Hesitant Fuzzy DeGroot Opinion Dynamics Model and Its Application in Multi-attribute Decision Making

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Abstract

The research on the evolution law of the opinions can help the decision makers (DMs) improve the decision-making efficiency, predict the trend of events and make the right decision. These opinions are always described by one number, which is inaccurate and incomplete. To solve such a problem, in this paper, the hesitant fuzzy DeGroot (HF-DeGroot) opinion dynamics model is proposed. In order to simulate the transformation of hesitant fuzzy opinions, we introduced the multiplications for real matrix and hesitant fuzzy matrix. Then three kinds of transformation matrices with the consideration of the similarity degree, self-confidence degree and authority degree are constructed based on the hesitant fuzzy data and the consensus condition for the model is discussed as well. Furthermore, the HF-DeGroot opinion dynamics decision-making method is proposed from a prediction perspective and is applied to the emergency decision for the public health events. Finally, the effectiveness, feasibility and practicability of this method are shown by the comparison and simulation results.

Keywords: Hesitant fuzzy set, Opinion dynamics, consensus, decision making, opinion transition matrix, DeGroot opinion dynamics model.

1 Introduction

Nowadays, it is difficult for individuals or groups to make decisions without being influenced by public opinions. However, public opinions are becoming more complicated due to the increasing ways of information transmission. People are difficult to distinguish the right and meaningful opinions and are easily affected by negative ones meanwhile. Therefore, it is very necessary for the decision-makers (DMs) to study the evolution law of opinions which can not only help DMs to understand a large number of opinions in a short time and improve the decision-making efficiency, but also provide targeted and predictive information to DMs to help them make the right decisions.

Based on the theories of mathematics, physics and cybernetics, the opinion dynamics [2, 17] simulates the communication, influence and diffusion process of opinions in the transmission process by establishing a model and predicts the tendency of evolution. It has been studied and applied in many fields, such as politics [4], marketing [18], and venture capital [25], etc. According to the opinion space being a discrete numerical set or a continuous interval, the opinion dynamic models can be divided into two types: discrete and continuous. Discrete models are performed well in simulating large-scale social activities such as mayor election [1] and the elections in Germany [19], mainly including Ising model [5, 12, 25], Voter model [23] and Sznajd model [4, 18, 19, 21], etc. Continuous models such as DeGroot opinion dynamics model [3, 9, 10], Deffuant model [13] and Hegselmann-Krause [11] model mainly study the evolution of the group's opinions from the perspective of individual social actors, henceforth called agents [17]. Among these models, the DeGroot model has always been very representative. It is expressed in a very concise form, using a stochastic matrix to represent the interactions between agents, and assuming that the interactions are time-invariant, it has been extensively studied these days. For example, Stamatelatos et al. [24] used it to reveal diverse aspects of the Nodes of Interest's political profile in the Twitter online social network. Askarzadeh et al. [2] extended it to describe the evolution of the probability distribution of interacting species.

However, both the DeGroot opinion dynamics model and other opinion dynamics models use a single value to describe the opinion. In fact, due to the cognitive ability's improvement and the increasing complexity, the traditional way of a single value can no longer meet the needs. In this case, professor Zadeh [31] proposed the concepts of membership degree and fuzzy set (FS), which enable people to describe the attributes of things rationally by using the uncertainty method. On this basis, Torra [26] proposed the hesitant fuzzy set (HFS), which specifies that the attributes can be represented by multiple membership degree values and shows the hesitant mentality of DMs. Due to its outstanding ability of expression, HFSs have been widely used in many fields [30], such as communication network [7], construction industry [8], venture investment [22].

For the common multi-attribute decision making (MADM) problems under the hesitant fuzzy environment, there are many mature solutions, such as hesitant fuzzy TOPSIS method [29] and hesitant fuzzy TODIM method [32] which are often used to deal with the energy policy selection problem [29] and service quality evaluation [32]. But those methods based on static information at a certain time will not be applicable when the actual situation is constantly changing. For example, in the early stage of outbreak of the Novel Coronavirus (COVID-19) in 2019[1], the virus was not particularly harmful and infectious, but with the virus spread rapidly, the corresponding problems related to politics, economy, culture and other aspects had also been changing rapidly. In this situation, the static decision-making methods cannot grasp the dynamic change of the events and the corresponding changes in the attribute values cannot be well reflected, by contrast, the dynamic MCDM methods is more appropriate.

If we look at the decision-making process from the perspective of opinion dynamics, the DMs are the agents whose hesitant fuzzy opinions form the group's opinions. With the changes of time and environment, DMs interact with each other through communication and learning. Finally, these opinions are aggregated into one or several opinions which represent the final decision. In the forming process of the group's opinions, there are some intersecting theories which can be taken advantage of each other although their research objectives are different. Therefore, in this paper, we will integrate

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[1]WHO, Coronavirus disease (COVID-19) Pandemic, https://www.who.int/emergencies/diseases/novel-coronavirus-2019, 2020.

the DMs' opinions and establish the corresponding decision-making methods based on the appropriate opinion dynamics model. Due to the good mathematical properties, the DeGroot opinion dynamics model is suitable to be used in this study. In order to make the hesitant fuzzy data fit the matrix operations better, we also define and discuss the operation rules of the hesitant fuzzy matrices.

The remainder of this paper is arranged as follows: In Section 2, basic concepts of HFSs and DeGroot model are presented in detail. Section 3 presents the hesitant fuzzy DeGroot (HF-DeGroot) opinion dynamics models and the corresponding decision-making algorithms are shown in Section 4. Next, we apply the HF-DeGroot decision-making algorithms to the emergency decision for the public health events in Section 5 and show the rationality of the proposed methods by a series of comparative analyses in Section 6. Section 7 ends the paper with some conclusions.
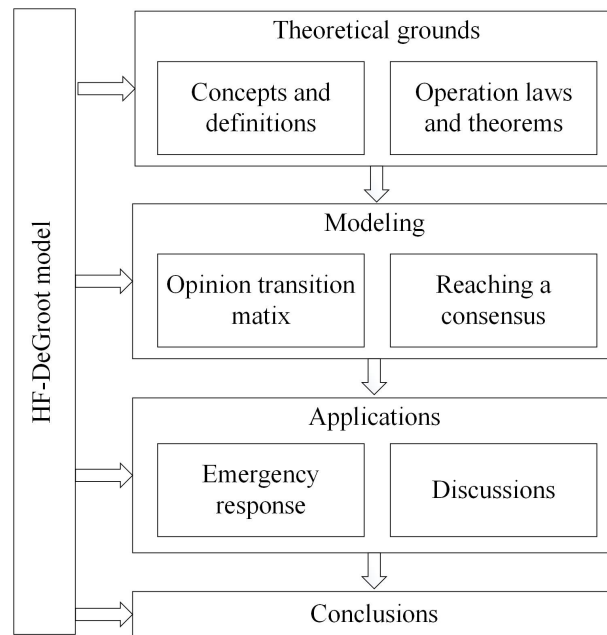
Figure 1: Organizational structure of this paper

2 The HFSs and opinion dynamics

This section introduces the necessary prior knowledge regarding HFSs and the DeGroot opinion dynamics model in opinion dynamics as the basis for the subsequent research.

2.1 HFSs

Torra [26] introduced the concept of HFS firstly. Compared with traditional FSs, the HFSs is superior in the expression of the uncertain information. In order to make HFS easily understood, Xia and Xu [27] proposed the mathematical expression of HFS and defined the hesitant fuzzy element (HFE) $h_A(x)$. So the HFS A on a fixed set X is described as: $A = \{\langle x, h_A(x)\rangle | x \in X\}$, where $h_A(x)$ includes several different values between 0 and 1, and represents the membership degrees of the element x to the set A.

The HFEs can't be compared directly so the score function

$$s\left(h\left(x\right)\right) = \frac{1}{l\left(h\right)} \sum_{k=1}^{l} \gamma_k \tag{1}$$

was proposed that the higher the score values is, the greater the HFE is where $l\left(h\right)$ is the number of membership degree in $h(x)$ which can be abbreviated as l. But they are difficult to be distinguished

when the HFEs have the same scores. So the deviation degree [6] of the HFE

$$d\left(h\left(x\right)\right) = \left[\frac{1}{l}\sum_{k=1}^{l}\left(\gamma_k - s\left(h\left(x\right)\right)\right)^2\right]^{\frac{1}{2}} \quad (2)$$

provides an effective solution that the larger the values, the smaller the HFE.

The basic operational laws and the aggregation operators proposed by Liao et al. [14] are of great importance for decision making.

Definition 1. Let h, h_1, h_2 be three HFEs and c be a positive real number, then

(1) $h^c = \bigcup\limits_{i=1}^{l}\left\{\left(\gamma^{\sigma(i)}\right)^c\right\}$;

(2) $ch = \bigcup\limits_{i=1}^{l}\left\{1-\left(1-\gamma^{\sigma(i)}\right)^c\right\}$;

(3) $h_1 \oplus h_2 = \bigcup\limits_{i=1}^{l_{\max}}\left\{\gamma_1^{\sigma(i)} + \gamma_2^{\sigma(i)} - \gamma_1^{\sigma(i)}\gamma_2^{\sigma(i)}\right\}$;

(4) $h_1 \otimes h_2 = \bigcup\limits_{i=1}^{l_{\max}}\left\{\gamma_1^{\sigma(i)}\gamma_2^{\sigma(i)}\right\}$.

where $\gamma_j^{\sigma(i)}$ is the largest value in h_j and $l_{\max} = \max\left\{l\left(h_1\right), l\left(h_2\right)\right\}$.

If the numbers of membership degree values contained in the HFEs are different, the pessimistic or optimistic principle introduced by Xu and Xia [28] can be adopted to extend the shorter ones. If there is no special explanation, the pessimistic principle is used in this article for extension.

Definition 2 [14]. An adjusted hesitant fuzzy weighted averaging (AHFWA) operator is a mapping $F^n \rightarrow F$,

$$AHFWA\left(h_1, h_2, ..., h_n\right) = \bigoplus_{k=1}^{n} w_k h_k = \bigcup_{i=1}^{l_{\max}}\left\{1 - \prod_{k=1}^{n}\left(1-\gamma_k^{\sigma(i)}\right)^{w_k}\right\} \quad (3)$$

Mo et al. [16] found and proved that when the AHFWA operator and pessimistic principle are adopted, the operation between the real number and HFE satisfies the distributive law. Besides the above conclusions there are some other properties which will be listed in Theorem 1:

Theorem 1. Suppose that λ and μ are two positive numbers and h, h_1, h_2, h_3 are HFEs, if the AHFWA operator is adopted, then the following conclusions can be got:

(1) $(\lambda + \mu)h = \lambda h \oplus \mu h$ [16];

(2) $h \oplus h = 2h$ [16];

(3) $h_1 \oplus h_2 = h_2 \oplus h_1$;

(4) $\lambda\left(h_1 \oplus h_2\right) = \lambda h_1 \oplus \lambda h_2$;

(5) $\left(h_1 \oplus h_2\right) \oplus h_3 = h_1 \oplus \left(h_2 \oplus h_3\right)$;

(6) $\lambda\left(\mu h\right) = \left(\lambda\mu\right)h$.

Proof. (1) and (2) have been proven by Mo et al. in Ref. [16].

(3) $h_1 \oplus h_2 = \bigcup\limits_{i=1}^{l_{\max}}\left\{\gamma_1^{\sigma(i)} + \gamma_2^{\sigma(i)} - \gamma_1^{\sigma(i)}\gamma_2^{\sigma(i)}\right\} = \bigcup\limits_{i=1}^{l_{\max}}\left\{\gamma_2^{\sigma(i)} + \gamma_1^{\sigma(i)} - \gamma_2^{\sigma(i)}\gamma_1^{\sigma(i)}\right\} = h_2 \oplus h_1$

(4) $\lambda\left(h_1 \oplus h_2\right) = \lambda\left(\bigcup\limits_{i=1}^{l_{\max}}\left\{\gamma_1^{\sigma(i)} + \gamma_2^{\sigma(i)} - \gamma_1^{\sigma(i)}\gamma_2^{\sigma(i)}\right\}\right) = \bigcup\limits_{i=1}^{l_{\max}}\left\{1 - \left(1 - \gamma_1^{\sigma(i)} + \gamma_2^{\sigma(i)} + \gamma_1^{\sigma(i)}\gamma_2^{\sigma(i)}\right)^{\lambda}\right\}$

$= \bigcup\limits_{i=1}^{l_{\max}}\left\{1 - \prod\limits_{k=1}^{2}\left(1 - \gamma_k^{\sigma(i)}\right)^{\lambda}\right\} = \lambda h_1 \oplus \lambda h_2$

(5) $\left(h_1 \oplus h_2\right) \oplus h_3 = \bigcup\limits_{i=1}^{l_{\max}}\left\{1 - \prod\limits_{k=1}^{2}\left(1 - \gamma_k^{\sigma(i)}\right)\right\} \oplus h_3 = \bigcup\limits_{i=1}^{l_{\max}}\left\{1 - \prod\limits_{k=1}^{3}\left(1 - \gamma_k^{\sigma(i)}\right)\right\}$

$h_1 \oplus \left(h_2 \oplus h_3\right) = h_1 \oplus \bigcup\limits_{i=1}^{l_{\max}}\left\{1 - \prod\limits_{k=2}^{3}\left(1 - \gamma_k^{\sigma(i)}\right)\right\} = \bigcup\limits_{i=1}^{l_{\max}}\left\{1 - \prod\limits_{k=1}^{3}\left(1 - \gamma_k^{\sigma(i)}\right)\right\} = \left(h_1 \oplus h_2\right) \oplus h_3$

(6) $\lambda\left(\eta h\right) = \lambda\left[\bigcup\limits_{i=1}^{l}\left\{1 - \left(1 - \gamma^{\sigma(i)}\right)^{\eta}\right\}\right]$

$= \bigcup\limits_{i=1}^{l}\left\{1 - \left[1 - 1 + \left(1 - \gamma^{\sigma(i)}\right)^{\eta}\right]^{\lambda}\right\} = \bigcup\limits_{i=1}^{l}\left\{1 - \left(1 - \gamma^{\sigma(i)}\right)^{\eta\lambda}\right\} = \left(\lambda\mu\right)h$

The proof is completed.

Note 1. In Theorem 1, HFE must be extended by pessimistic principle or optimistic principle. Because Mo et al. [16] found that if 0.5 or the average value of the previous several membership degree values is used to extension, the conclusion in Theorem 1 is not valid. In addition to the basic operational rules, similarity measure also occupies a very important place in the hesitant fuzzy theory. For two HFSs A and B, Xu and Xia [28] defined several similarity formulas based on the traditional Hamming distance and Euclidean distance.

(1) Normalized hesitant fuzzy weighted Hamming similarity:

$$s_1(A, B) = 1 - \frac{1}{n} \sum_{i=1}^{n} \left[\frac{1}{l(x_i)} \sum_{j=1}^{l(x_i)} \left| h_A^{\sigma(j)}(x_i) - h_B^{\sigma(j)}(x_i) \right| \right] \quad (4)$$

(2) Normalized hesitant fuzzy weighted Euclidean similarity:

$$s_2(A, B) = 1 - \left[\frac{1}{n} \sum_{i=1}^{n} \left(\frac{1}{l(x_i)} \sum_{j=1}^{l(x_i)} \left| h_A^{\sigma(j)}(x_i) - h_B^{\sigma(j)}(x_i) \right|^2 \right) \right]^{1/2} \quad (5)$$

These two formulas above can also be used in the calculation of the similarity measures between two HFEs.

2.2 DeGroot opinion dynamics model

The DeGroot opinion dynamics model [9] introduced in 1974 was used to study the simple procedure to reach consensus among the agents.

Assume that $\{1, 2, \cdots, n\}$ is a set of agents and the opinion of the ith agent at the time $t \in \mathbb{N}$ is $x_i(t) \in R$ $(i = 1, 2, \cdots, n)$. Then at the time $t+1$, the evaluation of the opinions can be expressed as the weighted average of its own and other neighbors' opinions:

$$x_i(t+1) = \sum_{j=1}^{n} p_{ij} x_j(t), t = 0, 1, 2, \cdots \quad (6)$$

where $p_{ij} \geq 0$ $(i = 1, 2, \cdots, k; j = 1, 2, \cdots, k)$ is the weight that the ith agent gives to the jth agent and $\sum_{j=1}^{n} p_{ij} = 1$.

If $X(t) = \{x_1(t), x_2(t), \cdots, x_n(t)\}^T$ and $P = (p_{ij})_{n \times n}$. then the ith agent's opinion $x_i(t)$ at the time $t+1$ in Eq.(6) can be rewritten as

$$X(t+1) = PX(t), t = 0, 1, 2, \cdots \quad (7)$$

where P is a stochastic matrix.

The consensus of the model as the core content in the model research should be considered because the convergence of the DeGroot opinion dynamics model cannot guarantee the formation of a unified final opinion.

Definition 3 (Consensus) [9]. If the model satisfies that for any initial opinion $x(0)$ the limit exists $\lim_{t \to \infty} x(t) = \lim_{t \to \infty} P^t x(0) = \mu^* 1_n$ (μ^* is a real number), then the model (7) can reach a consensus, where 1_n is the column vector whose elements are all 1.

That is, when the model reaches a consensus, there is $x_1(\infty) = x_2(\infty) = \cdots = x_n(\infty)$ for all the initial opinions.

The fully regularity of the stochastic matrix P is the most important factor to make the model reaches a consensus.

Definition 4 (Fully regularity) [20]. The matrix P is fully regular if the limit $P^\infty = \lim_{k \to \infty} P^k = 1_n(p_1, p_2, \cdots, p_n)$, $p_i \in R$ $(i = 1, 2, \cdots, n)$ exists and the rows of P^∞ are identical.

The DeGroot opinion dynamics model assumes that the opinion is time-invariant, so it can be analyzed by using the existing conclusions in Markov chain (MC) theory [20]. The function of stochastic

matrix P in the DeGroot opinion dynamics model is equivalent to the transition probability matrix in MC theory. If P is a doubly stochastic matrix, the following Theorem 2 can be concluded:

Theorem 2 [15]. Let a Markov chain with the finite state space $I = \{1, 2, \cdots, n\}$ be irreducible and aperiodic, the transition probability matrix $P = (p_{ij})_{n \times n}$ be doubly stochastic matrix, then $\lim_{n \to \infty} P^n = \frac{1}{n} I_n$ where I_n is the n order matrix whose elements are all 1.

In addition, DeGroot [9] gives the sufficient conditions for the model to reach a consensus by using the MC theory.

Theorem 3 [9]. If there exists a positive integer n and all the elements in at least one column of the matrix P^n are positive, the DeGroot opinion dynamics model reaches a consensus.

3 The evaluation of the hesitant fuzzy opinion

The main work of this section is to study the dynamic evolution of the hesitant fuzzy opinion by using the DeGroot opinion dynamics model. Because the transformation of hesitant fuzzy opinions needs to be expressed by the matrix operations, but there's a big difference between the hesitant fuzzy operators and the real arithmetic, so at the beginning of this section, we need to define the multiplication between the real number matrix and hesitant fuzzy matrix, and study the operation rules between real numbers and HFEs.

3.1 Multiplication between the real matrix and hesitant fuzzy matrix

The following is the definition of the multiplication between the real matrix and the hesitant fuzzy matrix:

Definition 5. $A = (a_{ij})$ is a $m \times s$ real matrix, $Z = (z_{ij})$ is a $s \times n$ hesitant fuzzy matrix, the hesitant fuzzy matrix Z is pre-multiplied by the real matrix A to get an $m \times n$ hesitant fuzzy matrix $C = (c_{ij})$ where $a_{ij} \in R$, z_{ij} and c_{ij} are the HFEs, and

$$c_{ij} = a_{i1}z_{1j} \oplus a_{i2}z_{2j} \oplus \cdots \oplus a_{is}z_{sj} = \overset{s}{\underset{q=1}{\oplus}} a_{iq}z_{qj}, i = 1, 2, \cdots, m; j = 1, 2, \cdots, n \qquad (8)$$

where $\gamma_{lj}^{\sigma(k)}$ denotes the kth largest value in z_{lj}, denoted as $C = AZ$.

Especially the product of a $1 \times s$ real row matrix and a $s \times 1$ hesitant fuzzy column matrix is a first-order square matrix or a HFE.

When the AHFWA operator is used in Eq.(8), the multiplication between the real matrix and the hesitant fuzzy matrix satisfies the associative law.

Theorem 4. Suppose that A is a $m \times s$ real matrix, B is a $s \times n$ real matrix, and Z is a $n \times p$ hesitant fuzzy matrix, then $(AB) Z = A (BZ)$.

Proof. Let $A = (a_{ij})_{m \times s}$ and $B = (b_{jk})_{s \times n}$ be real matrixes, and $Z = (z_{kr})_{n \times p}$ be a hesitant fuzzy matrix. Then $D = AB = (d_{ik})_{m \times n}$ be a real matrix, and $E = BZ = (e_{jr})_{s \times p}$ be a hesitant fuzzy matrix, where

$$d_{ik} = \sum_{j=1}^{s} a_{ij}b_{jk}, e_{jr} = \overset{n}{\underset{k=1}{\oplus}} b_{jk}z_{kr}.$$

According to Theorem 1, the operations between the real numbers and the HFEs satisfy the commutative law, the associative law and the distributive law [16], so the elements in DZ are

$$\overset{n}{\underset{k=1}{\oplus}} d_{ik}z_{kr} = \overset{n}{\underset{k=1}{\oplus}} \left(\sum_{j=1}^{s} a_{ij}b_{jk} \right) z_{kr} = \overset{n}{\underset{k=1}{\oplus}} \left(\overset{s}{\underset{j=1}{\oplus}} (a_{ij}b_{jk}) z_{kr} \right) = \overset{n}{\underset{k=1}{\oplus}} \left(\overset{s}{\underset{j=1}{\oplus}} a_{ij} (b_{jk}z_{kr}) \right) = \overset{s}{\underset{j=1}{\oplus}} a_{ij} \left(\overset{n}{\underset{k=1}{\oplus}} b_{jk}z_{kr} \right).$$

Based on the associative property of the operation between the real numbers and HFEs, the elements in AE are

$$\overset{s}{\underset{j=1}{\oplus}} a_{ij}e_{jr} = \overset{s}{\underset{j=1}{\oplus}} a_{ij} \left(\overset{n}{\underset{k=1}{\oplus}} b_{jk}z_{kr} \right),$$
$$DZ = AE.$$

The proof is completed.

3.2   HF-DeGroot opinion dynamics model

Hesitant fuzzy data allow the opinions to be expressed more accurately and truthfully. In this section, its evolution in the transmission is simulated.

Assume that n agents give their opinions $z_1(t), z_2(t), \cdots, z_n(t)$ at the time t which are in the form of HFEs respectively. Influenced by others, the agents adjust their opinions at the next moment. P is the $n \times n$ stochastic matrix comprising the elements $p_{ij} \in R\, (i = 1, 2, \cdots, n; j = 1, 2, \cdots, n)$ which represents the transition weight corresponding $z_i(t)$ to $z_j(t)$ at the time t. And each element p_{ij} is nonnegative and the sum of the elements in any given row is 1. The HF-DeGroot opinion dynamics model is time-invariant. Then, at the time $t+1$, the opinion of the ith agent can be expressed as:

$$z_i(t+1) = \sum_{j=1}^{n} p_{ij} z_j(t), t = 0, 1, 2, \cdots \tag{9}$$

Let $Z(t) = \{z_1(t), z_2(t), \cdots, z_n(t)\}^T$, then,

$$Z(t+1) = PZ(t), t = 0, 1, 2, \cdots \tag{10}$$

From Theorem 4 and the theory of the DeGroot opinion dynamics model, we can get

Theorem 5. Let $P = (p_{ij})_{n \times n}$ be a stochastic matrix, and $Z(t) = (z_1(t), z_2(t), \cdots, z_n(t))^T$ be a hesitant fuzzy column matrix, then

$$Z(t) = PZ(t-1) = P^t Z(0), t = 1, 2, 3, \cdots \tag{11}$$

Proof. According to Theorem 4, $Z(t) = PZ(t-1) = P^2 Z(t-2) = P^t Z(0)$.

The proof is completed.

Eq.(11) in Theorem 5 simplifies the operation of hesitant fuzzy matrix. The agents' opinions at the time t can be got by the multiplication of t power of the stochastic matrix P and the hesitant fuzzy matrix only one time.

Definition 6 (Consensus of the hesitant fuzzy opinions). If the model satisfies that for any initial hesitant fuzzy opinion $x(0)$ the limit exists $\lim_{t \to \infty} z(t) = \lim_{t \to \infty} P^t x(0) = 1_n h^*$ (h^* is a HFE), then the model of Eq.(10) can reach a consensus, where 1_n is the column vector whose elements are all 1.

There is $z_1(\infty) = z_2(\infty) = \cdots = z_n(\infty)$ for all the initial hesitant fuzzy opinions when the model reaches a consensus.

4   Decision making method based on the HF-DeGroot opinion dynamics model

In this section we will discuss the following questions in combination with the characteristics of the hesitant fuzzy opinions:
– How to explore the relationships between the hesitant fuzzy opinions?
– When will a group of hesitant fuzzy opinions reach consensus?

4.1   The structure of the transition matrix

In the research of DeGroot opinion dynamics model, the stochastic matrix P which represents the transition weights between opinions plays an important role. Due to the difficulty in obtaining information the stochastic matrix P is mostly obtained by subjective weighting, but this method is not objective enough and often lacks of mining the hidden information in the opinions. Therefore, in the following research, several methods of obtaining the stochastic matrix P will be introduced which take into account both the objective and subjective factors.

Since people's opinions are not always the same, the original opinions will be adjusted while learning others' opinions. Here we assume that there are three factors that affect the transition of opinions: (1) Similarity degree–Similar opinions are more easily transferred to each other. (2) Self-confident degree–Confident people are more likely to get others' trust and less likely to accept others' opinions.

(3) Authority degree–Authority figures are more likely to influence others. Now we will construct the opinion transition matrix from these three aspects.

Plan A: Opinion transition matrix based on the similarity degree

According to people's thinking habits, similar opinions are more acceptable to each other. And in the hesitant fuzzy environment, the similarity degree between opinions can be described by similarity measure. Based on this idea, the opinion transition matrix can be established.

Suppose that at the time $t = 0$, the initial opinions of the DMs are $z_1(0), z_2(0), \cdots, z_n(0)$ and s_{ij} is the similar measure between $z_i(0)$ and $z_j(0)$. Then we get the opinion transition matrix $P = (p_{ij})_{n \times n}$, where

$$p_{ij(1)} = \frac{s_{ij}}{\sum\limits_{j=1}^{n} s_{ij}} \tag{12}$$

is the opinion transition weight assigned by z_i to z_j and the HF-DeGroot opinion dynamics model is time-invariant.

Plan B: Opinion transition matrix based on the self-confident degree

In the hesitant fuzzy theory, the fluctuation of membership degrees in HFEs is measured by the deviation degree, which is similar to the concept of mean square deviation in statistics. The DM's self-confidence can be reflected by the deviation degree of the HFEs. The more confident the DM is, the smaller the deviation degree of the data will be. Meanwhile in real life, DMs with higher self-confidence are hard to transform their own opinions with others, so we should appropriately increase the weights of their own opinions. Based on the above analysis, we will add the factor of self-confidence, namely the deviation degree, to expand the transition weights of the opinions.

The values of deviation degrees $\alpha(z_i)$ for $z_i(0) \, (i = 1, 2, \cdots, n)$ are almost close to 10^{-2} since the membership degrees are within [0,1]. To be better compared with the similarity measures, the deviation degrees should be standardized and then adjusted by the importing parameter $\rho > 1$. The specific steps to calculate the transition weights are shown below:

Algorithm I

Input: A set of HFEs $z_1, z_2, \cdots, z_n$ and the parameters ρ.

Output: the opinion transition matrix P for $z_i \, (i = 1, 2, \cdots, n)$.

Step 1. Use Eq.(2) to get the deviation degrees $\alpha(z_i)$ for $z_i \, (i = 1, 2, \cdots, n)$.

Step 2. Standardize these deviation degrees by using Eq.(13):

$$\bar{\alpha}(z_i) = \frac{\max\limits_{i}(\alpha(z_i)) - \alpha(z_i)}{\max\limits_{i}(\alpha(z_i)) - \min\limits_{i}(\alpha(z_i))}, i = 1, 2, \cdots, n \tag{13}$$

Step 3. Combining with Eq.(12), determine the parameters ρ, and get the weight assignment of z_i in the transformation of opinion as follows:

$$p_{ij(2)} = \begin{cases} \dfrac{s_{ij}}{1 + \rho\bar{\alpha}(z_i) + \sum\limits_{j=1, j \neq i}^{n} s_{ij}}, i \neq j \\ \dfrac{1 + \rho\bar{\alpha}(z_i)}{1 + \rho\bar{\alpha}(z_i) + \sum\limits_{j=1, j \neq i}^{n} s_{ij}}, i = j \end{cases} \tag{14}$$

Step 4. End.

Note 2. The method mentioned in Eq.(13) is not the only formula for standardizing deviation degree. There are other valid formulas such as Eq.(15). This method is to standardize the deviation degree by its entire value range [0,0.5].

$$\bar{\alpha}'(z_i) = \frac{0.5 - \alpha(z_i)}{0.5}, i = 1, 2, \cdots, n \tag{15}$$

The two methods of standardization of deviation degrees have their own advantages. Eq.(15) still can be used when Eq.(13) is invalid for a group of deviation degrees are all equal. But Eq.(13)

considers more interaction between the opinions.

Plan C: Opinion transition matrix based on the authority degree

In the common decision-making process, since the authority in some areas is powerful enough to change people's opinions, it is reasonable to give more weights to their opinions. Based on the above analysis, the parameter $\eta > 1$ is introduced to simulate the influence of authority. For convenience, the qth DM is assumed to be the authority. In this way, the transition weight assigned by z_i to z_j is determined as follows:

$$p_{ij(3)} = \begin{cases} \dfrac{\frac{s_{ij}}{n}}{\eta s_{iq} + \sum\limits_{i=1, j \neq q}^{n} s_{ij}}, j \neq q \\ \dfrac{\frac{\eta s_{iq}}{n}}{\eta s_{iq} + \sum\limits_{i=1, j \neq q}^{n} s_{ij}}, j = q \end{cases} \tag{16}$$

4.2 Evolution analysis of the HF-DeGroot opinion dynamics model

Previously, we defined the HF-DeGroot model, mined the implicit information in hesitant fuzzy data, and determined the weight transition matrices. Next, the evolution of various opinions in the hesitant fuzzy environment will be specifically analyzed according to the hypothesis that the HF-DeGroot opinion dynamics model is time-invariant.

With the help of the opinion transition matrix established in Section 4.1, we consider the following two scenarios.

Scenario 1. There are only two types of opinions: $\{1\}$ ***and*** $\{0\}$

At this time, DMs have very different opinions, but whether they can reach consensus requires further analysis.

Because the similarity measures of $\{1\}$ $\{1\}$ and $\{0\}$ $\{0\}$ is 1 but the similarity measure of $\{0\}$ $\{1\}$ is 0. For the sake of convenience, the same opinions will be put together in the following, which will not affect the final results.

Let there be n DMs, among which m opinions are $\{1\}$ and n-m opinions are $\{0\}$. We discuss the evolution results of these opinions under the three different transfer modes mentioned in Plan A, Plan B and Plan C respectively:

(1) Evolution based on Plan A

According to Eq.(12) in Plan A, the opinion transition matrix of the opinion set

$$\left\{ \underbrace{\{1\}, \cdots, \{1\}}_{m}, \underbrace{\{0\}, \cdots, \{0\}}_{n-m} \right\}^{T} \quad \text{is:}$$

$$Q_1 = \begin{pmatrix} \frac{1}{m} & \cdots & \frac{1}{m} & 0 & \cdots & 0 \\ & \ddots & & & & \vdots \\ \frac{1}{m} & \cdots & \frac{1}{m} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \frac{1}{n-m} & & \frac{1}{n-m} \\ & \cdots & & & \ddots & \\ 0 & & 0 & \frac{1}{n-m} & & \frac{1}{n-m} \end{pmatrix}$$

Since $Q_1{}^t = Q_1$, then $\lim\limits_{t \to \infty} Q_1{}^t = Q_1$,

$$Z(\infty) = Q_1 \left\{ \underbrace{\{1\}, \cdots, \{1\}}_{m}, \underbrace{\{0\}, \cdots, \{0\}}_{n-m} \right\}^{T} = \left\{ \underbrace{\{1\}, \cdots, \{1\}}_{m}, \underbrace{\{0\}, \cdots, \{0\}}_{n-m} \right\}^{T}.$$

This means that these DMs can't reach consensus and no one changes their opinion.

(2) Evolution based on Plan B

Since these DMs only have two kinds of opinions $\{0\}$ and $\{1\}$, whose deviation degrees are both 0. They correspond to the highest degree of confidence. The opinion transition matrix obtained by Eqs.(14) and (15) is:

$$Q_2 = \begin{pmatrix} \frac{1+\rho}{m+\rho} & \cdots & \frac{1}{m+\rho} & 0 & \cdots & 0 \\ & \ddots & & & & \vdots \\ \frac{1}{m+\rho} & \cdots & \frac{1+\rho}{m+\rho} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \frac{1+\rho}{n-m+\rho} & & \frac{1}{n-m+\rho} \\ & \cdots & & & \ddots & \\ 0 & & 0 & \frac{1}{n-m+\rho} & & \frac{1+\rho}{n-m+\rho} \end{pmatrix} \qquad (17)$$

Let

$$Q_{21} = \begin{pmatrix} \frac{1+\rho}{m+\rho} & \cdots & \frac{1}{m+\rho} \\ & \ddots & \\ \frac{1}{m+\rho} & \cdots & \frac{1+\rho}{m+\rho} \end{pmatrix}, Q_{22} = \begin{pmatrix} \frac{1+\rho}{n-m+\rho} & \cdots & \frac{1}{n-m+\rho} \\ & \ddots & \\ \frac{1}{n-m+\rho} & \cdots & \frac{1+\rho}{n-m+\rho} \end{pmatrix},$$

Then in the matrix Q_2,

$$Q_2 = \begin{pmatrix} Q_{21} & O \\ O & Q_{22} \end{pmatrix},$$

both Q_{21} and Q_{22} are doubly stochastic matrixes. According to Theorem 2,

$$\lim_{t \to \infty} Q_{21}^t = \begin{pmatrix} \frac{1}{m} & \cdots & \frac{1}{m} \\ & \ddots & \\ \frac{1}{m} & \cdots & \frac{1}{m} \end{pmatrix}, \lim_{t \to \infty} Q_{22}^t = \begin{pmatrix} \frac{1}{n-m} & \cdots & \frac{1}{n-m} \\ & \ddots & \\ \frac{1}{n-m} & \cdots & \frac{1}{n-m} \end{pmatrix},$$

$$Q_2^t = \begin{pmatrix} Q_{21}^t & O \\ O & Q_{22}^t \end{pmatrix},$$

we can get $\lim_{t \to \infty} Q_2{}^t = Q_1$,

$$Z(\infty) = Q_1 \left\{ \underbrace{\{1\}, \cdots, \{1\}}_{m}, \underbrace{\{0\}, \cdots, \{0\}}_{n-m} \right\}^T = \left\{ \underbrace{\{1\}, \cdots, \{1\}}_{m}, \underbrace{\{0\}, \cdots, \{0\}}_{n-m} \right\}^T.$$

This suggests that when all DMs are confident, these opinions do not reach consensus.

(3) Evolution based on Plan C

Assume that the authority parameter of the first DM is $\eta > 1$ for the convenience. According to Eq.(16), the opinion transition matrix is obtained as:

$$Q_3 = \begin{pmatrix} \frac{\eta}{\eta+m-1} & \cdots & \frac{1}{\eta+m-1} & 0 & \cdots & 0 \\ & \ddots & & & & \vdots \\ \frac{\eta}{\eta+m-1} & \cdots & \frac{1}{\eta+m-1} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \frac{1}{n-m} & & \frac{1}{n-m} \\ & \cdots & & & \ddots & \\ 0 & & 0 & \frac{1}{n-m} & & \frac{1}{n-m} \end{pmatrix} \qquad (18)$$

Let

$$Q_3 = \begin{pmatrix} Q_{31} & \\ & Q_{32} \end{pmatrix}, Q_3^t = \begin{pmatrix} Q_{31}^t & O \\ O & Q_{32}^t \end{pmatrix},$$

$$Q_{31} = \begin{pmatrix} \frac{\eta}{\eta+m-1} & \frac{1}{\eta+m-1} & \cdots & \frac{1}{\eta+m-1} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\eta}{\eta+m-1} & \frac{1}{\eta+m-1} & \cdots & \frac{1}{\eta+m-1} \end{pmatrix}_{m \times m} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}_{m \times 1} \left(\frac{\eta}{\eta+m-1}, \frac{1}{\eta+m-1}, \cdots, \frac{1}{\eta+m-1} \right)_{1 \times m},$$

$$Q_{31}^t = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \left[\left(\frac{\eta}{\eta+m-1}, \frac{1}{\eta+m-1}, \cdots, \frac{1}{\eta+m-1} \right) \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \right]^{t-1}$$

$$\cdot \left(\frac{\eta}{\eta+m-1}, \frac{1}{\eta+m-1}, \cdots, \frac{1}{\eta+m-1} \right) = Q_{31}$$

$$Q_{32}{}^t = Q_{32},$$

Then $\lim_{t \to \infty} Q_3{}^t = Q_3$,

$$Z(\infty) = Q_3 \left\{ \underbrace{\{1\}, \cdots, \{1\}}_{m}, \underbrace{\{0\}, \cdots, \{0\}}_{n-m} \right\}^T = \left\{ \underbrace{\{1\}, \cdots, \{1\}}_{m}, \underbrace{\{0\}, \cdots, \{0\}}_{n-m} \right\}^T.$$

The results show that no one has changed his opinion.

From the perspective of MC theory [20], all the opinions compose the state space $D = \{1, 2, \cdots, n\}$ which is decomposed into two irreducible closed sets $D_1 = \{1, 2, \cdots, m\}$ and $D_2 = \{m+1, m+2, \cdots, n\}$. No matter what is the transfer probability between the internal states of D_1 and D_2, the states in D_1 and D_2 cannot communicate with each other. The principle can be simply expressed by the following Figure 2:
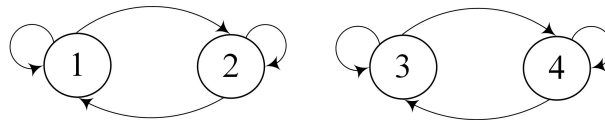

Figure 2: Transformation of opinions in Scenario 1. (States 1 and 2 represent the opinions of $\{1\}$, States 3 and 4 represent the opinions of $\{0\}$)

It can be seen from the above simulation that when there are only two diametrically opposed opinions, neither the influence of similarity, self-confidence, or authority will cause the DMs to change their original opinions. This refusal to transfer from one opinion to another is the cause of the failure to reaches a consensus, which conforms the fact. In addition, when there are only $\{0\}$ and $\{1\}$ opinions, these HFEs have been completely reduced to membership degrees, so Scenario 1 describes only the extreme situation of the evolution of the hesitant fuzzy opinions. In the following, the HF-DeGroot opinion dynamics model is used to study the transformation trend of the opinions of incompletely opposite groups, which is also the general situation of the evolution of opinions.

Scenario 2. There are not just $\{1\}$ and $\{0\}$ among DMs

The following is a discussion on the general situations in which multiple opinions coexist. For convenience, we replace the last DM in Scenario 1, whose opinion is neither $\{0\}$ nor $\{1\}$. Suppose that there are n DMs, the first m-bit opinion is $\{1\}$, the n-m-1 bit opinion is $\{0\}$, and the last 1-bit DM opinion is expressed by a HFE h, the similarity measure between his opinion and the previous DMs is as follows:

$$s_{in} = s_{ni} = \begin{cases} s_{1n}, & i = 1, 2, \cdots, m \\ s_{m+1,n}, & i = m+1, m+2, \cdots, n-1 \\ 1, & i = n \end{cases}.$$

Using Eq.(12), the opinion transition matrix is obtained as follows:

$$Q_4 = \begin{pmatrix} p_{11} & \cdots & p_{11} & 0 & \cdots & 0 & p_{1n} \\ & \vdots & & & \vdots & & \vdots \\ p_{11} & \cdots & p_{11} & 0 & \cdots & 0 & p_{1n} \\ 0 & \cdots & 0 & p_{m+1,m+1} & \cdots & p_{m+1,m+1} & p_{m+1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & p_{m+1,m+1} & \cdots & p_{m+1,m+1} & p_{m+1,n} \\ p_{n1} & \cdots & p_{n1} & p_{n,m+1} & \cdots & p_{n,m+1} & p_{n,n} \end{pmatrix} \tag{19}$$

Here

$$p_{11} = p_{ij} = \frac{1}{m + s_{1n}} \left(i = 1, 2, \cdots, m; j = 1, 2, \cdots, m \right),$$

$$p_{1n} = p_{in} = \frac{s_{1n}}{m + s_{1n}} \left(i = 1, 2, \cdots, m \right),$$

$$p_{m+1,m+1} = p_{ij} = \frac{1}{n - m - 1 + s_{m+1,n}} \left(i = m + 1, \cdots, n - m - 1; j = m + 1, \cdots, n - m - 1 \right),$$

$$p_{m+1,n} = p_{in} = \frac{s_{m+1,n}}{n - m - 1 + s_{m+1,n}} \left(i = m + 1, \cdots, n - 1 \right),$$

$$p_{n1} = p_{nj} = \frac{s_{1n}}{m s_{1n} + (n - m - 1) s_{m+1,n} + 1} \left(j = 1, 2, \cdots, m \right),$$

$$p_{n,m+1} = p_{nj} = \frac{s_{m+1,n}}{m s_{1n} + (n - m - 1) s_{m+1,n} + 1} \left(j = m + 1, m + 2, \cdots, n - 1 \right),$$

$$p_{nn} = \frac{1}{m s_{1n} + (n - m - 1) s_{m+1,n} + 1}.$$

Since $s_{in} > 0 \ (i = 1, 2, \cdots, n)$, then the elements of the n column of Q_4 are all positive, which satisfies the condition in Theorem 3, the DMs' opinions will finally reach consensus. That is to say, based on the similarity degree of opinions, as long as there is a third opinion besides agreement and disagreement, these opinions will reach consensus in the end. Similarly, by adding the influence of the self-confidence degree or authority degree, these opinions can also reach a consistent conclusion.

In Scenario 2, the opinions reach consensus because the transition matrix P is fully regular in which $\lim_{t \to \infty} P^t = 1_n (p_1, p_2, \cdots, p_n)$. And $p_i \ (i = 1, 2, \cdots, n)$ can be seen as the weight of the initial hesitant fuzzy opinion $z_i (0)$ in the final opinion. By referring to the theory of MC, the transformation relationship between the opinions shown in Scenario 2 can be expressed in Figure 3 as follows:
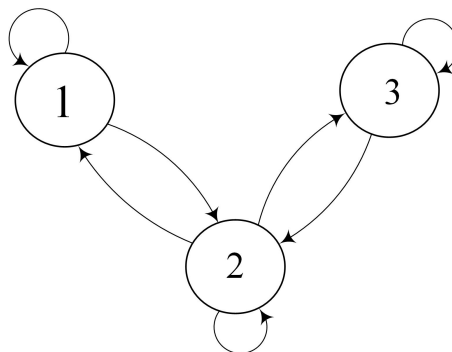
Figure 3: Transformation of opinions in Scenario 2. (State 1 represents the opinions of $\{1\}$, State 2 represents the opinions of h, and State 3 represents the opinions of $\{0\}$)

This is actually an irreducible aperiodic MC with the finite state space or ergodic MC.

The final consensus opinion is the weighted average of the weight vector $(p_1, p_2, \cdots, p_n)$ and the initial opinion matrix $(z_1 (0), z_2 (0), \cdots, z_n (0))^T$, which is also the prediction of the final opinion of the group and can provide help for the hesitant fuzzy decision-making problem.

4.3 The decision-making method based on the HF-DeGroot opinion dynamics model

Based on the above analysis in Section 4.2, when the initial opinions are presented, we can get the prediction of the final opinions from a dynamic perspective. Therefore, it can be applied to deal with the decision-making problems that change quickly. In the following, three corresponding algorithms will be given.

Algorithm II. HF-DeGroot decision-making method based on the similarity degree

Step 1. Identify the decision-making problem and the alternatives in $S = \left\{ S^{(1)}, S^{(2)}, \cdots, S^{(m)} \right\}$, and obtain the evaluation information of the alternatives in $S^{(k)}$ given by DMs $M_1, M_2, \cdots, M_n$ in the form of HFSs $\left\{ h_1^{(k)}, h_2^{(k)}, \cdots, h_n^{(k)} \right\} (k = 1, 2, \cdots m)$.

Step 2. Obtain the similarity measure matrix. This step includes the use of Eqs.(4) or (5) to calculate the similarity measures $s_{ij}^{(k)}$ between the opinions from M_i and M_j on the alternative $S^{(k)}$.

$$S^{(k)} = \begin{pmatrix} s_{11}^{(k)} & \cdots & s_{1n}^{(k)} \\ \vdots & \cdots & \vdots \\ s_{n1}^{(k)} & \cdots & s_{nn}^{(k)} \end{pmatrix} \tag{20}$$

Step 3. According to Eq.(12), we obtain the DM's opinion transition matrix $P^{(k)}$ on the alternative in $S^{(k)}$ as follows:

$$P^{(k)} = \begin{pmatrix} p_{11}^{(k)} & \cdots & p_{1n}^{(k)} \\ \vdots & \cdots & \vdots \\ p_{n1}^{(k)} & \cdots & p_{nn}^{(k)} \end{pmatrix} \tag{21}$$

Step 4. Calculate the limit of the matrix $\left(P^{(k)} \right)^t$. If the $P^{(k)}$ is fully regular,

$$\lim_{t \to \infty} \left(P^{(k)} \right)^t = \begin{pmatrix} w_1^{(k)} & w_2^{(k)} & \cdots & w_n^{(k)} \\ w_1^{(k)} & w_2^{(k)} & \cdots & w_n^{(k)} \\ \vdots & \vdots & \vdots & \vdots \\ w_1^{(k)} & w_2^{(k)} & \cdots & w_n^{(k)} \end{pmatrix} \tag{22}$$

then we extract $w^{(k)} = \left(w_1^{(k)}, w_2^{(k)}, \cdots, w_n^{(k)} \right)$ from the matrix in Eq.(22) which is DMs' final weight vector for $S^{(k)}$, then go to the next step, otherwise the algorithm stops.

Step 5. Use the operator (3) and $w^{(k)}$ to get the final opinions for $S^{(k)}(k = 1, 2, \cdots, m)$, and then sort them according to the score function in Eq.(1) by the descending order.

Algorithm III. HF-DeGroot decision-making method based on the self-confidence degree

Step 1-2. Same as Steps 1-2 in Algorithm II.

Step 3. Determine the parameter $\rho > 1$ and use Algorithm I to calculate the opinion transition matrix $P^{(k)}$.

Step 4-5. Same as Steps 4-5 in Algorithm II.

Algorithm IV. HF-DeGroot decision-making method based on the authority degree

Step 1-2. Same as Step 1-2 in Algorithm II.

Step 3. Determine the parameter $\eta > 1$ and use Eq.(16) to calculate the opinion transition matrix $P^{(k)}$.

Step 4-5. Same as Steps 4-5 in Algorithm II.

5 Numerical examples and analysis

This section presents a case of public health emergency with three different methods.

5.1 Background

Although modern medicine has been quite developed, there are still many public health emergencies such as the H1N1 Pandemic Influenza in 2009 and the Novel Coronavirus (COVID-19) in 2019[1]. In response to the emergence and international spread of the emerging infectious diseases, WHO has enacted the International Health Regulations (IHR)[2] that regulate the detection, assessment and response to the public health events.

Many countries have also established crisis response mechanisms to deal with the public health emergencies. The Chinese government has established and revised a series of laws[3] and regulations[4] which specify that when public health emergency occurs, the health administration department should organize experts to comprehensively evaluate the events, then the central government and local governments shall establish corresponding emergency response plans. The National Response Framework[5] established by the United States government is a guide for emergency response to all types of threats and hazards, including health emergencies. The response framework specify that the federal government, state and local governments shall form a three-level response system, and each of them shall play the required role in the process of incident assessment and graded emergency response.

The assessment of the events is an important step in the active response to the public health emergencies, both for WHO and for countries around the world. But making a reasonable assessment is often very difficult. Firstly, the assessment of such events is full of uncertainty because that many public health emergencies are unpredictable and lack of a basic understanding of hazard and transmission mode in the early stage. Secondly, in the process of evaluation, the DMs is hesitated to make a decision due to considering the event comprehensively from the economic, political, cultural and other aspects besides the medical perspective. Thirdly, the events are constantly changing. So the HF-DeGroot opinion dynamics model in this paper can be used to support DMs to make decision analysis from the perspective of dynamic development.

Suppose that a local government sets up a decision-making group $M = \{M_1, M_2, \cdots, M_5\}$ composed of government officials, medical experts, disease control experts, etc. They want to evaluate a public health event according to the pre-set four response levels $S = \left\{S^{(1)}, S^{(2)}, S^{(3)}, S^{(4)}\right\}$.

Table 1: The hesitant fuzzy decision matrix

	M_1	M_2	M_3	M_4	M_5
$S^{(1)}$	$\{0.2\}$	$\{0.4, 0.3\}$	$\{0.7, 0.2\}$	$\{0.9, 0.8\}$	$\{0.8, 0.7, 0.6\}$
$S^{(2)}$	$\{0.9, 0.8\}$	$\{0.4, 0.3, 0.2\}$	$\{0.7, 0.4\}$	$\{0.6, 0.4, 0.3\}$	$\{0.3, 0.2\}$
$S^{(3)}$	$\{0.7, 0.2\}$	$\{0.9, 0.8\}$	$\{0.2\}$	$\{0.4, 0.2, 0.1\}$	$\{0.7, 0.3\}$
$S^{(4)}$	$\{0.7, 0.6\}$	$\{0.8, 0.4\}$	$\{0.5, 0.4\}$	$\{0.2, 0.1\}$	$\{0.6, 0.5, 0.4\}$

The three decision-making algorithms introduced in Section 4.3 will be applied in the following problems.

5.2 Decision-making process based on Algorithm II

The detailed decision-making process based on the similarity measure mentioned in Section 4.3 will be shown as follows:

Step 1. According to the above analysis, the decision-making problem is to select the appropriate response level from the four alternatives. Here, the DM's opinions on the four response levels are listed in Table 1.

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[1]WHO, Emergencies, https://www.who.int/emergencies/crises/en/, 2020.

[2]WHO, Strengthening health security by implementing the International Health Regulations (2005), https://www.who.int/ihr/en/, 2019.

[3]The NPC Standing Committee, Law of the People's Republic of China on the Prevention and Treatment of Infections Diseases, http://www.npc.gov.cn/npc/c238/202001/099a493d03774811b058f0f0ece38078.shtml, 2020

[4]C. S. Council, Overall emergency plan for national public emergencies, http://www.gov.cn/yjgl/2006-01/08/ content_21048.htm, 2006.

[5]FEMA, National response framework, Fourth Edition, https://www.fema.gov/media-library/assets/documents /117791, 2020.

Step 2. Calculate the similarity degrees between these decision opinions for the response level $S^{(k)}$ $(k = 1, 2, 3, 4)$ by using Eq.(5). These results are listed in Table 2-5.

Table 2: Similarity degrees between DMs' opinions for $S^{(1)}$

$s_{ij}^{(1)}$	M_1	M_2	M_3	M_4	M_5
M_1	1.0000	0.8419	0.6464	0.3481	0.4934
M_2	0.8419	1.0000	0.7764	0.5000	0.6303
M_3	0.6464	0.7764	1.0000	0.5528	0.6258
M_4	0.3481	0.5000	0.5528	1.0000	0.8586
M_5	0.4934	0.6303	0.6258	0.8586	1.0000

Table 3: Similarity degrees between DMs' opinions for $S^{(2)}$

$s_{ij}^{(2)}$	M_1	M_2	M_3	M_4	M_5
M_1	1.0000	0.4646	0.6838	0.5918	0.4000
M_2	0.4646	1.0000	0.7840	0.8586	0.9184
M_3	0.6838	0.7840	1.0000	0.9184	0.6838
M_4	0.5918	0.8586	0.9184	1.0000	0.7840
M_5	0.4000	0.9184	0.6838	0.7840	1.0000

Table 4: Similarity degrees between DMs' opinions for $S^{(3)}$

$s_{ij}^{(3)}$	M_1	M_2	M_3	M_4	M_5
M_1	1.0000	0.5528	0.6464	0.8174	0.9000
M_2	0.5528	1.0000	0.3481	0.3945	0.4615
M_3	0.6464	0.3481	1.0000	0.8709	0.7085
M_4	0.8174	0.3945	0.8709	1.0000	0.8709
M_5	0.9000	0.4615	0.7085	0.8709	1.0000

Table 5: Similarity degrees between DMs' opinions for $S^{(4)}$

$s_{ij}^{(4)}$	M_1	M_2	M_3	M_4	M_5
M_1	1.0000	0.8419	0.8000	0.5000	0.8586
M_2	0.8419	1.0000	0.7879	0.5257	0.8709
M_3	0.8000	0.7879	1.0000	0.7000	0.9184
M_4	0.5000	0.5257	0.7000	1.0000	0.6303
M_5	0.8586	0.8709	0.9184	0.6303	1.0000

Step 3. According to Eq.(12), we calculate the DMs' transition weights for the response level $S^{(k)}$ $(k = 1, 2, 3, 4)$ listed in Table 6-9.

Table 6: The opinion transition matrix based on the similarity degrees for $S^{(1)}$

$p_{ij}^{(1)}$	M_1	M_2	M_3	M_4	M_5
M_1	0.3003	0.2528	0.1941	0.1045	0.1482
M_2	0.2246	0.2668	0.2071	0.1334	0.1681
M_3	0.1795	0.2156	0.2777	0.1535	0.1738
M_4	0.1068	0.1534	0.1696	0.3068	0.2634
M_5	0.1367	0.1747	0.1735	0.2380	0.2772

Table 7: The opinion transition matrix based on the similarity degrees for $S^{(2)}$

$p_{ij}^{(2)}$	M_1	M_2	M_3	M_4	M_5
M_1	0.3185	0.1480	0.2178	0.1884	0.1274
M_2	0.1154	0.2484	0.1948	0.2133	0.2281
M_3	0.1680	0.1926	0.2457	0.2256	0.1680
M_4	0.1425	0.2068	0.2211	0.2408	0.1888
M_5	0.1056	0.2426	0.1806	0.2071	0.2641

Table 8: The opinion transition matrix based on the similarity degrees for $S^{(3)}$

$p_{ij}^{(3)}$	M_1	M_2	M_3	M_4	M_5
M_1	0.2534	0.1401	0.1638	0.2072	0.2355
M_2	0.1897	0.3431	0.1194	0.1353	0.2125
M_3	0.1844	0.0993	0.2853	0.2485	0.1824
M_4	0.2114	0.1020	0.2252	0.2586	0.2027
M_5	0.2340	0.1559	0.1610	0.1974	0.2518

Table 9: The opinion transition matrix based on the similarity degrees for $S^{(4)}$

$p_{ij}^{(4)}$	M_1	M_2	M_3	M_4	M_5
M_1	0.2500	0.2104	0.2000	0.1250	0.2146
M_2	0.2091	0.2484	0.1957	0.1306	0.2163
M_3	0.1902	0.1873	0.2377	0.1664	0.2183
M_4	0.1490	0.1566	0.2086	0.2980	0.1878
M_5	0.2007	0.2036	0.2147	0.1473	0.2337

Table 10: Final weight assignment matrix based on Algorithm II

$w^{(i)}$	M_1	M_2	M_3	M_4	M_5
$S^{(1)}$	0.1898	0.2136	0.2052	0.1858	0.2056
$S^{(2)}$	0.1638	0.2099	0.2123	0.2166	0.1975
$S^{(3)}$	0.2168	0.1601	0.1925	0.2124	0.2182
$S^{(4)}$	0.2014	0.2027	0.2117	0.1689	0.2153

Step 4. Obtain the final opinion transition matrix for DMs shown in Table 10.
Step 5. Use the AHFWA operator expressed in Eq.(3) to obtain the final result and sort it as $S^{(1)} \succ S^{(2)} \succ S^{(4)} \succ S^{(3)}$, so $S^{(1)}$ is the emergency response level of the event.

5.3 Decision-making process based on Algorithm III

The study on how a person's self-confidence affects the final decision will be shown in this section. Assume that parameter $\rho = 5$ in the algorithm.
Step 1-2. The similarity matrix for $S^{(k)}$ $(k = 1, 2, 3, 4)$ is obtained in the same way shown in Step 1 and Step 2 in Section 5.2.
Step 3. Use Algorithm I to calculate the transition weight of the DM's opinion for $S^{(k)}$ $(k = 1, 2, 3, 4)$ listed in Table 11-14.
Step 4. Obtain the final opinion weight matrix shown in Table 15.
Step 5. Use Eq.(3) to aggregate evaluations and sort the four levels as: $S^{(1)} \succ S^{(2)} \succ S^{(3)} \succ S^{(4)}$. So $S^{(1)}$ is the final emergency response level.

Table 11: The opinion transition matrix based on self-confidence degree for $S^{(1)}$

$p_{ij}^{(1)}$	M_1	M_2	M_3	M_4	M_5
M_1	0.7203	0.1011	0.0776	0.0418	0.0592
M_2	0.1087	0.6453	0.1002	0.0645	0.0813
M_3	0.1795	0.2156	0.2777	0.1535	0.1738
M_4	0.0479	0.0689	0.0761	0.6888	0.1183
M_5	0.0707	0.0904	0.0897	0.1231	0.6261

Table 12: The opinion transition matrix based on self-confidence degree for $S^{(2)}$

$p_{ij}^{(2)}$	M_1	M_2	M_3	M_4	M_5
M_1	0.7371	0.0571	0.0840	0.0727	0.0491
M_2	0.0624	0.5935	0.1053	0.1154	0.1234
M_3	0.1680	0.1926	0.2457	0.2256	0.1680
M_4	0.1092	0.1585	0.1695	0.4180	0.1447
M_5	0.0455	0.1045	0.0778	0.0892	0.6829

Table 13: The opinion transition matrix based on self-confidence degree for $S^{(3)}$

$p_{ij}^{(3)}$	M_1	M_2	M_3	M_4	M_5
M_1	0.2534	0.1401	0.1638	0.2072	0.2355
M_2	0.0799	0.7231	0.0503	0.0570	0.0896
M_3	0.0760	0.0409	0.7055	0.1024	0.0752
M_4	0.1283	0.0619	0.1367	0.5501	0.1230
M_5	0.1869	0.1245	0.1286	0.1577	0.4023

Table 14: The opinion transition matrix based on self-confidence degree for $S^{(4)}$

$p_{ij}^{(4)}$	M_1	M_2	M_3	M_4	M_5
M_1	0.6666	0.0935	0.0889	0.0556	0.0954
M_2	0.2091	0.2484	0.1957	0.1306	0.2163
M_3	0.0869	0.0856	0.6517	0.0760	0.0998
M_4	0.0598	0.0629	0.0838	0.7180	0.0754
M_5	0.1044	0.1059	0.1117	0.0767	0.6014

Table 15: Final weight assignment matrix based on Algorithm III

$w^{(i)}$	M_1	M_2	M_3	M_4	M_5
$S^{(1)}$	0.2456	0.2285	0.1062	0.2140	0.2057
$S^{(2)}$	0.2404	0.2198	0.1202	0.1600	0.2595
$S^{(3)}$	0.1285	0.2252	0.2769	0.2075	0.1619
$S^{(4)}$	0.2319	0.1037	0.2372	0.2153	0.2119

5.4 Decision-making process based on Algorithm IV

This section discusses the role of the authority in decision making. As in Section 5.3, our calculation starts from Step 3 in Algorithm IV and it is assumed that the authoritative parameter $\eta = 4$.

Step 1-2. The similarity matrix for $S^{(k)}$ $(k = 1, 2, 3, 4)$ is obtained in the same way as shown in Step 1 and Step 2 in Section 5.2.

Step 3. According to Eq.(16), we calculate the transition weight of the DM's opinion corresponding to the response level $S^{(k)}$ $(k = 1, 2, 3, 4)$ listed in Table 16-20.

Step 4. Obtain the final opinion weight matrix shown in Table 20.

Step 5. Use Eq.(3) to obtain the aggregation result and get $S^{(2)} \succ S^{(4)} \succ S^{(3)} \succ S^{(1)}$, so the best

Table 16: The opinion transition matrix based on the authority degree for $S^{(1)}$

$p_{ij}^{(1)}$	M_1	M_2	M_3	M_4	M_5
M_1	0.6319	0.1330	0.1021	0.0550	0.0779
M_2	0.5367	0.1594	0.1237	0.0797	0.1005
M_3	0.4667	0.1401	0.1805	0.0998	0.1130
M_4	0.3235	0.1162	0.1284	0.2324	0.1995
M_5	0.3879	0.1239	0.1230	0.1687	0.1965

Table 17: The opinion transition matrix based on the authority degree for $S^{(2)}$

$p_{ij}^{(2)}$	M_1	M_2	M_3	M_4	M_5
M_1	0.6515	0.0757	0.1114	0.0964	0.0651
M_2	0.3429	0.1845	0.1447	0.1584	0.1695
M_3	0.4468	0.1281	0.1634	0.1500	0.1117
M_4	0.3993	0.1448	0.1549	0.1687	0.1323
M_5	0.3209	0.1842	0.1371	0.1572	0.2006

Table 18: The opinion transition matrix based on the authority degree for $S^{(3)}$

$p_{ij}^{(3)}$	M_1	M_2	M_3	M_4	M_5
M_1	0.5759	0.0796	0.0931	0.1177	0.1338
M_2	0.4835	0.2187	0.0761	0.0863	0.1354
M_3	0.4750	0.0639	0.1837	0.1600	0.1175
M_4	0.5174	0.0624	0.1378	0.1583	0.1241
M_5	0.5499	0.0916	0.0946	0.1160	0.1479

Table 19: The opinion transition matrix based on the authority degree for $S^{(4)}$

$p_{ij}^{(4)}$	M_1	M_2	M_3	M_4	M_5
M_1	0.5714	0.1203	0.1143	0.0714	0.1226
M_2	0.5140	0.1526	0.1202	0.0802	0.1329
M_3	0.4844	0.1193	0.1514	0.1060	0.1390
M_4	0.4119	0.1082	0.1442	0.2059	0.1298
M_5	0.5011	0.1271	0.1340	0.0920	0.1459

Table 20: Final weight assignment matrix based on Algorithm IV

$w^{(i)}$	M_1	M_2	M_3	M_4	M_5
$S^{(1)}$	0.5442	0.1349	0.1191	0.0925	0.1094
$S^{(2)}$	0.5224	0.1153	0.1302	0.1261	0.1061
$S^{(3)}$	0.5461	0.0899	0.1070	0.1242	0.1329
$S^{(4)}$	0.5296	0.1239	0.1250	0.0918	0.1296

response level is $S^{(2)}$.

6   Comparative analysis

By the discussions in Section 5, the decision-making method based on the proposed HF-DeGroot opinion dynamics model will be further studied, including the comparison of the influence of the three factors on the decision making, the comparison with other decision-making method, and the discussion on the sensitivity of the parameters.

To further analyze these algorithms, Algorithm V which gives the DMs the weight $w = \left\{ \frac{1}{2}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8} \right\}$

is added and uses the AHFWA operator to perform the operation. The results of the four algorithms are listed in the following Table 21:

Table 21: Comparison of sorting results by different methods

	$S^{(1)}$	$S^{(2)}$	$S^{(3)}$	$S^{(4)}$	Rank
Algorithm II	0.5578	0.5016	0.4625	0.4840	$S^{(1)} \succ S^{(2)} \succ S^{(4)} \succ S^{(3)}$
Algorithm III	0.5678	0.5315	0.4950	0.4681	$S^{(1)} \succ S^{(2)} \succ S^{(3)} \succ S^{(4)}$
Algorithm IV	0.4177	0.6901	0.4243	0.5538	$S^{(2)} \succ S^{(4)} \succ S^{(3)} \succ S^{(1)}$
Algorithm V	0.4551	0.6789	0.4482	0.5416	$S^{(2)} \succ S^{(4)} \succ S^{(1)} \succ S^{(3)}$

6.1 Comparison between algorithms

The following analysis is divided into two categories, including the comparison among these three decision-making methods and the comparison between the three methods and Algorithm V.

6.1.1 Comparison of Algorithms II, III and IV

To validate these proposed methods, we will provide some comparative analysis on the final decision results.

The results of Table 21 show that both Algorithms II and III choose the response level $S^{(1)}$, and the ranking results of the two algorithms are similar. This result shows that the influence of the self-confidence on the final decision-making result is not particularly obvious. Different from the previous two algorithms, after adding the authority degree in Algorithm III, a completely different order is obtained. It can be seen that the authority degree has a much greater impact on the final decision-making result than the similarity degree and self-confidence degree.

In terms of the speed to reach a consensus for the three algorithms, we compare the power $r \in R$ of the opinion transition matrix when the same criterion is reached for the first time. It should be noted that the criterion of reaching a consensus here is that the difference between all the elements in the power matrix and the corresponding elements in the limit matrix is less than 10^{-4}, which means that if the element $\tilde{P}_{ij}^{(k)}$ $(i, j = 1, 2, \cdots, 5; k = 1, 2, 3, 4)$ in $\left(P^{(k)}\right)^r$ satisfies $\left| \tilde{P}_{ij}^{(k)} - w_j^{(k)} \right| \leq 10^{-4}$, the smallest r that satisfies the inequation is the object by comparison. The speed to reaching a consensus for the three algorithms are shown in the following Figure 4.
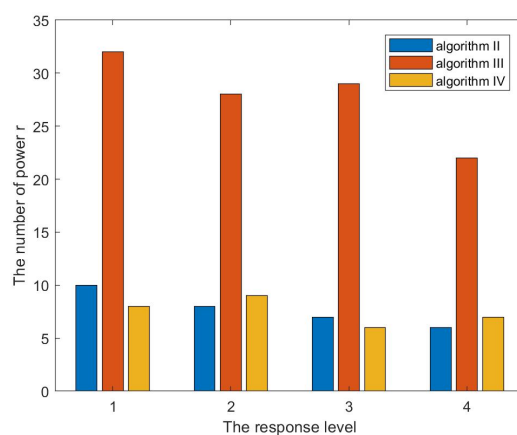

Figure 4: the speed to reaching a consensus for Algorithms II, III and IV

By using MATLAB2017b, the results are calculated and shown in Figure 4. It can be seen that in Algorithms II and IV, the group's opinions can reach consensus after at most 10 times of transmission, while Algorithm III needs at least 20 times. This shows that the similarity degree and the authority degree have little difference on the influence of consensus, while confidence is the most important factor to delay the consensus reaching.

6.1.2 Comparison between Algorithm IV and Algorithm V

To demonstrate the effectiveness of the proposed algorithms, a static decision algorithm V is introduced to compare with Algorithm IV. Because in Section 5.4, the parameter $\eta=4$, the corresponding weight of algorithm V is set as $w = \left\{\frac{4}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}\right\}$. It can be seen from Table 21 that although the final opinion ultimately chooses the response level $S^{(2)}$, the ranking results are different. The reason for this difference is that the original opinions are constantly learning and changing as they spread which exactly reflects the effectiveness and rationality of these dynamic algorithms.

6.2 Sensitivity analysis based on parameter values

In order to further understand the model, the following is mainly about the parameter sensitivity analysis for Algorithm III and Algorithm IV.

6.2.1 Parameter sensitivity analysis of Algorithm III

In Algorithm III, the parameter ρ is used to adjust the self-confidence degree. The following Figure 5 will show the change of the final score values of the response levers caused by ρ.
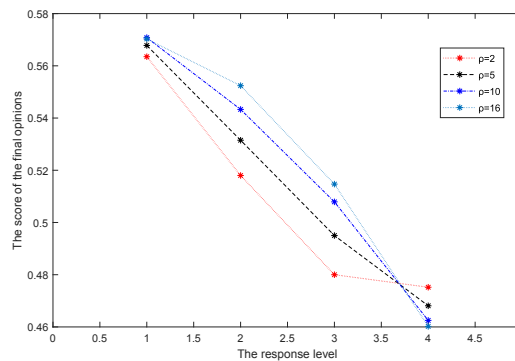

Figure 5: Scores of the response levels corresponding to different values of the parameter ρ

Here, four groups of data ($\rho = 2, 5, 10, 15$) are selected to compare the final scores of each response level. It can be seen that ρ has little impact on the final score and ranking results, especially when the values of ρ increase from 10 to 15, the scores of each response level do not increase rapidly, which indicates that Algorithm III has good robustness.

6.2.2 Parameter sensitivity analysis of Algorithm IV

In Algorithm IV, η is a parameter used to adjust the authority degree of DMs. The following Figure 6 will show the change of the final score values of the response levels caused by η.
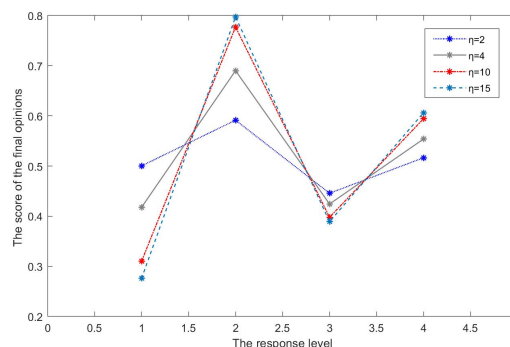

Figure 6: The score of each response level corresponding to different parameter values of η

Four groups of data ($\eta = 2, 4, 10, 15$) are selected to compare the final scores of each response level. It can be seen that η has a great impact on the final scores and ranking results and enlarge the gap between the scores of each response level, especially in $\eta = 10, 15$. Meanwhile, when η is small, such as $\eta = 2, 4$, the gap increases rapidly, and when η is large, such as $\eta = 10, 15$, the gap increases slowly. It can be concluded that the change of the authority degree has big impact on the final opinion obviously.

7   Conclusions

The study on the law of opinion dissemination can help DMs to understand information comprehensively, improve decision-making efficiency and predict the tendency of opinions to make correct decisions. With the help of DeGroot opinion dynamics model, this paper has studied the dynamic evolution of hesitant fuzzy information, and proposed the corresponding decision-making methods. Firstly, the multiplication of real matrix and hesitant fuzzy matrix have been defined for the description of the evolution law of the hesitant fuzzy opinions. Moreover, the corresponding operation properties have been discussed, which have provided the foundation for the application and extension of the hesitant fuzzy theory.

Secondly, the HF-DeGroot opinion dynamics model has been proposed. This includes combining the characteristics of the hesitant fuzzy data, constructing the transformation matrix from the three aspects: the similarity degree, self-confidence degree and authority degree, and simulating the learning and transformation process of the hesitant fuzzy opinions. The research has shown that, under the premise of the hypothesis in this paper, as long as there are not only two opinions of agreement and disagreement, these hesitant fuzzy opinions will eventually reach consensus. Therefore the HF-DeGroot opinion dynamics model can be used to predict the final consistent opinions based on DMs' initial opinions.

Finally, the HF-DeGroot decision-making method has been proposed and applied to the emergency decision making for the public health events. The comparison and simulation results have shown that this method based on opinion prediction takes full account of the development and change of the events. Compared with the traditional static decision-making method, it's more adaptive in making good decisions. At the same time, among the three factors, the similarity between the opinions and DM's self-confidence have less influence on the final result than the authority, and under the influence of self-confidence, a group of opinions will take more time to reach a consensus which is also consistent with the actual situation.

Acknowledgments

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