30
	100	28	28	28	29	29	29	30	30	30	30	30	31	31	31	31	31	31	31

* Values based on exact theory.

Table-5(Contd.)

P_s	n \ t	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
	1	2*	2*	2*	2*	2*	2*	2*	2*	2*	2*	2*	2*	2*	2*	2*	2	2	2
	2	4*	4*	4*	4*	4*	4	4	4	4	4	4	4	4	4	4	4	4	4
	3	5*	5	5	5	6	6	6	6	6	6	6	6	6	6	6	6	6	6
	4	6	6	6	6	6	6	6	6	7	7	7	7	7	7	7	7	7	7
	5	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	8	8	8
	6	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8
0.975	7	8	8	8	8	8	8	9	9	9	9	9	9	9	9	9	9	9	9
	8	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	10	10
	9	9	9	9	9	10	10	10	10	10	10	10	10	10	10	10	10	10	10
	10	10	10	10	10	10	10	10	10	10	10	10	10	10	11	11	11	11	11
	11	10	10	10	10	11	11	11	11	11	11	11	11	11	11	11	11	11	11
	12	11	11	11	11	11	11	11	11	11	11	11	11	11	12	12	12	12	12
	13	11	11	11	11	11	11	12	12	12	12	12	12	12	12	12	12	12	12
	14	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	13	13	13
	15	12	12	12	12	12	12	13	13	13	13	13	13	13	13	13	13	13	13
	16	12	12	13	13	13	13	13	13	13	13	13	13	13	13	13	13	13	13
	17	13	13	13	13	13	13	13	13	13	14	14	14	14	14	14	14	14	14
	18	13	13	13	13	14	14	14	14	14	14	14	14	14	14	14	14	14	14

Table-5(Contd.)

	19	13	14	14	14	14	14	14	14	14	14	14	14	14	15	15	15	15	15
	20	14	14	14	14	14	14	14	15	15	15	15	15	15	15	15	15	15	15
	25	15	16	16	16	16	16	16	16	16	16	16	17	17	17	17	17	17	17
	30	17	17	17	17	17	18	18	18	18	18	18	18	18	18	18	18	18	18
	35	18	18	19	19	19	19	19	19	19	19	19	20	20	20	20	20*	20	20
	40	19	20	20	20	20	20	20	21	21	21	21	21	21	21	21	21	21	21
0.975	45	21	21	21	21	21	22	22	22	22	22	22	22	22	22	22	23	23	23
	50	22	22	22	22	23	23	23	23	23	23	23	23	23	24	24	24	24	24
	60	24	24	24	24	25	25	25	25	25	25	26	26	26	26	26	26	26	26
	70	26	26	26	26	27	27	27	27	27	27	28	28	28	28	28	28	28	28
	80	27	28	28	28	28	29	29	29	29	29	29	30	30	30	30	30	30	30
	90	29	30	30	30	30	30	31	31	31	31	31	31	31	32	32	32	32	32
	100	31	31	31	32	32	32	32	32	33	33	33	33	33	33	33	34	34	34

* Values based on exact theory.

Table-5(Contd.)

p	s n \ t	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
	1	2*	2*	2*	2*	2*	2*	2*	2*	2*	2*	2*	2*	2*	2*	2*	2	2	2
	2	4*	4*	4*	4*	4*	4	4	4	4	4	4	4	4	4	4	4	4	4
	3	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6
	4	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7
	5	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8
	6	8	8	8	8	8	8	8	9	9	9	9	9	9	9	9	9	9	9
.99	7	9	9	9	9	9	9	9	9	9	9	9	9	10	10	10	10	10	10
	8	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10
	9	10	10	10	10	10	10	11	11	11	11	11	11	11	11	11	11	11	11
	10	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	12	12
	11	11	11	11	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12
	12	12	12	12	12	12	12	12	12	12	12	12	12	12	13	13	13	13	13
	13	12	12	12	13	13	13	13	13	13	13	13	13	13	13	13	13	13	13
	14	13	13	13	13	13	13	13	13	13	13	13	13	13	14	14	14	14	14
	15	13	13	13	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14
	16	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	15	15	15
	17	14	14	14	14	14	14	15	15	15	15	15	15	15	15	15	15	15	15
	18	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15

....7

Table-6 partition of scores and their frequencies in F.P. design with lies <u>t=3</u> <u>n=1</u>

Partition			Frequency	Y	X
2	1	0	6	0	0
1	1	1	2	0	0
2	½	½	3	0	1
1½	1	½	6	1	1
1½	1½	0	3	1	1
1½	1	½	6	1	2
1	1	1	1	2	3
		Total	27		

<u>t=3</u> <u>n=2</u>

4	2	0	6	0	0
4	1	1	6	0	0
2	2	2	10	0	0
3	2	1	36	0	0
3	3	0	6	0	0
4	3/2	½	12	0	1
3	3/2	3/2	24	0	1
3	5/2	½	24	0	1
4	1	1	3	0	2
3	2	1	12	0	2
2	2	2	8	0	2
7/2	3/2	1	24	1	1
5/2	2	3/2	60	1	1
7/2	2	½	12	1	1
7/2	5/2	0	12	1	1
5/2	5/2	1	24	1	1

Table-6 (Contd.)

7/2	3/2	1	24	1	2
7/2	2	½	24	1	2
5/2	5/2	1	24	1	2
5/2	2	3/2	72	1	2
7/2	3/2	1	12	1	3
5/2	5/2	1	12	1	3
5/2	2	3/2	24	1	3
3	2	1	18	2	2
3	3	0	3	2	2
3	5/2	½	24	2	2
3	3/2	3/2	24	2	2
3	3/2	3/2	12	2	3
3	5/2	½	12	2	3
3	2	1	48	2	3
2	2	2	16	2	3
3	2	1	6	2	4
2	2	2	10	2	4
3	3/2	3/2	12	2	4
5/2	2	3/2	24	3	3
5/2	5/2	1	12	3	4
5/2	2	3/2	24	3	4
5/2	2	3/2	12	3	5
2	2	2	1	4	6
					729

Table-7 Number of replications 'n' to implement Selection rule R_3 .

P	ϕ_3	ϕ'	t \ $\bar{\Pi}$	.55	.6	.65	.7	.75	.8	.85	.9
			3	27	10	5	3	2	2	1	1
			4	29	10	5	3	2	2	1	1
			5	28	10	5	3	2	2	1	1
			6	26	10	5	3	2	2	1	1
.75	.1	.05	7	25	9	5	3	2	2	1	1
			8	23	9	5	3	2	1	1	1
			9	22	8	4	3	2	1	1	1
			10	22	8	4	3	2	1	1	1
			3	16	7	4	2	2	1	1	
			4	16	7	4	3	2	1	1	
			5	15	7	4	3	2	1	1	
.75	.15	.1	6	14	6	4	2	2	1	1	
			7	13	6	4	2	2	1	1	
			8	12	6	3	2	2	1	1	
			9	11	5	3	2	2	1	1	
			10	10	5	3	2	2	1	1	

Table-7 (Contd.)

P	ϕ	ϕ'	$t \backslash \bar{\Pi}$	.5	.55	.6	.65	.7	.75	.8
			3	25	9	5	3	2	1	1
			4	24	9	5	3	2	1	1
			5	23	9	5	3	2	1	1
.75	.2	.15	6	23	9	4	3	2	1	1
			7	22	8	4	3	2	1	1
			8	21	8	4	3	2	1	1
			9	20	7	4	2	2	1	1
			10	19	7	4	2	2	1	1
			3	13	6	3	2	1	1	
			4	14	6	3	2	2	1	
			5	13	6	3	2	2	1	
.75	.25	.2	6	12	6	3	2	2	1	
			7	12	5	3	2	2	1	
			8	11	5	3	2	1	1	
			9	11	5	3	2	1	1	
			10	10	5	3	2	1	1	

P	ϕ	ϕ'	$t \backslash \bar{\Pi}$	.45	.5	.55	.6	.65	.7
			3	21	8	4	2	2	1
			4	23	8	4	3	1	1
.75	.3	.25	5	22	8	4	3	1	1

Table-7(Contd.)

P	Ø	Ø'	t \ Π	.45	.5	.55	.6	.65	.7
			6	21	8	4	2	1	1
			7	19	8	4	2	1	1
.75	.3	.25	8	18	7	4	2	1	1
			9	18	7	3	2	1	1
			10	17	7	3	2	1	1

P	Ø	Ø'	t \ Π	.45	.5	.55	.6	.65
			3	11	5	3	2	1
			4	12	5	3	2	1
			5	11	5	3	2	1
.75	.35	.3	6	11	5	3	2	1
			7	10	5	3	2	1
			8	10	4	3	2	1
			9	10	4	3	2	1
			10	10	4	2	2	1

			t \ Π	.4	.45	.5	.55	.6
			3	19	7	3	2	1
			4	19	7	4	2	2
			5	19	7	4	2	2
.75	.4	.35	6	18	7	3	2	2
			7	17	6	3	2	1
			8	16	6	3	1	1
			9	15	6	3	1	1
			10	14	5	3	1	1

Table-7(Contd.)

P	Ø	Ø'	t \ $\bar{\Pi}$	.55	.6	.65	7.	.75	.8	.85	.9
			3	66	23	12	7	4	3	2	1
			4	60	21	11	6	4	3	2	1
.9	.1	.05	5	54	19	10	6	4	3	2	1
			6	49	18	9	5	3	3	2	1
			7	45	16	8	5	3	2	2	1
			8	41	15	8	5	3	2	2	1
			9	39	14	7	4	3	2	2	1
			10	36	13	7	4	3	2	2	1

			3	35	15	8	5	3	2	1
			4	32	14	8	5	3	2	2
			5	29	13	7	4	3	2	1
.9	.15	.1	6	26	12	6	4	3	2	1
			7	24	11	6	4	3	2	1
			8	22	10	6	4	2	2	1
			9	20	9	5	3	2	2	1
			10	19	9	5	3	2	2	1

			t \ $\bar{\Pi}$	.5	.55	.6	.65	.7	.75	.8
			3	59	21	10	6	4	2	2
			4	54	19	9	6	4	2	2
			5	48	17	9	5	3	2	2
.9	.20	.15	6	44	16	8	5	3	2	2

Table-7(Contd.)

P	Ø	Ø'	t | Π	.5	.55	.6	.65	.7	.75	.8
			7	40	14	7	4	3	2	2
			8	37	13	7	4	3	2	1
.9	.20	.15	9	34	12	6	4	3	2	1
			10	32	12	6	4	2	2	1
			3	31	13	7	4	3	2	
			4	28	12	7	4	3	2	
			5	26	11	6	4	3	2	
			6	23	10	6	4	2	2	
.9	.25	.2	7	21	9	5	3	2	2	
			8	20	9	5	3	2	2	
			9	18	8	5	3	2	2	
			10	17	8	4	3	2	1	

P	Ø	Ø'	t | Π	.45	.5	.55	.6	.65	.7
			3	52	18	9	5	3	2
			4	47	17	8	5	3	2
			5	42	15	8	4	3	2
			6	39	14	7	4	3	2
			7	35	13	6	4	3	2
.9	.3	.25	8	32	12	6	4	2	2
			9	30	11	6	4	2	2
			10	27	10	5	3	2	2

Table-7(Contd.)

P	Ø	Ø'	t \ $\bar{\Pi}$	.45	.5	.55	.6	.65
			3	27	12	6	4	2
			4	24	11	6	4	2
			5	22	10	5	3	2
.90	.35	.3	6	20	9	5	3	2
			7	18	8	5	3	2
			8	17	8	4	3	2
			9	16	7	4	3	2
			10	15	7	4	3	2
			t \ $\bar{\Pi}$	.4	.45	.5	.55	.6
			3	45	16	8	4	3
			4	41	15	7	4	3
			5	37	13	6	4	2
.90	.4	.35	6	33	12	6	4	2
			7	30	11	6	3	2
			8	28	10	5	3	2
			9	26	19	5	3	2
			10	24	9	5	3	2

Table-7 (Contd.)

P	Ø	Ø'	t \ Π	.55	.6	.65	.7	.75	.8	.85	.9
			3	97	34	17	10	6	4	3	2
			4	94	30	15	9	5	4	2	2
			5	75	26	13	8	5	3	2	2
.95	.1	.05	6	67	24	12	7	5	3	2	2
			7	60	21	11	6	4	3	2	2
			8	55	20	10	6	4	3	2	1
			9	51	18	9	6	4	3	2	1
			10	47	17	9	5	3	2	2	1

			3	51	22	13	7	4	3	2
			4	45	19	10	6	4	3	2
			5	39	17	9	6	4	3	2
.95	.15	.1	6	35	15	8	5	3	2	2
			7	32	14	8	5	3	2	2
			8	29	13	7	4	3	2	2
			9	27	12	7	4	3	2	2
			10	25	11	6	4	3	2	2

			t \ Π	.5	.55	.6	.65	.7	.75	.8
			3	87	30	15	8	5	3	2
.95	.2	.15	4	76	27	13	8	5	3	2
			5	67	24	12	7	4	3	2

Table-7(Contd.)

P	Ø	Ø'	t \ Π	.5	.55	.6	.65	.7	.75	.8
			6	59	21	11	6	4	3	2
.95	.2	.15	7	54	19	10	6	4	3	2
			8	49	17	9	5	3	2	2
			9	45	16	8	5	3	2	2
			10	42	15	8	5	3	2	2
			3	45	19	10	6	4	2	
			4	40	16	9	6	4	2	
			5	35	15	8	5	3	2	
			6	31	14	7	5	3	2	
.95	.25	.2	7	28	12	7	4	3	2	
			8	26	11	6	4	3	2	
			9	24	10	6	4	3	2	
			10	22	10	5	3	2	2	

P	Ø	Ø'	t \ Π	.45	.5	.55	.6	.65	.7
			3	76	27	13	7	4	3
			4	66	23	11	7	4	3
			5	58	21	10	6	4	2
.95	.3	.25	6	52	18	9	5	3	2
			7	47	17	8	5	3	2
			8	43	15	8	5	3	2
			9	40	14	7	4	3	2
			10	37	13	7	4	3	2

Table-7 (Contd.)

P	Ø	Ø'	t \ $\bar{\Pi}$	.45	.5	.55	.5	.65
			3	39	17	9	5	3
			4	34	15	8	5	3
			5	30	12	7	4	3
.95	.35	.3	6	27	11	6	4	3
			7	25	11	6	4	2
			8	22	10	5	3	2
			9	21	9	5	3	2
			10	19	9	5	3	2

P	Ø	Ø'	t \ $\bar{\Pi}$	.4	.45	.5	.55	.6
			3	66	23	11	6	4
			4	57	20	10	6	3
			5	50	18	9	5	3
			6	45	16	8	5	3
.95	.4	.35	7	41	14	7	4	3
			8	37	13	7	4	3
			9	34	12	6	4	2
			10	32	11	6	3	2

Table-7(Contd.)

P	Ø	Ø'	t ¦ Π	.55	.6	.65	.7	.75	.8	.85	.9
			3	164	57	28	16	10	6	4	2
			4	137	48	24	13	8	5	4	2
			5	117	41	20	12	7	5	3	2
			6	102	36	18	10	6	4	3	2
			7	91	32	16	9	5	4	3	2
.99	.1	.05	8	81	29	15	9	5	4	3	2
			9	75	28	13	8	5	4	2	2
			10	69	25	12	7	5	3	2	2
			3	86	37	19	11	7	4	3	
			4	72	31	17	10	6	4	3	
			5	61	27	14	9	6	4	3	
			6	54	23	13	8	5	3	2	
.99	.15	.1	7	48	21	11	7	5	3	2	
			8	44	19	10	6	4	3	2	
			9	40	17	10	6	4	3	2	
			10	37	16	9	6	4	3	2	

P	Ø	Ø'	t ¦ Π	.5	.55	.6	.65	.7	.75	.8
			3	146	51	25	14	8	5	3
.99	.2	.15	4	122	43	21	12	7	5	3
			5	104	37	18	9	6	4	3

Table-7(Contd.)

P	Ø	Ø'	t \ $\bar{n}$	.5	.55	.6	.65	.7	.75	.8
.99	.2	.15	6	91	32	16	9	6	4	3
			7	81	29	14	8	5	4	2
			8	74	26	13	8	5	3	2
			9	67	24	12	7	5	3	2
			10	62	22	11	6	4	3	2

P	Ø	Ø'	t \ $\bar{n}$	.5	.55	.6	.65	.7	.75
			3	76	32	17	10	6	4
			4	63	27	14	9	5	3
			5	54	23	13	8	5	3
.99	.25	.2	6	48	21	11	7	4	3
			7	43	18	10	6	4	3
			8	39	17	9	6	4	3
			9	35	15	8	5	3	2
			10	32	14	8	5	3	2

P	Ø	Ø'	t \ $\bar{n}$	.45	.5	.55	.6	.65	.7
			3	129	45	21	12	7	4
			4	107	37	18	10	6	4
			5	92	32	16	9	6	4
.99	.3	.25	6	80	28	14	8	5	3
			7	71	25	12	7	5	3
			8	65	23	11	7	4	3
			9	59	21	10	6	4	3
			10	54	19	10	6	4	2

Table-7 (Contd.)

p	Ø	Ø'	t \ $\overline{\Pi}$	.45	.5	.55	.6	.65
			3	66	29	[illegible]	[illegible]	5
			4	55	23	12	7	4
			5	47	20	11	6	4
.99	.35	.3	6	42	18	10	6	4
			7	37	16	9	5	3
			8	34	15	8	5	3
			9	31	13	7	4	3
			10	28	12	7	4	3

p	Ø	Ø'	t \ $\overline{\Pi}$	.4	.45	.5	.55	.6
			3	111	38	18	10	6
			4	92	32	15	9	5
			5	79	28	13	8	5
.99	.4	.35	6	69	24	12	7	4
			7	62	22	10	6	4
			8	56	20	10	6	4
			9	51	18	9	5	3
			10	47	17	7	5	3

Table -8

Comparison of number of repetitions (Rule R_3)

P	$\overline{\Pi}1$	t	φ' 0 / φ 0	.05 / .1	0.1 / .15	.15 / .2	.2 / .25	.25 / .3	.3 / .35
		3	45	28	19	14	10	7	5
		4	38	24	17	12.	9	6	4
		5	33	20	14	9	8	6	4
		6	29	18	13	9	7	5	4
.99	.65	7	26	16	11	8	6	5	3
		8	23	15	10	8	6	4	3
		9	21	13	10	7	5	4	3
		10	20	12	9	6	5	4	3
		3	26	17	13	8	6	4	3
		4	23	15	10	8	6	4	3
		5	20	13	9	7	5	4	3
		6	18	12	8	6	5	3	3
.95	.65	7	16	11	8	6	4	3	2
		8	15	10	7	5	4	3	2
		9	14	9	7	5	4	3	2
		10	13	9	6	5	3	3	2
		3	18	12	8	6	4	3	2
		4	16	11	8	6	4	3	2
		5	15	10	7	5	4	3	2
		6	13	9	6	5	4	3	2
		7	12	8	6	4	3	3	2
.90	.65	8	11	8	6	4	3	2	2
		9	11	7	5	4	3	2	2
		10	10	7	5	4	3	2	2

Table - 9

Value of ν for the decision rule R_4

P*	Ø	n \ t	3.	4.	5.	6.	7.	8.	9.	10.
		1	1	1.5	2	2.5	2.5	3	3	3.5
		2	1.5	2.5	3	3.5	3.5	4	4.5	5
		3	2	3	3.5	4	4.5	5	5.5	6
.75	0.05	4	2.5	3	4	4.5	5.5	6	6.5	7
		5	2.5	3.5	4.5	5	6	6.5	7	8
		6	3	4	5	5.5	6.5	7	8	8.5
		7	3	4.5	5.5	6	7	8	8.5	9
		8	3.5	4.5	5.5	6.5	7.5	8.5	9	10
		9	3.5	5	6	7	8	9	9.5	10.5
		10	4	5	6.5	7.5	8.5	9.5	10	11

Table-9(Contd.)

P*	∅	n	t 3	4	5	6	7	8	9	10
		1	1	1.5	2	2.5	2.5	3	3	3.5
		2	1.5	2	2.5	3	3.5	4	4.5	5
		3	2	2.5	3.5	4.0	4.5	5	5.5	6
		4	2.5	3	4	4.5	5	5.5	6	7
.75	.1	5	2.5	3.5	4.5	5	5.5	6.5	7	7.5
		6	3	4	5	5.5	6.5	7	7.5	8.5
		7	3	4	5	6	7	7.5	8.5	9
		8	3.5	4.5	5.5	6.5	7.5	8	9	9.5
		9	3.5	4.5	6	7	7.5	8.5	9.5	10
		10	3.5	5	6	7	8	9	10	11

Table-9(Contd.)

P*	∅	n ¦ t 3	4	5	6	7	8	9	10
		1	1	1.5	2	2	2.5	3	3	3.5
		2	1.5	2	2.5	3	3.5	4	4.5	4.5
		3	2	2.5	3.5	4	4.5	5	5.5	5.5
		4	2	3	4	4.5	5	5.5	6	6.5
.75	.15	5	2.5	3.5	4	5	5.5	6	7	7.5
		6	2.5	4	4.5	5.5	6	7	7.5	8
		7	3	4	5	6	6.5	7.5	8	8.5
		8	3	4.5	5.5	6	7	8	8.5	9.5
		9	3.5	4.5	5.5	6.5	7.5	8.5	9.[illegible]	10
		10	3.5	5	6	7	8	9	9.5	10.5

Table-9 (Contd.)

P*	Ø	n; t	3	4	5	6	7	8	9	10
		1	1	1.5	2	2	2.5	2.5	3	3
		2	1.5	2	2.5	3	3.5	4	4	4.5
		3	2	2.5	3	3.5	4	4.5	5	5.5
		4	2	3	3.5	4.5	5	5.5	6	6.5
.75	.2	5	2.5	3.5	4	5	5.5	6	6.5	7
		6	2.5	3.5	4.5	5	6	6.5	7	8
		7	3	4	5	5.5	6.5	7	8	8.5
		8	3	4	5	6	7	7.5	8.5	9
		9	3.5	4.5	5.5	6.5	7.5	8	9	9.5
		10	3.5	4.5	6	7	7.5	8.5	9.5	10

Table-9 (Contd.)

P^*	Ø	n ¦ t	3	4	5	6	7	8	9	10
		1	1	1.5	2	2	2.5	2.5	3	3
		2	1.5	2	2.5	3	3.5	3.5	4	4.5
		3	2	2.5	3	3.5	4	4.5	5	5.5
		4	2	3	3.5	4	4.5	5	5.5	6
.75	.25	5	2.5	3	4	4.5	5	6	6.5	7
		6	2.5	3.5	4.5	5	5.5	6.5	7	7.5
		7	3	4	4.5	5.5	6	7	7.5	8
		8	3	4	5	6	6.5	7.5	8	8.5
		9	3	4.5	5.5	6	7	8	8.5	9
		10	3.5	4.5	5.5	6.5	7.5	8	9	9.5

Table-9(Contd.)

P*	θ	n \ t	3	4	5	6	7	8	9	IO
		1	1	1.5	1.5	2	2.5	2.5	3	3
		2	1.5	2	2.5	3	3	3.5	4	4
		3	2.0	2.5	3	3.5	4	4.5	5	5
		4	2.0	3	3.5	4	4.5	5	5.5	6
.75	.3	5	2.5	3	4	4.5	5	5.5	6	6.5
		6	2.5	3.5	4	5	5.5	6	7	7.5
		7	2.5	3.5	4.5	5.5	6	6.5	7.5	8
		8	3	4	5	5.5	6.5	7	8	8.5
		9	3	4	5	6	7	7.5	8.5	9
		10	3	4.5	5.5	6.5	7	8	8.5	9.5

Table-9(Contd.)

P^*	θ	n \ t	3	4	5	6	7	8	9	10
		1	1	1.5	1.5	2	2	2.5	2.5	3
		2	1.5	2	2.5	2.5	3	3.5	4	4
		3	1.5	2.5	3	3.5	4	4	4.5	5
.75	.35	4	2	2.5	3.5	4	4.5	5	5.5	5.5
		5	2	3	3.5	4.5	5	5.5	6	6.5
		6	2.5	3.5	4	4.5	5.5	6	6.5	7
		7	2.5	3.5	4.5	5	6	6.5	7	7.5
		8	3	4	4.5	5.5	6	7	7.5	8
		9	3	4.0	5	6	6.5	7.5	8	8.5
		10	3	4.0	5	6	7	7.5	8.5	9

Table-9 (Contd.)

P*	∅	n | t	3	4	5	6	7	8	9	10
		1	2	2.5	3	3	3.5	4	4	4.5
		2	2.5	3.5	4	4.5	5	5.5	6	6.5
		3	3	4	5	5.5	6	7	7.5	8
.90	.05	4	3.5	4.5	5.5	6.5	7	8	8.5	9
		5	4	5.5	6.5	7	8	9	9.5	10
		6	4.5	6	7	8	8.5	9.5	10.5	11
		7	5	6	7.5	8.5	9.5	10.5	11.0	12
		8	5	6.5	8	9	10	11	12	13
		9	5.5	7	8.5	9.5	10.5	11.5	12.5	13.5
		10	6	7.5	9	10	11.5	12.5	13.5	14.5

Table-9(Contd.)

P*	Ø	n¦t	3	4	5	6	7	8	9	10
		1	2	2.5	2.5	3	3.5	4	4	4.5
		2	2.5	3	4	4.5	5	5.5	6	6.5
		3	3	4	4.5	5.5	6	6.5	7	7.5
		4	3.5	4.5	5.5	6	7	7.5	8.5	9
		5	4	5	6	7	8	8.5	9	10
.90	.1	6	4.5	5.5	6.5	7.5	8.5	9.5	10	11
		7	5	6	7	8	9	10	11	11.5
		8	5	6.5	7.5	9	10	11	11.5	12.5
		9	5.5	7	8	9.5	10.5	11.5	12.5	13.5
		10	5.5	7.5	8.5	10	11	12	13	13.5

Table-9(Contd.)

P*	Ø	n; t	3	4	5	6	7	8	9	10
		1	1.5	2	2.5	3	3.5	3.5	4	4.5
		2	2.5	3	3.5	4.5	5	5	5.5	6
		3	3	4	4.5	5	6	6.5	7	7.5
		4	3.5	4.5	5.5	6	7	7.5	8	8.5
.9	.15	5	4	5	6	7	7.5	8.5	9	9.5
		6	4.5	5.5	6.5	7.5	8.5	9	10	10.5
		7	4.5	6	7	8	9	10	10.5	11.5
		8	5	6.5	7.5	8.5	9.5	10.5	11.5	12.5
		9	5.5	6.5	8	9	10	11	12	13
		10	5.5	7	8.5	9.5	10.5	12	12.5	13.5

Table-9(Contd.)

P^*	ϕ	n¦t	3	4	5	6	7	8	9	10
		1	1.5	2	2.5	3	3	3.5	4	4
		2	2.5	3	3.5	4	4.5	5	5.5	5.5
		3	3	3.5	4.5	5	5.5	6	6.5	7
		4	3.5	4	5	5.5	6.5	7	7.5	8
.90	.25	5	3.5	4.5	5.5	6.5	7	8	8.5	9
		6	4	5	6	7	8	8.5	9	10
		7	4.5	5.5	6.5	7.5	8.5	9	10	10.5
		8	4.5	6	7	8	9	10	10.5	11.5
		9	5	6.5	7.5	8.5	9.5	10.5	11.5	12
		10	5	6.5	8	9	10	11	12	13

Table-9(Contd.)

P*	Ø	n\t	3	4	5	6	7	8	9	10
		1	1.5	2	2.5	3	3.5	4	4	4
		2	2	3	3.5	4	4.5	5	5	5.5
		3	2.5	3.5	4	5	5.5	6	6.5	7
		4	3	4	5	5.5	6	6.5	7.5	8
		5	3.5	4.5	5.5	6	7	7.5	8	8.5
.90	.3	6	4	5	6	6.5	7.5	8	9	9.5
		7	4	5.5	6.5	7.5	8	9	9.5	10
		8	4.5	5.5	7	8	8.5	9.5	10.5	11
		9	5	6	7	8	9	10	11	11.5
		10	5	6.5	7.5	8.5	9.5	10.5	11.5	12.5

Table-9 (Contd.)

P^*	ϕ	n|t	3	4	5	6	7	8	9	10
		1	1.5	2	2.5	2.5	3.0	3	3.5	4
		2	2	3	3.5	3.5	4	4.5	5	5.5
		3	2.5	3.5	4	4.5	5	5.5	6	6.5
		4	3	4	4.5	5.5	6	6.5	7	7.5
.90	.35	5	3.5	4.5	5	6	6.5	7.5	8	8.5
		6	4	5	5.5	6.5	7	8	8.5	9
		7	4.0	5.0	6.0	7.0	8.0	8.5	9.5	10
		8	4.5	5.5	6.5	7.5	8.5	9	10	10.5
		9	4.5	6	7	8	9	9.5	10.5	11.5
		10	5	6	7.5	8.5	9.5	10	11	12

Table-9 (Contd.)

P*	ϕ	n ¦ t	3	4	5	6	7	8	9	10
		1	2	3	3.5	3.5	4	4.5	5	5.
		2	3	4	4.5	5.5	6	6.5	7	7.5
		3	4	5	5.5	6.5	7	8	8.5	9
.95	.05	4	4.5	5.5	6.5	7.5	8.5	9	9.5	10.5
		5	5	6.5	7.5	8.5	9	10	11	11.5
		6	5.5	7	8	9	10	11	12	13
		7	6	7.5	8.5	10	11	12	13	14
		8	6.5	8	9.5	10.5	11.5	13	14	15
		9	6.5	8.5	10	11.5	12.5	13.5	14.5	15.5
		10	7	9	10.5	12	13	14.5	15.5	16.5

Table-9(Contd.)

I*	Ø	n\t	3	4	5	6	7	8	9	10
		1	2	2.5	3	3.5	4	4.5	[illegible].5	5
		2	3	4	4.5	5	5.5	6	6.5	7
		3	4	4.5	5.5	6.5	7	7.5	8	9
		4	4.5	5.5	6.5	7.5	8	9	9.5	10
.95	.1	5	5	6.0	7	8	9	10	10.5	11.5
		6	5.5	6.5	8	9	10	11	11.5	12.5
		7	6	7	8.5	9.5	10.5	11.5	12.5	13.5
		8	6	7.5	9	10.5	11.5	12.5	13.5	14.5
		9	6.5	8	9.5	11	12	13	14	15
		10	7	8.5	10	11.5	12.5	14	15	16

Table-9(Contd.)

P*	Ø	n|t	3	4	5	6	7	8	9	10
		1	2	2.5	3	3.5	4	4.5	4.5	5
		2	3.0	3.5	4.5	5	5.5	6	6.5	7
		3	3.5	4.5	5.5	6	7	7.6	8	8.5
.95	.15	4	4.5	5.5	6	7	8	8.5	9	10
		5	5	6	7	8	8.5	9.5	10.5	11
		6	5	6.5	7.5	8.5	9.5	10.5	11.5	12
		7	5.5	7	8	9.5	10.5	11.5	12	13
		8	6	7.5	9	10	11	12	13	14
		9	6.5	8	9.5	10.5	11.5	13	14	15
		10	6.5	8.5	10	11	12.5	13.5	14.5	15.5

Table-9(Contd.)

P*	Ø	n ¦ t	3	4	5	6	7	8	9	10
		1	2	2.5	3	3.5	4	4	4.5	5
		2	3	3.5	4.5	5	5.5	6	6.5	7
		3	3.5	4.5	5.0	6.0	6.5	7	7.5	8.5
.95	.2	4	4	5	6	7	7.5	8.5	9	9.5
		5	4.5	5.5	6.5	7.5	8.5	9.5	10	10.5
		6	5	6.5	8	8.5	9.5	10	11	11.5
		7	5.5	7	8	9	10	11	12	12.5
		8	6	7.5	8.5	9.5	10.5	11.5	12.5	13.5
		9	6.5	7.5	9	10.5	11.5	12.5	13.5	14.5
		10	6.5	8	9.5	11	12	13	14	15.5

Table-9 (Contd.)

P*	Ø	n|t	3	4	5	6	7	8	9	10
		1	1.5	2	2.5	3	3.5	3.5	4	4
		2	2.5	3	3.5	4	4.5	5	5.5	6
		3	3	3.5	4.5	5	5.5	6	6.5	7
		4	3.5	4.5	5	6	6.5	7	8	8.5
.90	.20	5	4	5	5.5	6.5	7.5	8	8.5	9.5
		6	4	5.5	6.5	7	8	9	9.5	10
		7	4.5	5.5	7	8	8.5	9.5	10.5	11
		8	5	6	7.5	8.5	9.5	10	11	12
		9	5	6.5	7.5	9	10	11	11.5	12.5
		10	5.5	7	8	9.5	10.5	11.5	12.5	13

Table-9(Contd.)

r*	∅	n|t	3	4	5	6	7	8	9	10
		1	2	2.5	3	3.5	3.5	4	4.5	4.5
		2	3	3.5	4	4.5	5	5.5	6	6.5
		3	3.5	4.5	5	5.5	6.5	7	7.5	8
		4	4	5	6	6.5	7.5	8	8.5	9.5
		5	4.5	5.5	6.5	7.5	8	9	9.5	10.5
.95	.25	6	5	6	7	8	9	10	10.5	11.5
		7	5.5	6.5	7.5	9	9.5	10.5	11.5	12.5
		8	5.5	7	8.5	9.5	10.5	11.5	12	13
		9	6	7.5	9	10	11	12	13	14
		10	6.5	8	9	10.5	11.5	12.5	13.5	14.5

6	4.5	6	7	8	8.5	9.5	10	11
7	5	6.5	7.5	8.5	9.5	10	11	12
8	5.5	7	8	9	10	11	12	12.5
9	6	7	8.5	9.5	10.5	11.5	12.5	13.5
10	6	7.5	9	10	11	12	13	14

Table-9(Contd.)

P^*	Ø	n \ t	3	4	5	6	7	8	9	10
		1	2.0	2.5	2.5	3	3.5	3.5	4	4.5
		2	2.5	3.5	4	4.5	5	5.5	5.5	6
		3	3	4	4.5	5.5	6	6.5	7	7.5
		4	3.5	4.5	5.5	6	7	7.5	8	8.5
.95	.35	5	4	5	6	7	7.5	8.5	9	9.5
		6	4.5	5.5	6.5	7.5	8.5	9	10	10.5
		7	5	6	7	8	9	10	10.5	11.5
		8	5.5	6.5	7.5	8.5	9.5	10.5	11.5	12
		9	5.5	7	8	9	10.5	11	12	13
		10	6	7.5	8.5	9.5	11	12	12.5	13.5

Table-9(Contd.)

P*	Ø	n | t	3	4	5	6	7	8	9	10
		1	3	3.5	4	4.5	5	5.5	6	6.5
		2	4	5	6	6.5	7	8	8.5	9
		3	5	6	7	8	9	9.5	10.5	11
.99	0.05	4	6	7	8	9	10	11	12	12.5
		5	6.5	8	9	10.5	11.5	12.5	13.5	14
		6	7	8.5	10	11.5	12.5	13.5	14.5	15.5
		7	7.5	9.5	11	12	13.5	14.5	15.5	17
		8	8.5	10	11.5	13	14.5	15.5	17	18
		9	9	10.5	12.5	14	15.5	16.5	18	19
		10	9	11	13	14.5	16	17.5	19	20

Table-9 (Contd.)

P*	Ø	n\t	3	4	5	6	7	8	9	10
		1	3	3.5	4	4.5	5	5.5	6	6
		2	4	5	5.5	6	7	7.5	8	8.5
		3	5	6	7	8	8.5	9.5	10	10.5
		4	5.5	7	8	9	10	11	11.5	12.5
.99	.1	5	6.5	7.5	9	10	11	12	13	14
		6	7	8.5	10	11	12	13	14	15
		7	7.5	9	10.5	12	13	14.5	15.5	16.5
		8	8	10	11.5	12.5	14	15	16.5	17.5
		9	8.5	10.5	12	13.5	15	16	17.5	18.5
		10	9	11	12.5	14	15.5	17	18.5	19

Table-9(Contd.)

P*	Ø	n ¦ t	3	4	5	6	7	8	9	10
		1	3	3.5	4	4.5	5	5	5.5	6
		2	4	4.5	5.5	6	7	7.5	8	8.5
		3	5	6	6.5	7.5	8.5	9	9.5	10.5
.99	.15	4	5.5	6.5	8	8.5	9.5	10.5	11	12
		5	6	7.5	8.5	10	11	11.5	12.5	13.5
		6	7	8	9.5	10.5	12	13	14	14.5
		7	7.5	9	10.5	11.5	12.5	14	15	16
		8	8	9.5	11	12.5	13.5	15	16	17
		9	8.5	10	11.5	13	14.5	15.5	17	18
		10	8.5	10.5	12.5	13.5	15	16.5	18	19

Table-9(Contd.)

P*	Ø	n ¦ t	3	4	5	6	7	8	9	10
		1	2.5	3.5	4	4	4.5	5	5.5	6
		2	4	[illegible].5	5.5	6	6.5	7	7.5	8
		3	4.5	5.5	6.5	7.5	8	9	9.5	10
.99	.2	4	5.5	6.5	7.5	8.5	9.5	10	11	11.5
		5	6	7.5	8.5	9.5	10.5	11.5	12	13
		6	6.5	8	9	10.5	11.5	12.5	13.5	14.5
		7	7	8.5	10	11	12.5	13.5	14.5	15
		8	7.5	9	10.5	12	13	14.5	15.5	16.5
		9	8	10	11.5	12.5	13.5	15.5	16.5	17.5
		10	8.5	10.5	12	13.5	15	16	17.5	18.5

Table-9(Contd.)

P*	Ø	n \ t	3	4	5	6	7	8	9	10
		1	2.5	3	3.5	4	4.5	5	5.5	5.5
		2	3.5	4.5	5	6	6.5	7	7.5	8
		3	4.5	5.5	6.5	7	8	8.5	9	10
		4	5	6	7.5	8	9	10	10.5	11
.99	.25	5	6	7	8	9	10	11	12	12.5
		6	6.5	7.5	9	10	11	12	13	14
		7	7	8.5	9.5	10.5	12	13	14	15
		8	7.5	9	10.5	11.5	13	14	15	16
		9	8	9.5	11	12.5	13	15	16	17
		10	8	10	11.5	13	14.5	15.5	16.5	18

Table-9(Contd.)

P^*	$\emptyset$	n/t	3	4	5	6	7	8	9	10
		1	2.5	3.0	3.5	4	4.5	4.5	5	5.5
		2	3.5	4.5	5	5.5	6	6.5	7	7.5
		3	4	5.5	6	7	7.5	8	8.5	9.5
.99	.3	4	5	6	7	8	8.5	9.5	10	11
		5	5.5	7	8	9	10	10.5	11.5	12
		6	6	7.5	8.5	9.5	10.5	11.5	12.5	13.5
		7	6.5	8	9.5	10.5	11.5	12.5	13.5	14.5
		8	7	8.5	10	11	12.5	13.5	14.5	15.5
		9	7.5	9	10.5	12	13	14.5	15.5	16.5
		10	8	9.5	11	12.5	14	15	16	17

Table-9(Contd.)

P*	Ø	n|t	3	4	5	6	7	8	9	10
		1	2.5	3.0	3.5	4	4.0	4.5	5	5
		2	3.5	4	5	5.5	6	6.5	7	7.5
		3	4	5	6	6.5	7.5	8	8.5	9
		4	5	5.5	7	7.5	8.5	9	9.5	10.5
.99	.95	5	5.5	6.5	7.5	8.5	9.5	10	11	11.5
		6	6	7	8.5	9.5	10.5	11	12	13
		7	6.5	8	9	10	11	12	13	14
		8	6.5	8.5	9.5	11	12	13	13.5	15
		9	7	8.5	10	11.5	12.5	13.5	15	15.5
		10	7.5	9	11	12	13.5	14.5	15.5	16.5

Table 10 Partition of scores and their frequencies in SP design with ties

t=4, n=i

Partition				Frequency	y'	x'
2	1	1	0	12	0	0
1	1	1	1	2	0	0
2	2	0	0	2	0	0
2	1	½	½	8	0	1
2	1½	½	0	8	0	1
2	1	½	½	4	0	2
1½	1	1	½	8	1	1
1½	1½	1	0	8	1	1
1½	1	1	½	8	1	2
1½	1½	1	0	4	1	2
1½	1½	½	½	8	1	2
1½	1	1	½	8	1	3
1	1	1	1	1	2	4
				81		

t=3, n=i

Partition			Frequency	y'	x'
2	1	0	6	0	0
1	1	1	2	0	0
2	½	½	3	0	1
1½	1	½	6	1	1
1½	1½	0	3	1	1
1½	1	½	6	1	2
1	1	1	1	2	3
			27		

Symemetrical pairs with ties

Table 11 Least number 'n' of replications required to implement the section sale .5

P	θ	θ'	t \ π	.55	.6	.65	.7	.75	.8	.85	.9
			3	58	20	10	5	3	2	1	1
.75	.1	.05	4	86	30	14	8	5	3	2	1
			5	108	37	18	10	6	4	3	2
			6	126	43	21	12	7	5	3	2
			7	139	48	23	13	8	5	4	2
			8	152	52	26	15	9	6	4	3
			9	161	56	28	16	10	6	4	3
			10	169	60	29	17	10	7	5	3
			3	30	13	7	4	2	1	1	1
			4	45	19	10	6	4	2	1	1
			5	56	24	13	7	5	3	2	1
.75	.15	.1	6	66	28	15	9	6	4	2	1
			7	73	31	16	10	6	4	3	2
			8	79	34	18	11	7	4	3	2
			9	84	36	19	11	7	5	3	2
			10	90	38	21	12	8	5	3	2

Table 11 contd.

P	θ	θ'	t \ π	.5	.55	.6	.65	.7	.75	.8	.85
			3	52	18	8	5	3	2	1	1
			4	76	26	13	7	4	3	2	1
			5	97	33	16	9	5	3	2	1
.75	.2	.15	6	110	38	19	10	6	4	3	2
			7	124	43	21	12	7	5	3	2
			8	135	47	23	13	8	5	3	2
			9	145	50	24	14	8	5	4	2
			10	152	52	26	15	9	6	4	3
			3	26	11	6	3	2	1	1	
			4	39	17	9	5	3	2	1	
			5	49	21	11	6	4	2	1	
.75	.25	.2	6	58	24	13	8	5	3	2	
			7	64	27	14	8	5	3	2	
			8	70	29	16	9	6	4	2	
			9	74	32	17	10	6	4	3	
			10	79	34	18	11	7	4	3	

p	θ	θ	t \ π	.45	.5	.55	.6	.65	.7	.75
			3	45	15	7	4	2	1	1
			4	68	23	11	6	4	2	1
			5	83	29	14	8	5	3	2
.75	.3	.25	6	98	33	16	9	5	3	2
			7	108	37	18	10	6	4	2
			8	118	41	19	11	7	4	3
			9	125	43	21	12	7	5	3
			10	133	46	22	12	8	5	3

Table 11 contd.

p	θ	θ'	t\π	.45	.5	.55	.6	.65	.7	.75
			3	23	9	5	3	1	1	
			4	34	14	7	4	22	1	
			5	43	18	9	5	3	2	
.75	.35	.3	6	50	21	11	6	4	2	
			7	55	23	12	7	4	3	
			8	60	25	13	8	5	3	
			9	64	27	14	8	5	3	
			10	64	29	15	9	5	3	

			π	.4	.45	.5	.55	.6	.65
			3	39	13	6	3	2	1
			4	57	20	9	5	3	2
			5	71	24	12	6	4	2
.75	.4	.35	6	84	28	13	7	4	3
			7	93	32	15	8	5	3
			8	101	35	16	9	5	3
			9	107	37	18	9	6	4
			10	114	39	19	10	6	4

P	θ	θ'	t\π	.55	.6	.65	.7	.75	.8	.85	.9
			3	99	34	16	9	5	3	2	1
			4	101	48	23	13	8	5	3	2
			5	108	57	28	16	10	6	4	3
.9	.1	.05	6	126	65	31	18	11	7	5	3
			7	138	70	34	20	12	8	5	4
			8	152	75	37	21	13	9	6	4
			9	161	79	39	22	14	9	6	4
			10	171	82	40	23	15	10	6	4

Table 11 contd.

P	θ	θ'	t\π	.55	.6	.65	.7	.75	.8	.85	.9
			3	52	21	11	7	4	2	1	1
			4	72	30	16	7	6	4	2	1
			5	86	37	20	11	7	5	3	2
.9	.15	.1	6	97	41	22	13	8	5	4	2
			7	105	45	24	14	9	6	4	3
			8	112	48	25	15	10	7	4	3
			9	118	51	27	16	10	7	5	3
			10	123	53	28	17	12	7	5	3
			π	.5	.55	.6	.65	.7	.75	.8	.85
			3	88	30	14	8	5	3	2	1
			4	122	42	20	11	7	4	3	2
.9	.2	.15	5	146	51	24	14	8	5	4	2
			6	164	57	28	16	10	6	4	3
			7	179	62	30	17	11	7	5	3
			8	192	66	32	18	11	7	5	3
			9	202	70	34	19	12	8	5	3
			10	209	73	36	20	13	8	5	4
			3	45	19	10	5	3	2	1	
			4	63	27	14	8	5	3	2	
			5	76	32	17	10	6	4	3	
.9	.25	.2	6	85	36	19	11	7	5	3	
			7	94	40	21	12	8	5	3	
			8	100	42	22	13	8	5	4	
			9	104	44	24	14	9	6	4	
			10	109	46	25	15	9	6	4	

Table 11 contd.

P	θ	θ'	t \ π	.45	.5	.55	.6	.65	.7	.75
			3	76	26	12	7	4	2	1
			4	107	36	17	10	6	3	2
			5	128	44	21	12	7	4	3
.9	.3	.25	6	144	50	24	13	8	5	3
			7	156	54	26	15	9	6	4
			8	166	58	28	16	10	6	4
			9	177	61	29	17	10	6	4
			10	183	63	31	17	11	7	4
			3	39	16	8	5	3	1	
			4	55	23	12	7	4	2	
			5	65	27	14	8	5	3	
			6	74	31	16	9	6	4	
.9	.35	.3	7	80	34	18	10	6	4	
			8	85	36	19	11	7	4	
			9	91	38	20	12	7	5	
			10	95	40	21	12	8	5	

P	θ	θ'	t \ π	.4	.45	.5	.55	.6	.65
			3	66	22	11	5	3	2
			4	93	31	15	8	5	3
			5	111	38	18	10	6	3
.9	.4	.35	6	123	42	20	11	7	4
			7	134	46	22	12	7	4
			8	144	50	24	13	8	5
			9	152	52	25	14	8	5
			10	158	55	26	15	9	5

Table 11 contd.

P	θ	θ'	t \ π	.45	.5	.55	.6	.65	.7	.75
			3	76	26	12	7	4	2	1
			4	107	36	17	10	6	3	2
			5	128	44	21	12	7	4	3
.9	.3	.25	6	144	50	24	13	8	5	3
			7	156	54	26	15	9	6	4
			8	166	58	28	16	10	6	4
			9	177	61	29	17	10	6	4
			10	183	63	31	17	11	7	4
			3	39	16	8	5	3	1	
			4	55	23	12	7	4	2	
			5	65	27	14	8	5	3	
			6	74	31	16	9	6	4	
.9	.35	.3	7	80	34	18	10	6	4	
			8	85	36	19	11	7	4	
			9	91	38	20	12	7	5	
			10	95	40	21	12	8	5	
			π	.4	.45	.5	.55	.6	.65	
			3	66	22	11	5	3	2	
			4	93	31	15	8	5	3	
			5	111	38	18	10	6	3	
.9	.4	.35	6	123	42	20	11	7	4	
			7	134	46	22	12	7	4	
			8	144	50	24	13	8	5	
			9	152	52	25	14	8	5	
			10	158	55	26	15	9	5	

Table 11 contd.

P	θ	θ'	t \ π	.55	.6	.65	.7	.75	.8	.85	.9
			3	129	45	21	12	7	4	3	2
			4	176	62	30	17	10	7	4	3
			5	208	72	35	20	13	8	6	4
.95	.1	.05	6	231	81	39	23	14	9	6	4
			7	247	86	42	24	15	10	7	5
			8	264	91	45	26	16	11	7	5
			9	274	96	47	27	17	11	7	5
			10	285	100	49	28	18	12	8	5
			3	67	29	15	9	5	3	2	1
			4	93	39	21	12	8	5	3	2
			5	109	46	25	15	9	6	4	3
.95	.15	.1	6	121	51	28	16	10	7	5	3
			7	131	56	30	18	11	8	5	3
			8	137	59	32	19	12	8	5	4
			9	144	62	33	20	13	8	6	4
			10	149	65	34	21	13	9	6	4
			π	.5	.55	.6	.65	.7	.75	.8	.85
			3	115	40	19	10	6	4	2	1
			4	158	54	26	15	9	6	4	2
			5	185	65	31	18	11	7	5	3
.95	.2	.15	6	205	72	35	20	12	8	5	3
			7	221	77	37	21	13	9	6	4
			8	232	82	40	23	14	9	6	4
			9	244	85	42	24	15	10	6	4
			10	253	88	43	24	15	10	7	4

Table 11 contd.

p	θ	θ'	t \ π	.5	.55	.6	.65	.7	.75	.8	.85
			3	59	25	13	7	4	2	1	
			4	82	35	18	11	7	4	3	
			5	96	41	21	13	8	5	3	
.95	.25	.2	6	107	46	24	14	9	6	4	
			7	114	49	26	15	10	6	4	
			8	122	52	27	16	10	7	4	
			9	128	54	29	17	11	7	5	
			10	132	56	30	18	11	7	5	
			π	.45	.5	.55	.6	.65	.7	.75	
			3	102	34	16	9	5	3	2	
			4	138	47	23	13	8	5	3	
			5	162	56	27	15	9	6	4	
.95	.3	.25	6	180	62	30	17	10	6	4	
			7	192	67	32	18	11	7	5	
			8	203	71	34	19	12	7	5	
			9	213	74	36	20	12	8	5	
			10	222	77	37	21	13	8	5	
			3	51	21	11	6	3	2		
			4	71	30	15	9	5	3		
			5	83	35	18	11	6	4		
.95	.35	.3	6	92	39	20	12	7	5		
			7	100	42	22	13	8	5		
			8	104	44	23	14	9	5		
			9	110	47	25	14	9	6		
			10	114	48	26	15	9	6		

Table 11 contd.

P	θ	θ'	t \ π	.4	.45	.5	.55	.6	.65
			3	101	30	14	7	4	2
			4	119	40	19	10	6	4
			5	139	47	23	12	7	5
.95	.4	.35	6	154	53	25	14	8	5
			7	165	57	27	15	9	6
			8	176	61	29	16	10	6
			9	183	64	30	17	10	6
			10	190	66	32	18	11	7

P	θ	θ'	t \ π	.55	.6	.65	.7	.75	.8	.85	.9
			3	196	67	33	18	11	7	4	2
			4	261	91	45	25	16	10	7	5
			5	300	104	51	29	18	12	8	5
.99	.1	.05	6	325	114	56	32	20	13	9	6
			7	343	120	59	34	21	14	10	6
			8	360	125	62	36	22	15	10	6
			9	372	131	64	37	23	15	10	6
			10	385	134	66	38	24	16	11	6
			3	102	43	23	13	8	5	3	1
			4	137	58	31	19	12	8	5	3
			5	157	68	36	22	14	9	6	4
.99	.15	.1	6	170	73	39	24	15	10	7	4
			7	180	77	42	25	16	11	7	4
			8	189	81	43	26	17	11	8	4
			9	196	84	45	27	17	12	8	4
			10	202	86	46	28	18	12	8	4

Table 11 contd.

P	θ	θ'	t\π	.5	.55	.6	.65	.7	.75	.8	.85
			3	173	60	28	16	9	6	3	2
			4	234	81	39	22	14	9	6	4
.99	.2	.15	5	267	92	46	26	16	10	7	5
			6	289	101	49	28	17	11	8	5
			7	308	108	52	30	18	12	8	5
			8	318	112	54	31	18	13	8	5
			9	331	116	57	32	20	13	9	5
			10	342	120	59	34	21	13	9	5
			3	90	38	20	11	7	4	2	
			4	121	51	27	16	10	6	4	
.99	.25	.2	5	139	59	31	19	12	8	5	
			6	150	64	34	20	13	8	6	
			7	159	68	36	22	14	9	6	
			8	167	71	38	23	14	9	6	
			9	172	74	39	24	15	10	6	
			10	177	76	40	25	15	10	6	
			π	.45	.5	.55	.6	.65	.7	.75	
			3	152	51	25	13	8	4	2	
			4	203	71	34	19	11	7	5	
.99	.3	.25	5	234	81	39	22	14	9	6	
			6	255	88	42	24	15	9	6	
			7	268	93	45	25	16	10	6	
			8	281	98	48	27	16	11	7	
			9	290	101	49	28	17	11	7	
			10	300	104	50	29	18	11	7	

Table 11 contd.

P	θ	θ'	t \ π	.45	.5	.55	.6	.65	.7	.75
			3	78	32	16	9	5	3	
			4	104	44	23	14	8	5	
			5	120	51	27	16	10	6	
.99	.35	.3	6	130	55	29	17	11	7	
			7	139	59	31	18	11	7	
			8	143	61	32	19	12	8	
			9	149	64	34	20	12	8	
			10	153	66	35	20	13	8	

P	θ	θ'	t \ π	.40	.45	.5	.55	.6	.65	.7
			3	132	45	21	11	6	3	
			4	176	60	29	16	9	6	
.99	.4	.35	5	200	69	33	18	11	7	
			6	217	75	36	20	12	7	
			7	231	79	38	21	13	8	
			8	241	83	40	22	13	8	
			9	249	86	42	23	14	9	
			10	256	89	43	24	14	9	

Table 12
Size of experiment N for rule R_3 and R_5

P	θ	θ'	t \ π	.65 FP	.65 SP	.7 FP	.7 SP	.75 FP	.75 SP	.8 FP	.8 SP	.85 FP	.85 SP	.9 FP	.9 SP
			4					48	64	30	40	24	28	12	20
.99	.1	.05	6					90	120	60	78	45	54	30	36
			8					140	176	112	120	84	80	56	48
			10					225	240	135	160	90	110	90	60
			4					36	48	24	32	18	20		
.99	.15	.1	6					75	90	45	60	30	42		
			8					112	136	84	88	56	64		
			10					180	180	135	120	90	80		
			4	72	88	42	56	30	36	18	24				
.99	.2	.15	6	135	168	90	102	60	66	45	48				
			8	224	248	140	144	84	104	56	64				
			10	270	340	180	210	135	130	90	90				

Table 13

Comparison of number of repetitions

P^*	$\overline{\Pi}$	θ	0	.1	.15	.2	.25	.3	.35	.4
		θ'	0	.05	.1	.15	.2	.25	.3	.35
		t								
		3	16	10	7	5	3	2	1	1
		4	23	14	10	7	5	4	2	2
		5	30	18	13	9	6	5	3	2
.75	.65	6	34	21	15	10	8	5	4	3
		7	38	23	16	12	8	6	4	3
		8	41	26	18	13	9	7	5	3
		9	44	28	19	14	10	7	5	4
		10	46	29	21	15	11	8	5	4
		3	26	16	11	8	5	4	3	2
		4	36	23	16	11	8	6	4	3
		5	44	28	20	14	10	7	5	3
.90	.65	6	49	31	22	16	11	8	6	4
		7	54	34	24	17	12	9	6	4
		8	57	37	25	18	13	10	7	5
		9	60	39	27	19	14	10	7	5
		10	63	40	28	20	15	11	8	5
		3	35	21	15	10	7	5	3	2
		4	48	30	21	15	11	8	5	4
		5	57	35	25	18	13	9	6	5
.95	.65	6	63	39	28	20	14	10	7	5
		7	68	42	30	21	15	11	8	6
		8	72	45	32	23	16	12	9	6
		9	74	47	33	24	17	12	9	6
		10	77	49	34	24	18	13	9	7

Table 13 contd.

P^*	π	θ / θ' / t	0 / 0	.1 / .05	.15 / .1	.2 / .15	.25 / .2	.3 / .25	.35 / .3	.4 / .35
		3	54	33	23	16	11	8	5	3
		4	74	45	31	22	16	11	8	6
		5	84	51	36	26	19	14	10	7
		6	92	56	39	28	20	15	11	7
.99	.65	7	97	59	42	30	22	16	11	8
		8	101	62	43	31	23	16	12	8
		9	104	64	45	32	24	17	12	9
		10	107	66	46	34	25	18	13	9

Table 14

Values of ϑ for selection rule R 6

P	θ	n \ t	3	4	5	6	7	8	9	10
		1	2	2	2	2	2	2	2	2
		2	2.5	2.5	3.0	3	3	3	3	3
		3	3	3	3.5	3.5	3.5	3.5	4	4
		4	3.5	3.5	4	4	4	4	4.5	4.5
.75	.05	5	4	4	4.5	4.5	4.5	5	5	5
		6	4.5	4.5	5	5	5	5	5.5	5.5
		7	4.5	5	5	5.5	5.5	5.5	5.5	6
		8	5	5.5	5.5	5.5	6	6	6	6
		9	5.5	5.5	6	6	6.5	6.5	6.5	6.5
		10	5.5	6	6	6.5	6.5	7	7	7
		1	1.5	2	2	2	2	2	2	2
		2	2.5	2.5	2.5	3	3	3	3	3
		3	3	3	3.5	3.5	3.5	3.5	3.5	3.5
.75	.10	4	3.5	3.5	4	4	4	4	4	4.5
		5	4	4	4.5	4.5	4.5	4.5	4.5	5
		6	4	4.5	4.5	5	5	5	5	5.5
		7	4.5	5	5	5	5.5	5.5	5.5	5.5
		8	5	5	5.5	5.5	5.5	6	6	6
		9	5	5.5	5.5	6	6	6	6.5	6.5
		10	5.5	6	6	6	6.5	6.5	6.5	7

Table 14 contd.

P	θ	n \ t	3	4	5	6	7	8	9	10
		1	1.5	2	2	2	2	2	2	2
		2	2.5	2.5	2.5	2.5	3	3	3	3
		3	3	3	3	3.5	3.5	3.5	3.5	3.5
		4	3.5	3.5	3.5	4	4	4	4	4
.75	.15	5	3.5	4	4	4.5	4.5	4.5	4.5	4.5
		6	4	4.5	4.5	4.5	5	5	5	5
		7	4.5	4.5	5	5	5	5.5	5.5	5.5
		8	4.5	5	5	5.5	5.5	5.5	6	6
		9	5	5.5	5.5	6	6	6	6	6.5
		10	5	5.5	6	6	6	6.5	6.5	6.5
		1	1.5	1.5	2	2	2	2	2	2
		2	2.5	2.5	2.5	2.5	2.5	3	3	3
		3	3	3	3	3	3.5	3.5	3.5	3.5
		4	3	3.5	3.5	3.5	4	4	4	4
.75	.20	5	3.5	4	4	4	4.5	4.5	4.5	4.5
		6	4	4	4.5	4.5	4.5	5	5	5
		7	4.5	4.5	5	5	5	5	5.5	5.5
		8	4.5	5	5	5.5	5.5	5.5	5.5	5.5
		9	5	5	5.5	5.5	5.5	6	6	6
		10	5	5.5	5.5	6	6	6	6.5	6.5

Table 14 contd.

P	θ	n \ t	3	4	5	6	7	8	9	10
		1	1.5	1.5	1.5	2	2	2	2	2
		2	2	2.5	2.5	2.5	2.5	2.5	2.5	3
		3	2.5	3	3	3	3	3.5	3.5	3.5
		4	3	3	3.5	3.5	3.5	4	4	4
.75	.25	5	3.5	3.5	4	4	4	4	4.5	4.5
		6	4	4	4.5	4.5	4.5	4.5	4.5	5
		7	4	4.5	4.5	5	5	5	5	5
		8	4.5	4.5	5	5	5	5.5	5.5	5.5
		9	4.5	5	5	5.5	5.5	5.5	6	6
		10	5	5.5	5.5	5.5	6	6	6	6
		1	1.5	1.5	1.5	1.5	2	2	2	2
		2	2	2.5	2.5	2.5	2.5	2.5	2.5	2.5
		3	2.5	3	3	3	3	3	3	3.5
.75	.30	4	3	3	3.5	3.5	3.5	3.5	3.5	4
		5	3.5	4	4	4	4	4	4	4
		6	3.5	4	4	4.5	4.5	4.5	4.5	4.5
		7	4	4.5	4.5	4.5	4.5	5	5	5
		8	4.5	4.5	5	5	5	5	5.5	5.5
		9	4.5	5	5	5	5.5	5.5	5.5	5.5
		10	5	5	5.5	5.5	5.5	6	6	6

Table 14 contd.

P^*	θ	n \ t	3	4	5	6	7	8	9	10
		1	1.5	1.5	1.5	1.5	1.5	2	2	2
		2	2	2	2.5	2.5	2.5	2.5	2.5	2.5
		3	2.5	2.5	3	3	3	3	3	3
		4	3	3	3.5	3.5	3.5	3.5	3.5	3.5
.75	.35	5	3.5	3.5	3.5	4	4	4	4	4
		6	3.5	4	4	4	4	4.5	4.5	4.5
		7	4	4	4.5	4.5	4.5	4.5	5	5
		8	4	4.5	4.5	4.5	5	5	5	5
		9	4.5	4.5	5	5	5	5.5	5.5	5.5
		10	4.5	5	5	5.5	5.5	5.5	5.5	6
		1	2.5	2.5	2.5	2.5	2.5	2.5	2.5	2.5
		2	3	3.5	3.5	3.5	3.5	3.5	3.5	3.5
		3	4	4	4	4.5	4.5	4.5	4.5	4.5
.9	.05	4	4.5	4.5	5	5	5	5	5	5
		5	5	5.5	5.5	5.5	5.5	5.5	6	6
		6	5.5	6	6	6	6	6	6.5	6.5
		7	6	6.5	6.5	6.5	6.5	6.5	7	7
		8	6.5	6.5	7	7	7	7	7.5	7.5
		9	7	7	7.5	7.5	7.5	7.5	7.5	8
		10	7	7.5	7.5	8	8	8	8	8

Table 14 contd.

P^*	θ	n \ t	3	4	5	6	7	8	9	10
		1	2	2.5	2.5	2.5	2.5	2.5	2.5	2.5
		2	3	3.5	3.5	3.5	3.5	3.5	3.5	3.5
		3	4	4	4	4	4	4.5	4.5	4.5
		4	4.5	4.5	4.5	5	5	5	5	5
.9	.10	5	5	5	5	5.5	5.5	5.5	5.5	5.5
		6	5.5	5.5	5.5	6	6	6	6	6
		7	6	6	6	6.5	6.5	6.5	6.5	6.5
		8	6.5	6.5	6.5	7	7	7	7	7
		9	6.5	7	7	7	7.5	7.5	7.5	7.5
		10	7	7.5	7.5	7.5	7.5	8	8	8
		1	2	2	2.5	2.5	2.5	2.5	2.5	2.5
		2	3	3	3	3.5	3.5	3.5	3.5	3.5
		3	4	4	4	4	4	4	4	4
		4	4.5	4.5	4.5	4.5	4.5	5	5	5
.9	.15	5	5	5	5	5	5.5	5.5	5.5	5.5
		6	5.5	5.5	5.5	5.5	6	6	6	6
		7	5.5	6	6	6	6.5	6.5	6.5	6.5
		8	6	6.5	6.5	6.5	7	7	7	7
		9	6.5	6.5	7	7	7	7	7.5	7.5
		10	7	7	7	7	7.5	7.5	7.5	8

Table 14 contd.

P^*	θ	n \ t	3	4	5	6	7	8	9	10
		1	2	2	2	2.5	2.5	2.5	2.5	2.5
		2	3	3	3	3	3.5	3.5	3.5	3.5
		3	3.5	4	4	4	4	4	4	4
.9	.20	4	4	4.5	4.5	4.5	4.5	4.5	4.5	5
		5	4.5	5	5	5	5	5	5.5	5.5
		6	5	5.5	5.5	5.5	5.5	5.5	6	6
		7	5.5	5.5	6	6	6	6	6	6.5
		8	6	6	6.5	6.5	6.5	6.5	6.5	7
		9	6.5	6.5	6.5	7	7	7	7	7
		10	6.5	7	7	7	7.5	7.5	7.5	7.5
		1	2	2	2	2	2	2.5	2.5	2.5
		2	3	3	3	3	3	3	3	3.5
		3	3.5	3.5	3.5	4	4	4	4	4
		4	4	4	4.5	4.5	4.5	4.5	4.5	4.5
.9	.25	5	4.5	4.5	5	5	5	5	5	5
		6	5	5	5	5.5	5.5	5.5	5.5	5.5
		7	5.5	5.5	5.5	6	6	6	6	6
		8	6	6	6	6	6.5	6.5	6.5	6.5
		9	6	6.5	6.5	6.5	6.5	7	7	7
		10	6.5	6.5	7	7	7	7	7	7.5

Table 14 contd.

$\hat{P}$	θ	n \ t	3	4	5	6	7	8	9	10
		1	2	2	2	2	2	2	2	2
		2	3	3	3	3	3	3	3	3
		3	3.5	3.5	3.5	3.5	3.5	4	4	4
		4	4	4	4	4	4.5	4.5	4.5	4.5
.9	.30	5	4.5	4.5	4.5	4.5	5	5	5	5
		6	5	5	5	5	5.5	5.5	5.5	5.5
		7	5	5.5	5.5	5.5	5.5	6	6	6
		8	5.5	5.5	6	6	6	6	6	6.5
		9	6	6	6	6.5	6.5	6.5	6.5	6.5
		10	6	6.5	6.5	6.5	7	7	7	7
		1	2	2	2	2	2	2	2	2
		2	2.5	3	3	3	3	3	3	3
		3	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5
.9	.35	4	4	4	4	4	4	4	4	4.5
		5	4	4.5	4.5	4.5	4.5	4.5	4.5	5
		6	5	5	5	5	5	5	5	5.5
		7	5	5	5.5	5.5	5.5	5.5	5.5	5.5
		8	5.5	5.5	5.5	6	6	6	6	6
		9	5.5	6	6	6.5	6	6.5	6.5	6.5
		10	6	6	6.5	6.5	6.5	6.5	6.5	7

Table 14 contd.

P^*	θ	n \ t	3	4	5	6	7	8	9	10
		1	2.5	2.5	2.5	2.5	3	3	3	3
		2	3.5	3.5	4	4	4	4	4	4
		3	4.5	4.5	4.5	5	5	5	5	5
.95	.05	4	5.5	5.5	5.5	5.5	5.5	5.5	5.5	5.5
		5	6	6	6	6	6	6.5	6.5	6.5
		6	6.5	6.5	6.5	7	7	7	7	7
		7	7	7	7	7.5	7.5	7.5	7.5	7.5
		8	7.5	7.5	7.5	8	8	8	8	8
		9	8	8	8	8	8.5	8.5	8.5	8.5
		10	8.5	8.5	8.5	8.5	9	9	9	9
		1	2.5	2.5	2.5	2.5	2.5	2.5	3	3
		2	3.5	3.5	3.5	4	4	4	4	4
		3	4.5	4.5	4.5	4.5	4.5	4.5	5	5
		4	5	5	5.5	5.5	5.5	5.5	5.5	5.5
.95	.10	5	5.5	6	6	6	6	6	6	6
		6	6.5	6.5	6.5	6,5	6.5	6.5	6.5	7
		7	7	7	7	7	7	7.5	7.5	7.5
		8	7.5	7.5	7.5	7.5	7.5	8	8	8
		9	7.5	8	8	8	8	8	8.5	8.5
		10.	8	8	8.5	8.5	8.5	8.5	8.5	9

Table 14 contd.

P	θ	n \ t	3	4	5	6	7	8	9	10
		1	2.5	2.5	2.5	2.5	2.5	2.5	2.5	2.5
		2	3.5	3.5	3.5	3.5	3.5	4	4	4
		3	4.5	4.5	4.5	4.5	4.5	4.5	4.5	4.5
		4	5	5	5	5	5.5	5.5	5.5	5.5
.95	.15	5	5.5	5.5	6	6	6	6	6	6
		6	6	6	6.5	6.5	6.5	6.5	6.5	6.5
		7	6.5	6.5	7	7	7	7	7	7
		8	7	7	7	7.5	7.5	7.5	7.5	7.5
		9	7.5	7.5	7.5	8	8	8	8	8
		10	8	8	8	8	8.5	8.5	8.5	8.5
		1	2.5	2.5	2.5	2.5	2.5	2.5	2.5	2.5
		2	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5
		3	4	4.5	4.5	4.5	4.5	4.5	4.5	4.5
		4	5	5	5	5	5	5	5	5
.95	.20	5	5.5	5.5	5.5	5.5	5.5	6	6	6
		6	6	6	6	6	6.5	6.5	6.5	6.5
		7	6.5	6.5	6.5	6.5	7	7	7	7
		8	7	7	7	7	7	7.5	7.5	7.5
		9	7.5	7.5	7.5	7.5	7.5	7.5	8	8
		10	7.5	8	8	8	8	8	8	8.5

Table 14 contd.

P	θ	n \ t	3	4	5	6	7	8	9	10
		1	2.5	2.5	2.5	2.5	2.5	2.5	2.5	2.5
		2	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5
		3	4	4	4	4	4.5	4.5	4.5	4.5
		4	4.5	5	5	5	5	5	5	5
.95	.25	5	5	5.5	5.5	5.5	5.5	5.5	5.5	5.5
		6	5.5	6	6	6	6	6	6	6
		7	6	6.5	6.5	6.5	6.5	6.5	6.5	6.5
		8	6.5	6.5	7	7	7	7	7	7
		9	7	7	7	7.5	7.5	7.5	7.5	7.5
		10	7.5	7.5	7.5	7.5	8	8	8	8
		1	2.5	2.5	2.5	2.5	2.5	2.5	2.5	2.5
		2	3	3	3.5	3.5	3.5	3.5	3.5	3.5
		3	4	4	4	4	4	4	4	4
		4	4.5	4.5	4.5	4.5	5	5	5	5
.95	.30	5	5	5	5	5.5	5.5	5.5	5.5	5.5
		6	5.5	5.5	5.5	6	6	6	6	6
		7	6	6	6	6	6.5	6.5	6.5	6.5
		8	6.5	6.5	6.5	6.5	7	7	7	7
		9	7	7	7	7	7	7.5	7.5	7.5
		10.	7	7.5	7.5	7.5	7.5	7.5	7.5	8

Table 14 contd.

p^*	θ	n \ t	3	4	5	6	7	8	9	10
		1	2	2	2.5	2.5	2.5	2.5	2.5	2.5
		2	3	3	3	3	3.5	3.5	3.5	3.5
		3	3.5	4	4	4	4	4	4	4
		4	4.5	4.5	4.5	4.5	4.5	4.5	4.5	4.5
.95	.35	5	5	5	5	5	5	5	5	5.5
		6	5.5	5.5	5.5	5.5	5.5	5.5	5.5	6
		7	6	6	6	6	6	6	6	6
		8	6	6.5	6.5	6.5	6.5	6.5	6.5	6.5
		9	6.5	6.5	7	7	7	7	7	7
		10	7	7	7	7	7.5	7.5	7.5	7.5
		1	3	3	3	3	3	3	3	3.5
		2	4.5	4.5	4.5	4.5	4.5	4.5	4.5	4.5
		3	5.5	5.5	5.5	5.5	5.5	6	6	6
		4	6.5	6.5	6.5	6.5	6.5	6.5	6.5	6.5
.99	.05	5	7	7	7.5	7.5	7.5	7.5	7.5	7.5
		6	8	8	8	8	8	8	8	8
		7	8.5	8.5	8.5	8.5	8.5	8.5	9	9
		8	9	9	9	9.5	9.5	9.5	9.5	9.5
		9	9.5	9.5	9.5	10	10	10	10	10
		10	10	10	10.5	10.5	10.5	10.5	10.5	10.5

Table 14 contd.

P^*	θ	n \ t	3	4	5	6	7	8	9	10
		1	3	3	3	3	3	3	3	3.5
		2	4.5	4.5	4.5	4.5	4.5	4.5	4.5	4.5
		3	5.5	5.5	5.5	5.5	5.5	5.5	5.5	5.5
		4	6.5	6.5	6.5	6.5	6.5	6.5	6.5	6.5
.99	.10	5	7	7	7	7	7	7	7	7.5
		6	7.5	7.5	7.5	8	8	8	8	8
		7	8	8.5	8.5	8.5	8.5	8.5	8.5	8.5
		8	9	9	9	9	9	9	9	9
		9	9.5	9.5	9.5	9.5	9.5	9.5	9.5	10
		10	10	10	10	10	10	10	10.5	10.5
		1	3	3	3	3	3	3	3	3
		2	4.5	4.5	4.5	4.5	4.5	4.5	4.5	4.5
		3	5	5.5	5.5	5.5	5.5	5.5	5.5	5.5
		4	6	6	6	6	6	6.5	6.5	6.5
.99	.15	5	7	7	7	7	7	7	7	7
		6	7.5	7.5	7.5	7.5	7.5	7.5	7.5	7.5
		7	8	8	8	8	8	8.5	8.5	8.5
		8	8.5	8.5	8.5	9	9	9	9	9
		9	9	9	9	9.5	9.5	9.5	9.5	9.5
		10	9.5	9.5	9.5	10	10	10	10	10

Table 14 contd.

P	θ	n \ t	3	4	5	6	7	8	9	10
		1	3	3	3	3	3	3	3	3
		2	4	4	4	4	4.5	4.5	4.5	4.5
		3	5	5	5	5	5	5.5	5.5	5.5
.99	.20	4	6	6	6	6	6	6	6	6
		5	6.5	6.5	6.5	6.5	7	7	7	7
		6	7	7	7.5	7.5	7.5	7.5	7.5	7.5
		7	8	8	8	8	8	8	8	8
		8	8.5	8.5	8.5	8.5	8.5	8.5	8.5	8.5
		9	9	9	9	9	9	9	9	9
		10	9.5	9.5	9.5	9.5	9.5	9.5	9.5	9.5
		1	3	3	3	3	3	3	3	3
		2	4	4	4	4	4	4	4	4
		3	5	5	5	5	5	5	5	5
		4	5.5	5.5	6	6	6	6	6	6
.99	.25	5	6.5	6.5	6.5	6.5	6.5	6.5	6.5	6.5
		6	7	7	7	7	7	7	7	7.5
		7	7.5	7.5	7.5	7.5	7.5	8	8	8
		8	8	8	8	8	8.5	8.5	8.5	8.5
		9	8.5	8.5	8.5	9	9	9	9	9
		10	9	9	9	9	9.5	9.5	9.5	9.5

Table 14 contd.

P*	θ	n \ t	3	4	5	6	7	8	9	10
		1	2.5	3	3	3	3	3	3	3
		2	4	4	4	4	4	4	4	4
		3	5	5	5	5	5	5	5	5
		4	5.5	5.5	5.5	5.5	5.5	5.5	5.5	5.5
.99	.30	5	6	6	6.5	6.5	6.5	6.5	6.5	6.5
		6	7	7	7	7	7	7	7	7
		7	7.5	7.5	7.5	7.5	7.5	7.5	7.5	7.5
		8	8	8	8	8	8	8	8	8
		9	8	8.5	8.5	8.5	8.5	8.5	8.5	8.5
		10	8.5	8.5	9	9	9	9	9	9
		1	2.5	2.5	2.5	2.5	2.5	2.5	2.5	3
		2	4	4	4	4	4	4	4	4
		3	4.5	4.5	4.5	4.5	4.5	4.5	5	5
.99	.35	4	5.5	5.5	5.5	5.5	5.5	5.5	5.5	5.5
		5	6	6	6	6	6	6	6	6
		6	6.5	6.5	6.5	6.5	6.5	6.5	6.5	7
		7	7	7	7	7	7.0	7	7.5	7.5
		8	7.5	7.5	7.5	7.5	7.5	7.5	8	8
		9	8	8	8	8	8	8	8	8.5
		10	8.5	8.5	8.5	8.5	8.5	8.5	8.5	8.5