

Approximation algorithms for general cluster routing problem

Xiaoyan Zhang¹, Donglei Du², Gregory Gutin³, Qiaoxia Ming¹, and Jian Sun¹ (✉)*

¹ School of Mathematical Science & Institute of Mathematics, Nanjing Normal University, Jiangsu 210023, P. R. China

sunjian199203@126.com

² Faculty of Management, University of New Brunswick, Fredericton, New Brunswick, Canada, E3B 5A3

ddu@unb.ca

³ Department of Computer Science Royal Holloway, University of London Egham, Surrey, TW20 0EX, UK

g.gutin@rhul.ac.uk

Abstract. Graph routing problems have been investigated extensively in operations research, computer science and engineering due to their ubiquity and vast applications. In this paper, we study constant approximation algorithms for some variations of the general cluster routing problem. In this problem, we are given an edge-weighted complete undirected graph $G = (V, E, c)$, whose vertex set is partitioned into clusters C_1, \dots, C_k . We are also given a subset V' of V and a subset E' of E . The weight function c satisfies the triangle inequality. The goal is to find a minimum cost walk T that visits each vertex in V' only once, traverses every edge in E' at least once and for every $i \in [k]$ all vertices of C_i are traversed consecutively.

Keywords: Routing problem · Approximation algorithm · General routing problem.

1 Introduction

Graph routing problems have been studied extensively since the early 1970s. Most of these problems are NP-hard, and hence no polynomial-time exact algorithms exist for most of them unless $P=NP$. In a typical routing problem, a salesman starts from a home location, visits a set of prescribed cities exactly once, and returns to the original location with minimum total distance travelled.

Arguably the most well-known routing problem is the *travelling salesman problem (TSP)* (see [7] for a compendium of results on the problem). We are

* This research is supported or partially supported by the National Natural Science Foundation of China (Grant Nos. 11871280, 11371001, 11771386 and 11728104), the Natural Sciences and Engineering Research Council of Canada (NSERC) Grant 06446 and Qinglan Project.

given a weighted graph $G = (V, E, c)$ (directed or undirected) with vertex set V , edge set E , and cost $c(e)$ for each edge $e \in E$. The TSP's goal is to find a Hamiltonian cycle with minimum total cost. Without loss of generality, we may assume that G is a complete graph (digraph); otherwise, we could replace the missing edges with edges of very large cost.

Unfortunately, the TSP is NP-hard even for metric arc costs [12]. Therefore, one approach for solving the TSP (and other NP-hard problems) is using (polynomial-time) approximative algorithm whose performance is measured by the *approximation ratio*, which is the maximum ratio of the approximative solution value to the optimum value among all problem instances. The best known approximation algorithm for the TSP with triangle inequality is by Christofides [3] with ratio 1.5. For the general TSP where the triangle inequality does not hold, there is no (polynomial-time) approximation algorithm with a constant approximation ratio, unless $P=NP$ [20]. TSP along with its variations have been extensively investigated in the literature. Here are two generalizations of TSP studied in the literature.

The *general routing problem (GRP)*: Let $G = (V, E, c)$ be an edge-weighted complete undirected graph such that the triangle inequality holds for the weight function c . The goal is to find a minimum cost walk that visits each vertex in a required subset $V' \subseteq V$ exactly once and traverses every edge in a required subset $E' \subseteq E$ at least once. For this problem, Jansen [10] gave a 1.5-approximation algorithm.

The *cluster travelling salesman problem (CTSP)*: Let $G = (V, E, c)$ be an edge-weighted complete undirected graph such that the triangle inequality holds for the weight function c . The vertex set V is partitioned into clusters C_1, \dots, C_k . The goal is to compute a minimum cost Hamiltonian cycle T that visits all vertices of each cluster consecutively (and thus for each cluster we have starting and finishing vertices on T). Arkin et al. [1] designed a 3.5-approximation algorithm for the problem with given starting vertices in each cluster. Guttmann-Beck et al. [8] proposed a 1.9091-approximation algorithm for the problem in which the starting and ending vertices of each cluster are specified and gave a 1.8-approximation algorithm if for each cluster two vertices are given such that one of the them can be a starting vertex and the other the finishing vertex.

In this paper, we introduce and study the *general cluster routing problem (GCRP)* which generalizes both GRP and CTSP. We provide approximation algorithms of constant approximation ratio for variations of this problem. In GCRP, we are given an edge-weighted undirected graph $G = (V, E, c)$ such that the triangle inequality holds for the weight function c . The vertex set V is partitioned into clusters C_1, \dots, C_k . For any given vertex subset $V' \subseteq V$ and edge subset $E' \subseteq E$, the aim is to find a minimum cost walk T (hereafter a walk will be called a *tour*) that visits each vertex in V' exactly once and traverses each edge in E' at least once such that for every $i \in [k]$ all vertices of T belonging to C_i are visited consecutively in T . Depending on whether or not the starting and finishing vertices of a cluster are specified or not, we consider two cases. When every cluster has a pair of specified starting and finishing vertices, we offer a

2.4-approximation combinatorial algorithm. When every cluster has unspecified starting and finishing vertices, depending on whether the *required* edges (i.e., those in E') are incident with different clusters or not, we further consider two subcases. If all required edges are only distributed in the clusters, we get a 3.25-approximation combinatorial algorithm. On the other hand, if there exist edges from E' incident with two different clusters, we get a 2.25-approximation combinatorial algorithm.

The remainder of this paper is organized as follows. We provide some preliminaries in Section 2. We study algorithms for the GCRP in Sections 3. We conclude in Section 4. Every theorem whose proof is omitted is marked by (\star) . The omitted proofs can be found in Appendix of the full version of the paper [24] in arXiv.

2 Preliminaries

In this section, we recall some algorithms for three problems along with some preliminary results, which will be used as subroutines in our algorithms later.

2.1 The Travelling Salesman Path Problem

The *traveling salesman path problem (TSPP)* [5, 8, 9, 21–23] is a generalization of the TSP, but received much less attention than TSP in the literature. In TSPP, given an edge-weighted undirected graph $G = (V, E, c)$ and two vertices $s, t \in V$, the aim is to find a minimum cost Hamiltonian path from s to t . Note that vertices s and t need not be distinct. However, when $s = t$ TSPP is equivalent to the TSP. Let $MST(G)$ be a minimum spanning tree of G . For simplicity, $MST(G)$ will also denote the cost of this tree.

Hoogeveen [9] considered three variations of the travelling salesman path problem (TSPP), where as part of the inputs, the following constraints are placed on the end vertices of the resulting Hamiltonian path:

- (1) both the source and the destination are specified;
- (2) one of the the source and the destination is specified;
- (3) neither the source nor the destination are specified.

Property 1. For Cases (2) and (3), it was shown in [9] that a straightforward adaptation of Christofide’s algorithm can yield an algorithm with a performance ratio of $\frac{3}{2}$.

However, Case (1) is more difficult, for which many results exist in the literature. On the positive side, a $\frac{5}{3}$ -approximation algorithm is proposed in [9], followed by an improved $\frac{8}{5}$ -approximation in [22]. Sebo [21] gave a strongly polynomial algorithm and improved the analysis of the metric $s - t$ path TSP. He found a tour of cost less than 1.53 times the optimum of the subtour elimination LP. On the negative side, the usual integer linear programming formulation has an integrality gap at least 1.5.

Let $c(P)$ be the sum of all the edge costs of a given path or tour P . The following result from [9] will be used later.

Theorem 1. [9] *There exists a polynomial-time algorithm for travelling salesman path problem with given end vertices s and t , and we can find two solutions S_1 and S_2 for the problem which satisfy the following inequalities:*

$$\begin{aligned} c(S_1) &\leq 2MST(G) - c(s, t) \leq 2OPT - c(s, t), \\ c(S_2) &\leq MST(G) + \frac{1}{2}(OPT + c(s, t)) \leq \frac{3}{2}OPT + \frac{1}{2}c(s, t). \end{aligned}$$

Corollary 1. [9] *The shorter of the tours S_1 and S_2 is at most $\frac{5}{3}OPT$.*

Proof. By Theorem 1, if $c(s, t) \geq \frac{1}{3}OPT$, then $c(S_1) \leq \frac{5}{3}OPT$. Otherwise (i.e. $c(s, t) \leq \frac{1}{3}OPT$) we have $c(S_2) \leq \frac{5}{3}OPT$. \square

Below, we consider a more general problem, called the *travelling general path problem (TGPP)*. Let $G = (V, E, c)$ be a weighted connected graph with two specified ending vertices $s, t \in V$. For any given vertex subset $V' \subseteq V$ and edge subset $E' \subseteq E$, the objective is to find a minimum cost path from s to t in G that visits all vertices in V' exactly once and traverses all edges in E' . Note that when $s = t$, this problem becomes the general routing problem introduced in [2] which was discussed earlier. We focus on the case $s \neq t$ in the remainder of this paper.

Note that this is a minimum cost problem and the edge costs satisfy the triangle inequality. Thus, we can reduce the visits of vertices and edges not in V' and E' . Namely, we can create a new reduced graph as follow in the problem:

$$G' = (\{v | v \in e, e \in E'\} \cup \{s\} \cup \{t\} \cup V', E').$$

We assume that s and t are two different vertices in the new graph G' . First, we compute the connected components of G' via depth-first search in polynomial time. Then, contracting each component to a vertex, we construct a new complete graph G^* , where each edge cost between vertices is the longest edge cost between each pair of components, which is defined as the distance of each pair of component. This can be done in polynomial time. But we only consider those edges between the vertices with degree $d(v) \in \{0, 1\}$. Finally from the graph G^* , we create a feasible solution as described in Algorithm 1.

Theorem 2. (\star) *Let S be the path output by Algorithm 1. Then we have*

$$c(S) \leq \min \left\{ 3OPT - c(s, t), \frac{3}{2}OPT + \frac{1}{2}c(s, t) \right\}.$$

Corollary 2. *The length of the tour output by Algorithm 1 is at most $2OPT$.*

Proof. By Theorem 2, if $c(s, t) \geq OPT$, then $c(S) \leq 3OPT - c(s, t) \leq 2OPT$. Otherwise, if $c(s, t) \leq OPT$, we have $c(S) \leq \frac{3}{2}OPT + \frac{1}{2}c(s, t) \leq 2OPT$. \square

Algorithm 1 Algorithm of TGPP with specified vertice

Input:

- 1: An edge-weighted undirected graph $G = (V, E, c)$.
- 2: Starting vertex s and ending vertex t of G .
- 3: $V' \subseteq V, E' \subseteq E$ are required vertex subset and edge subset, respectively.

Output: A travelling general salesman path.

begin:

- 4: Construct a new graph $G' = (\{v|v \in e, e \in E'\} \cup \{s\} \cup \{t\} \cup V', E')$.
 - 5: Compute the connected components K_1, \dots, K_k of G' .
 - 6: Let U be the set of vertices v with degree $d(v) \in \{0, 1\}$. Define a complete graph $G_k = ([k], E_k)$ with the cost $c(e)$ of edge $e = (i, j)$ with $i \neq j$ equal to the longest link between a vertex in $K_i \cap U$ and a vertex in $K_j \cap U$.
 - 7: Copy the edges of $MST(G^*)$ except for those on s - t path.
 - 8: Find an Eulerian walk between s and t .
 - 9: Turn the Eulerian walk into a Hamilton path S .
 - 10: **output** S .
- end**
-

2.2 The Stacker Crane Problem

Given a weighted graph $G = (V, E, c)$ whose edge costs satisfy the triangle inequality. Let $D = \{(s_i, t_i) : i = 1, \dots, k\}$ be a given set of special directed arcs, each with length l_i . The arc $\overrightarrow{(s_i, t_i)}$ denotes an object that is at vertex s_i and needs to be moved to vertex t_i using a vehicle (called the stacker crane). The problem is to compute a shortest walk that traverses each directed arc $\overrightarrow{(s_i, t_i)}$ at least once in the specified direction (from s_i to t_i). Let $D = \sum_i l_i$ and $A = OPT - D$.

This problem is a generalization of the TSP, which can be viewed as an instance of this problem where each vertex is replaced by an arc of zero-length. Frederickson et al. presented a 1.8-approximation algorithm for this problem [4]. This algorithm applies two subroutines and then selects the better of the two solutions generated. The main ideas of these two subroutines are summarized below for convenience (see [4, 8] for details):

- Algorithm Short-Arcs 1: Shrink the directed arcs and reduce the problem to an instance of TSP. Use an approximation algorithm for the TSP instance, and then recover a solution for the original problem. This algorithm works well when $D \leq \frac{3}{5}OPT$.
- Algorithm Long-Arcs 1: Complete the set of directed arcs into a directed cycle cover. Then find a set of edges of minimum total weight to connect the cycles together. Add two copies of each one of these edges, and orient the copies in opposite directions to each other. The resulting graph is Eulerian, and the algorithm outputs an Euler walk of this solution. The algorithm performs well when $D > \frac{3}{5}OPT$.

The following theorem can be derived from [4].

Theorem 3. [4] Consider an instance of the Stacker Crane Problem where the sum of the lengths of the special directed arcs is D . Let OPT be an optimal solution, and let $A = OPT - D$. The walk returned by Algorithm Short-Arcs 1 has length at most $\frac{3}{2}A + 2D$. The walk returned by Algorithm Long-Arcs 1 has length at most $3A + D$.

2.3 The Rural Postman Problem

Let $E' \subseteq E$ be a specified subset of special edges. We use $c(e)$ to denote the edge cost of e . The rural postman problem (RPP) is to compute a shortest walk that visits all the edges in E' . The Chinese Postman Problem is a special case of RPP in which $E' = E$, i.e., the walk must include all the edges. The Chinese Postman Problem is solvable in polynomial time by reducing it to weighted matching, whereas RPP is NP-hard. Let $D = \sum_i l_i$ be the total length of the paths in all clusters. We recall the algorithms in [4, 8].

- Algorithm Short-Arcs 2: Consider the line graph $c(G)$ of original graph G . This algorithm works well when $D \leq \frac{3}{5}OPT$.
- Algorithm Long-Arcs 2: Complete the set of undirected arcs into a cycle cover. Then find a set of edges of minimum total weight to connect the cycles together. Add two copies of each one of these edges. The resulting graph is Eulerian, and the algorithm outputs an Euler walk of this solution. The algorithm performs well when D is large. Note that Algorithm Long-Arcs 2 is similar to Long-Arcs, but in this case, D is a set of undirected edges. The algorithm performs well when $D > \frac{3}{5}OPT$.

The two algorithms defined above for SCP can be modified to solve RPP. It is easy to see that the second part of Theorem 3 holds for this case as well, i.e. the walk returned by Algorithm Long-Arcs 2 has length at most $3A + D$.

Remark 1. As indicated by Frederickson et al. [4], it is easy to show that the above algorithms produce a $\frac{3}{2}$ performance ratio for RPP.

3 The general cluster routing problem

3.1 The general cluster routing problem with pre-specified starting and ending vertices

Note that there may exist two subcases in this case. First, each edge in E' is fully contained in its cluster. Second, some edges may be incident with more than one cluster.

Let s_i and t_i be pre-specified starting and ending vertices of cluster C_i , $i \in [k]$. Since the goal is to find a minimal total edge cost and the edge costs satisfy the triangle inequality, we can ignore the vertices not in V' and edges not in E' from

graph G to consider a new graph instead. Namely, for every cluster C_i , $i \in [k]$, consider the GCRP in the following new graph $\overline{G} = \cup \overline{C}_i$, where

$$\overline{C}_i = (\overline{V}_i, \overline{E}_i) = \left(\{v | v \in e, e \in E'_i\} \cup V'_i \cup \{s_i\} \cup \{t_i\}, E'_i \right).$$

Our algorithm is based on the following idea. First, within each cluster \overline{C}_i , we find a path p_i , starting with s_i and ending at t_i , visits all the vertices in V' and edges of each cluster \overline{C}_i . This can be done by Algorithm 1. Second, we need to connect the paths by adding some edges to make the resulting graph into a single cycle.

Let $G = (V, E)$ be a complete graph with vertex set V and edge set E , the vertex set is partitioned into clusters C_1, \dots, C_k . The starting and ending vertices in each cluster are specified. Let $\overline{C}_i = (\overline{V}_i, \overline{E}_i)$ be the new graph as described above. Clearly, the desired tour in G does not always exist, e.g., when there exists a required edge $e \in E'$ between cluster C_i and cluster C_j , $i \neq j$, and this required edge is not a (t_i, s_j) edge (in such a case, at least one of the clusters must be visited more than one time). Henceforth, we will assume that the desired tour does exist.

Algorithm 2 Algorithm of given starting and ending vertices

Input:

- 1: An edge-weighted graph $G = (V, E, c)$.
- 2: A partition of V into clusters C_1, \dots, C_k .
- 3: Each cluster C_i with starting and ending vertices s_i and t_i , respectively, $i = 1, \dots, k$.

Output: A cluster general routing tour.

begin:

- 4: Construct a new graph $\overline{G} = \cup_{i=1}^k \overline{C}_i$.
- 5: For $i = 1, \dots, k$, apply Algorithm 1 to get a path p_i and orient the (s_i, t_i) edge a direction, from s_i to t_i , to obtain the arc $\overrightarrow{(s_i, t_i)}$.
- 6: Apply Algorithm Short-Arcs 1 and Algorithm Long-Arcs 1 for SCP on special arc $\overrightarrow{(s_i, t_i)}$, $i = 1, \dots, k$, and output the shorter solution T .
- 7: In T , replace the special directed arc (s_i, t_i) by the path p_i , for $i = 1, \dots, k$.
- 8: **Output** the resulting tour T_s .

end

The main idea of Algorithm 2 is illustrated as follows:

In Step 1, we first consider the number of connected components of \overline{C}_i . If the number is 1, it means that there exists a path from s_i to t_i that visits all the required edges in E' and vertices in V' . When the number is more than 2, shrinking the connected components to be vertices and finding a path to connect all these vertices lead to a feasible solution.

In Step 2, by applying Algorithm 1, we can get a path from the starting vertex s_i to ending vertex t_i .

In Step 3, we only need to connect these clusters to form a tour. In this progress, we can shrink the directed arc $\overrightarrow{(s_i, t_i)}$ and reduce the problem to an instance of TSP. Use Christofides' algorithm [3] for the TSP instance.

In Step 4 by replacing the special directed arc (s_i, t_i) by path p_i , we obtain a solution to the original graph.

Our algorithm is a combinational algorithm, which deals with the condition of the pre-specified starting and ending vertices carefully. Let OPT be the cost of the optimal solution. Let L be the sum of lengths of the paths of OPT through each cluster and let A be the length of the other edges of OPT that are not in L . Let D be the total length of the directed arcs $(s_i, t_i), i = 1, \dots, k$. Then, we have the following theorem:

Theorem 4. (\star) *Let T be the tour output by Algorithm 2. Then*

$$c(T_s) \leq 2.4 OPT.$$

For the second case, there exists required edges in E' between different clusters. If there exist required edges incident with two different clusters, they must be (t_i, s_j) edges. First, we need to compute the number of (t_i, s_j) edges. Suppose the number is k . If $k = 0$, it is just Case 1. If $k \geq 2$, we then get $k+1$ components and we can shrink the components and go back to Case 1 again.

According to Theorem 4, for the general cluster routing problem with pre-specified vertices, we now get a 2.4-approximation combinatorial algorithm.

3.2 The general cluster routing problem without specifying starting and ending vertices

In this section, we consider the version of GCRP where, for each cluster C_i we are free to choose the starting and ending vertices. We consider the two cases again. In the first case, all required edges in E' are only distributed within the clusters. In the second case, there exist some required edges incident with some different clusters.

For every cluster $C_i, i \in [k]$, we consider GCRP on the new graph $\overline{G} = \cup \overline{C}_i$ defined as before:

$$\overline{C}_i = (\overline{V}_i, \overline{E}_i) = (\{v | v \in e \in E'_i\} \cup V'_i, E'_i).$$

We first consider the connected components of \overline{C}_i . In order to obtain the resulted tour, the degree of every vertex of the tour must be even. Therefore, there also exist some cases that the tour cannot exist, i.e., there exists a vertex $v \in \overline{V}_i$ with degree $d(v) > 2$ (in such a case, at least one of the clusters must be visited more than once). Henceforth, we will assume that the desired tour exists.

To solve the first case when all required edges in E' are only distributed within the clusters, we propose an algorithm which computes two different solutions. Then we select the shorter one of these two tours. To get the first solution, by using Algorithm 1 with unspecified ends, we can find paths within each cluster. Then we can view this as a Rural Postman Problem instance. To get the second

solution, for each cluster, we select two vertices s_i and t_i such that $c(s_i, t_i)$ is maximized. Let them be the end vertices of each cluster. Then we can apply Algorithm 2 to get the second tour. Finally, we select the shorter tour.

The algorithm for the case when the tour exists can be described as follows:

Algorithm 3 Algorithm of unspecified ending vertices

Input:

- 1: An edge-weighted graph $G = (V, E, c)$, $V' \subseteq V$, $E' \subseteq E$.
- 2: A partition of V into clusters C_1, \dots, C_k .

Output: A cluster general routing tour.

begin:

- 3: Consider the new graph \overline{C}_i , for $i \in [k]$.
- 4: Apply Algorithm 1 with unspecified end vertices in each cluster $\overline{C}_1, \dots, \overline{C}_k$. Let path p_i be the resulting path on \overline{C}_i , and denote its end vertices by a_i and b_i . Apply Algorithm Long-Arc 2 and Algorithm Short-Arc 2 to output the shorter solution for RPP with special edges (a_i, b_i) and let T_1 be the tour obtained by replacing special edge (a_i, b_i) by path p_i , for $i \in [k]$.
- 5: In each cluster find vertices s_i and t_i that maximize $c(s_i, t_i)$, for $i \in [k]$. Apply Algorithm 2 with the end vertices $\{s_i, t_i\}$ to output a tour T_2
- 6: **Output** the shorter of T_1 and T_2 .

end

We will analyze the approximation ratio of Algorithm 3. We first introduce some notations. As in the previous section, let L denote the sum of the lengths of the Hamiltonian paths within the clusters in OPT , and let A denote the sum of the lengths of the remaining edges of OPT . Let $D = \sum_{i=1}^k c(s_i, t_i)$ denote the sum cost of required edges. The first algorithm works well when D is small, and the second works well when D is large.

Theorem 5. (\star) *Let T_1 be the tour computed in Step 2 of Algorithm 3. Then we have*

$$c(T_1) \leq \frac{3}{2}OPT + \frac{1}{2}L + 2D.$$

Theorem 6. (\star) *Let T_2 be the tour computed in Step 3 of Algorithm 3. Then we have*

$$c(T_2) \leq \frac{3}{2}OPT + 3L - 2D.$$

Now we can get the following theorem:

Theorem 7. *Let T be the tour returned by Algorithm 3, then*

$$c(T) \leq \frac{13}{4}OPT.$$

Proof. Note that $L \leq OPT$. If $2D \leq \frac{5}{4}L$, Theorem 5 implies that

$$c(T_1) \leq \frac{3}{2}OPT + \frac{7}{4}L \leq \frac{13}{4}OPT.$$

Otherwise, when $2D \geq \frac{5}{4}L$, Theorem 6 implies that

$$c(T_2) \leq \frac{13}{4}OPT.$$

Since the algorithm chooses the shorter one between the tours T_1 and T_2 , the proof is completed. \square

Next, we will consider Case 2 when there exist required edges between clusters.

We consider the problem in three different cases. In the first case, the number of required edges incident with different clusters is k . In the second case, some clusters have two required edges incident with other clusters. In the third case, the number of clusters with required edges incident with other cluster is 0.

In the first case, we only need to find paths between each specified vertices. This can be seen as an instance of Travelling Salesman Path Problem as we described before. So the performance ratio of this case is 1.5.

In the second case, for the clusters which have two required edges incident with other clusters, we find paths in them and it becomes the third case.

Without loss of generality, we consider the third case: the number of clusters with required edges incident to other cluster is 0. For every cluster C_i , we denote the specified vertex as a_i . First, in each cluster, by computing the distance between each component, we select the longest one; that is, we find the vertex b_i such that $c(a_i, b_i)$ is maximum. This can be done in polynomial time, because the number of vertices in each cluster is no more than n . Then we can find the path p_i in each cluster C_i by Algorithm 1. Since this problem has no direction, we apply Algorithm Long-Arc 2 and Algorithm Short-Arc 2 to output the shorter solution for RPP and find the tour with the edge (a_i, b_i) . At last, we replace the edge (a_i, b_i) by path p_i . The whole algorithm can be described as follows:

Algorithm 4 Algorithm of existing required edges between clusters

Input:

- 1: An edge-weighted graph $G = (V, E, c)$, $V' \subseteq V$, $E' \subseteq E$.
- 2: A partition of V into clusters C_1, \dots, C_k .

Output: A cluster general routing tour.

begin:

- 3: Let the vertex adjacent to required edges between different cluster C_i be a_i . Find a vertex that maximize $c(a_i, b_i)$, for $i = 1, \dots, k$.
- 4: For each C_i , compute a path p_i , a Hamiltonian path with end vertices a_i and b_i , for $i = 1, \dots, k$.
- 5: Apply Algorithm Long-Arc 2 and Algorithm Short-Arc 2 to output the shorter solution for RPP with the special edges $\{(a_i, b_i) | i = 1, \dots, k\}$ to obtain tour S , for $i = 1, \dots, k$.
- 6: In T , replace the special edge (a_i, b_i) by the path p_i , for $i = 1, \dots, k$.
- 7: **return** the resulting tour T .

end

Theorem 8. (\star) *Let T be the tour output by Algorithm 4. Then $c(T) \leq \frac{9}{4}OPT$.*

Algorithm 4 is a 2.25-approximation algorithm for the general cluster routing problem with unspecified end vertices, in which some required edges may be incident with different clusters. Therefore, the performance ratio of approximation algorithm for the problem with unspecified vertices is 3.25.

4 Conclusion

In this paper, we present constant approximation algorithms for two variations of the cluster general routing problem. However, the two presented algorithms have different approximation ratio, and in our future work we will consider whether we can design approximation algorithms with the same approximation ratio for these two problems.

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