Mechanism Design With Budget Constraints and a Population of Agents

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Abstract

This paper finds welfare- and revenue-maximizing mechanisms for assigning a divisible good to a population of budget-constrained agents who have independently distributed private valuations and budgets *without unit-demand*. Both optimal mechanisms feature a linear price for the good. The welfare-maximizing mechanism additionally has a uniform lump-sum transfer to all agents and a higher linear price than the revenue-maximizing mechanism. This transfer increases welfare because it ameliorates the key difficulty in the aforementioned setting: agents with high valuations cannot purchase an efficient amount of the good due to their budget constraints. Finally, in an extension, I relax the independence between valuations and budgets. In an online appendix, I consider production and large finite markets.

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1 Introduction

What is the optimal incentive-compatible mechanism for allocating a good to a population with private valuations and private budget constraints? Similar questions have been considered in several recent papers, but in all of them until this point, there has either been a finite number of agents or a finite discrete set of types. The innovation of this paper is that it considers the problem with a continuum of agents and types and here the optimal mechanisms derived are uniquely simple, linear prices and possibly lump-sum transfers. This result is partially enabled by the fact that the current paper does not make the standard unit-demand assumption. I say "partially" because sometimes the optimal mechanisms found here may additionally satisfy the unit demand assumption, in which case they would be optimal even if that assumption were present. On the other hand, in those cases where the unit-demand assumption is not satisfied by the optimal mechanism discovered here, the structure of the optimal mechanism with unit demand remains an open question.

To illustrate the problem and its solution, consider a finite example: Suppose that there are two agents with unit valuations $v_1,\ v_2$ and a common budget constraint w such that $w < v_1 < 2w < v_2$. To maximize welfare, the principal would like to sell the good to agent 2. However, agent 2 can only afford the price w, and at that price the principal faces the problem that agent 1 would also like to purchase the good. Since the good is divisible, one possible mechanism is for the principal to sell one half of the good to each agent for the price $w/2$. While this is incentive compatible, the principal can do better and achieve the first-best by giving each agent w and selling the good at a price of $2w$. Both agents can afford this price because of the transfer, but only agent 2 will want to purchase the good at this price. Moreover, this mechanism produces a balanced budget as each agent receives w and the high-valuation agent pays $2w$. An alternative implementation would be for the principal to give each agent half a unit of the good and then allowing agent 2 to buy agent 1's allocation for the price w, thus achieving the first-best outcome as above.

While the above example differs from the setting considered here, it illustrates a key principle, namely that transfers to agents can be used to weaken budget constraints and improve welfare. In this paper, an agent's utility is linear in both the quantity of the good and money while the agents' valuations and budget constraints are independently distributed and private. The principal has a finite supply of the good and must satisfy a weak balanced budget constraint. In both the welfare- and revenue-maximizing settings, the optimal mechanisms feature a linear price p for the good. In line with the previous example, the utilitarian welfare-maximizing mechanism also features a uniform lump-sum transfer from the principal to the agents, whereas the revenue-maximizing mechanism does not.[1] The welfare-maximizing principal can implement this mechanism via cash transfers or alternatively in-kind transfers of the good and then allow resale.[2] The principal's transfers to the agents are uniform because types are unobservable, and therefore high-value or low-budget agents cannot be profitably targeted to receive higher transfers.

Economic situations where a principal wishes to distribute a divisible good to budget-constrained agents abound. In the case of welfare maximization, such settings may be: the provision of healthcare or education in a government regulated system or the privatization of a government-owned enterprise. In each of these settings, budget constraints may be significant and may stand in the way of an efficient allocation of resources.[3] For the case of revenue maximization, examples include: a monopolist facing a budget-constrained population, the privatization of companies, and the sale of government land or the sale of bonds. In particular, the uniform auction method of selling bonds is a dominant strategy implementation of the optimal mechanism derived in this paper. Furthermore, if a principal wishes to maximize societal welfare from some larger planning problem, there may be a preference to maximize revenue or

[1]This is in line with Pai and Vohra (2014) who find that a revenue-maximizing principal does not make transfers to the agents in a finite auction setting.

[2]One may consider two versions of the model presented. One is a continuum population model and the other is of a single agent model with ex-ante constraints. Under the former interpretation, the good is transferred to the agents who then buy and resell among themselves. In the second interpretation, the good is transferred to the single agent who then would buy and resells it across different realizations and this implementation is fundamentally dependent upon the prior. In general, while it is typically impossible to achieve a prior free implementation in a single agent setting, see Devanur et al. (2013) for discussions of approximately optimal mechanisms without priors in other contexts.

[3]For example, in the mass privatization auctions in Central and Eastern Europe, vouchers were distributed which could then be used for bidding for different companies. There was significant variations in implementation across countries and in some countries, including Russia, resale of these vouchers was permitted (Boycko et al. (1994)). Furthermore, budget constraints were significant in that setting, Estrin (1991) states that the total sum of private savings (330 billion crowns) was 10% of the value of companies (3.3 billion crowns) being privatized in Czechoslovakia and 1% in Poland (78 million zloty / 64 billion zloty). For a more thorough overview, see Border (1991) and Boycko et al. (1996).

a joint welfare/revenue objective when selling the good so that the revenue generated can be applied to other welfare-improving endeavors. The proof of the main theorem also establishes that linear mechanisms are optimal for joint objectives.

The structure of the paper is as follows: First, I discuss related literature. In section 2, I introduce the framework and the essential assumptions. Section 3 contains the necessary lemmas and the main result which demonstrates the optimality of a linear pricing function in achieving utilitarian efficiency or revenue maximization. In section 4, I consider a relaxation of the independence assumption. Section 5 concludes. The appendix includes all proofs and an example of how the theorem fails when the given assumptions do not hold. Finally, an online supplement covers dominant strategy implementations, large approximately optimal mechanisms, and production.

Summary

While details can be found throughout the paper, I now present a full summary of the main results upfront without any excess distraction. The main result follows (with some twists) a standard approach to multi-dimensional mechanism design pioneered by Armstrong (1996). There, a multi-dimensional mechanism design problem is broken up into a collection of one-dimensional problems, which are then solved, and then it is demonstrated that these solutions together solve the original multi-dimensional problem.

Along these lines, due to a difference in settings (continuum population and no unit-demand), I solve again the common-budget mechanism design problems of Laffont and Robert (1996) and Maskin (2000) for the particular setting considered here. This is Lemma 1. This yields a class of what I call “take-it-or-leave-it mechanisms”. Such a mechanism, for each budget level, offers a transfer and then makes a take-it-or-leave-it offer for a fixed quantity of the good in return for all of the agent’s budget. The main innovation of this paper is then Lemma 2 which finds the optimal mechanism in this class. It uses a linear price. For technical reasons due to transfers, this solution may not be incentive compatible across different budgets, but then Lemma 3 smooths these transfers in order to obtain full incentive compatibility without disturbing the welfare optimality of Lemma 2’s solution. Crucial to Lemma 2 are both the continuum population and the lack of the unit-demand assumption. As mentioned earlier, there may be cases (such as a small supply) where the unit-demand assumption ends up being incidentally satisfied by the solution found here, but the approach fundamentally ignores this condition when solving for the optimal mechanism. The overall solution turns out to be linear pricing, with lump-sum transfers for the welfare objective (in order to relax budget constraints) and without any transfers for the revenue objective.

Related Literature

The addition of budget constraints to standard mechanism design problems has been studied widely in the literature. For example, Che and Gale (1998) showed that the standard revenue equivalence results of Myerson (1981) do not hold when agents have budget constraints. Specifically, they find that in a standard auction setting, the first price auction outperforms the second price auction because agents hedge their bids in a first-price setting and therefore budget constraints are less binding. Burkett (2015) shows that this performance gap can be eliminated if the principal optimally constrains the agents' budgets. Other work that also takes place in a finite auction setting include: a demonstration of a possible failure of the linkage principle (Fang and Parreiras (2002)), equilibrium analysis in first-price auctions (theoretically, Kotowski (2013) and experimentally Kotowski and Li (2010)), affiliated second-price auctions (Fang and Parreiras (2003)), welfare-maximization with externalities in a school design setting (Mestieri (2010)), and a setting without quasilinear utility where Baisa (2016) demonstrated the superior performance of probabilistic mechanisms. In general, I take a divisible good interpretation, but an indivisible good interpretation is possible as well with the good being distributed probabilistically.

The case of a common public budget constraint has been considered by Laffont and Robert (1996) and Maskin (2000) who have solved for revenue-maximizing and welfare-maximizing mechanisms, respectively.

When budget constraints are individual and unobservable, agents' types become two-dimensional. The multi-dimensional mechanism design literature is large and features two common approaches: one is to assume that each of the agent's two characteristics can take on one of two different values and the problem becomes one of finite parameter programming (see Armstrong and Rochet (1999)), while the other is to show that there is a one-dimensional formulation and solve that problem (for example, see Armstrong (1996), Jehiel et al. (1996), and Che and Gale (2000)). In Armstrong (1996), agents have homogenous preferences specified by a vector in $\mathbb{R}^k_+$ and therefore their preferences can be expressed by a direction and a radius. He solves for the optimal mechanism along each direction and in certain cases, while the interpath IC constraints are relaxed, they are nonetheless satisfied and therefore an admissible optimal mechanism is found. Here, as mentioned earlier, I take the approach of Armstrong (1996) with a minor twist.

There are several papers in the literature on mechanism design with budget constraints that are closely related to the present work. The first is Che and Gale (2000) who consider revenue maximization with one buyer. They find a convex pricing function in terms of probabilities to be revenue maximizing. While the objective of their mechanism is the same as in this paper, the difference in the found optimal mechanisms is due to a difference in single-agent versus continuum constraints.

Other closely related papers are Pai and Vohra (2014) and Che, Gale, Kim (2013a, 2013b). The first focuses on a single-good auction with a finite number of bidders. The fact that a single good is being sold rather than a supply differentiates that model from the current one. Such an optimal design problem degenerates as the number of agents grows because with high probability there exists an agent with both a high valuation and a high budget. The latter two papers focus on welfare maximization with a continuum of agents. A key difference though is that they generally focus on a 2×2 type setting, that is two possible valuations and two possible budgets and the former has a per-unit budget constraint rather than an overall budget constraint. The optimal mechanisms found there are quite different featuring a system of taxes and subsidies. The simplicity of the current model's optimal solutions is reminiscent of Azevedo and Leshno (2016) who find that a supply and demand framework applies to a continuum matching setting, unlike the standard finite setting.

Finally, there is a literature on envy-free auctions where agents have private budget constraints (as in the current paper), see Kempe et al. (2009), Feldman et al. (2012), and Colini-Baldeschi et al. (2014). An auction is envy-free if for every profile of bidders, no agent envies the allocation of any other agent, ex-post. While they differ in finite settings, in a continuum setting, such as the one studied here, the envy-free condition exactly coincides with ex-post incentive-compatibility Azevedo and Budish (2018). This contrasts with finite settings, where there is a difference between the two concepts because in an IC mechanism, one agent may envy the other and yet would not be able to profitably make a different declaration because that would change the overall profile of declarations. Nevertheless, in finite settings, there is often a close connection between the revenue generated by the optimal envy-free mechanism and the optimal incentive-compatible mechanism (see Hartline and Yan (2011)). Kempe et al. (2009) considers the case of budget constrained envy-free unit demand mechanism design, Feldman et al. (2012) considers the case of budget constrained multi-unit demand (as in the current paper), and Colini-Baldeschi et al. (2014) considers both unit and multi-unit demand problems. In all of these works, the objective is to find an envy-free auction whose performance approximates that of the optimal envy-free auction. Of these, Feldman et al. (2012) is closest to the current paper, in particular their envy-free item pricing mechanism. In that mechanism, there is a unit price p and each agent purchases as much of the good as they want given that price. In a general multi-agent framework, they show that such a mechanism performs within a constant fraction of the optimal envy-free mechanism. Interestingly, the same mechanism shows up here as a revenue-optimal incentive-compatible mechanism for the single agent case with ex-ante constraints.

2 Framework

2.1 Setup

There is a single good to be distributed by a principal with finite aggregate supply S. There is a unit measure of agents, each of whom is defined by two attributes, budget, w, and value, v. Note that I use the letter w for budgets for notational convenience and which may remind the reader of the term "wealth". Agents are risk-neutral with linear utility in the quantity of the good and money. An agent's per-unit value for the good is determined by her value type, v. If an agent has a quantity of the good x and pays a transfer $t(v)$, then her utility is $U(v, x) = xv - t(v)$. Budgets constraint the agent in that her transfers cannot exceed her budget.

Agents' attributes are distributed according to two independent distributions, F

over V, the space of possible values and G over W, the space of possible budgets. Each distribution is continuous with continuously differentiable densities and both spaces V, W are bounded and non-negative. A consequence of their independence is that knowing an agent's budget or value type reveals no information about the other attribute. Thus, there is no incentive to favor high- or low-budget agents from a purely correlation point of view.

Note: The distributions F, G define the aggregate makeup of the population. The formulation here has no aggregate uncertainty and integrals are with respect to an aggregation over all agents rather than taking an expectation with respect to some underlying uncertainty. Aggregate uncertainty is considered in the online supplement.

I consider two different mechanism design problems. In the first, a principal (perhaps the government or some other public institution) wishes to maximize the welfare of the agents. To achieve this, he wishes to assign the good to the agents with the highest valuation of the good. The mechanism design problem is to find assignment and transfer rules $x, t : V \times W \to \mathbb{R}$ that maximize the total utilitarian welfare of society.

In the second problem, the goal of the principal (perhaps the government or a monopolist) is to find the revenue-maximizing incentive-compatible assignment and transfer rules. In both problems, x, t are taken to be measurable functions. Notice that x and t are deterministic allocation and transfer rules which is without loss of generality because utility is linear and due to the specific incentive compatibility constraints imposed.

The following definition of the *welfare maximization* problem takes a utilitarian efficiency criterion and uses the revelation principle in formulating the problem as a direct mechanism.

Definition: Welfare Maximization Problem

Maximize

$$\mathcal{W}(x,t) := \int_W \int_V x(v,w) v f(v) dv g(w) dw \tag{1}$$

s.t.

$$\int_W \int_V t(v,w) f(v)\ dv\ g(w)\ dw \geq 0 \tag{BB}$$

$$\int_W \int_V x(v,w) f(v)\ dv\ g(w)\ dw \leq S \tag{LS}$$

$$0 \leq x(v,w) \qquad \forall v, w \tag{NN}$$

$$t(v,w) \leq w \qquad \forall v, w \tag{BC}$$

$$\mathbb{1}_{\{w' \leq w\}}(vx(v',w') - t(v',w')) \leq vx(v,w) - t(v,w) \qquad \forall v, w, v', w' \tag{IC}$$

$$vx(v,w) - t(v,w) \geq 0 \qquad \forall v, w \tag{IR}$$

The above conditions are respectively: (BB) budget balance, (LS) limited supply, (NN) non-negative consumption, (BC) budget constraints, (IC) incentive compatibility, and (IR) individual rationality. The budget balance condition states that the principal cannot inject money into the system. If he were able to do so, then he could distribute near infinite amounts and relieve every agent's budget constraint. The non-negativity condition states that only the principal supplies the good, i.e. agents cannot be allocated a negative quantity of the good.

Before discussing the budget constraint (BC) and incentive compatibility conditions (IC), I define the *revenue maximization* problem.

Definition: Revenue Maximization Problem

Replace the welfare objective function (1) to be maximized with the following revenue function:

$$\mathcal{R}(x,t) := \int_W \int_V t(v,w) f(v)\ dv\ g(w)\ dw \tag{1'}$$

The revenue maximization problem has a different objective function, but retains all the constraints of the welfare maximization problem (the budget balance condition (BB) could be omitted as it would never bind).

2.2 Budget Constraints and Incentive Compatibility

The budget constraint condition (BC) states that agents cannot be asked to make a transfer $t(v,w)$ strictly larger than their budget w. This condition differentiates the problem from an unconstrained budget setting.

The incentive compatibility condition that I impose above states that rich agents *can* imitate poorer agents, but poor agents *cannot* imitate richer ones. This type of condition has been explained in the literature as being appropriate in a setting where agents can post bonds. In this case, agents cannot exaggerate their budget because they are unable to post larger bonds. Another justification would be if the principal can charge a random price equal to the stated budget with non-zero probability. An alternate incentive compatibility condition is:

$$\mathbb{1}_{\{t(v',w')\leq w\}}(vx(v',w') - t(v',w')) \leq vx(v,w) - t(v,w) \qquad \forall v, w, v', w' \qquad \text{(IC')}$$

This condition is less restrictive on agents and hence more restrictive on the class of admissible mechanisms since agents can imitate any type whose transfers they can afford. In what follows, I solve for the optimal mechanism according to (IC) and find a variation of it that satisfies (IC'), which therefore will be optimal for either formulation.

Note: The incentive compatibility conditions (IC) and (IC') have both been defined in terms of a simultaneous deviation in the declaration of budget and value. These incentive compatibility constraints could instead be expressed in terms of one-dimensional deviations as follows:

$$x(v',w) - t(v'w) \leq vx(v,w) - t(v,w) \qquad \forall v, v' \qquad \text{(Value-IC)}$$
$$\mathbb{1}_{w'\leq w}(vx(v,w') - t(v,w')) \leq vx(v,w) - t(v,w) \qquad \forall w, w' \qquad \text{(Budget-IC)}$$

These one-dimensional ICs imply the two-dimensional IC for the following reason: If type (v,w) does not want to pretend to be (v,w') and type (v,w') does not want to pretend to be (v',w'), then type (v,w) does not want to pretend to be (v',w'), since both (v,w) and (v,w') have the same preferences over outcomes. In other words, the only determinant of an agent's preferences is his value type, while his budget type only determines feasibility. If an agent's budget also affects his preferences, then the two one-dimensional ICs may not imply the single two-dimensional IC constraint. If a mechanism only satisfies the value-IC constraint or the budget-IC constraint, it will be referred to as *value-incentive compatible* or *budget-incentive compatible*. Unless otherwise stated, incentive compatibility always refers to (IC) rather than (IC').

Note: Referring back to the formulation of the welfare objective function $\mathcal{W}$, notice that it is defined without regard to aggregate transfers. The budget balance constraint requires that the principal cannot introduce money into the system, but there can be a positive net transfer of money from the agents to the principal, which reduces the agents' utility. However, for any admissible mechanism, this money could then be disbursed back to the agents in the form of a uniform lump-sum transfer without affecting any of the given constraints. Thus, any solution to the problem in which transfers are included in the welfare function will correspond to a solution of the above formulation and vice versa. As it turns out, the optimal mechanism is exactly budget balanced, so deducting aggregate transfers from the welfare function will not change its optimality.

2.3 Assumptions

The following two assumptions on the distribution of value types will be imposed for the duration of the paper.

Assumption 1: $v - \frac{1-F(v)}{f(v)}$ is weakly increasing. (Regularity)
Assumption 2: $f(v)$ is weakly decreasing. (Decreasing Density)

Assumption 1 is standard regularity assumption and ties together agents' valuations and virtual valuations and implies that transferring the good from lower value types to higher value types will lead to an increase in revenue, as in Myerson (1981). This assumption is important because then both the welfare-maximizing planner and revenue-maximizing planner would like to target the same agents, those with high valuations.[4]

Assumption 2 states that higher value types are weakly less likely than lower value types. It is satisfied by the uniform distribution, the exponential distribution,

[4]Part 2 of Theorem 4 of the Online Supplement needs a stronger assumption, that of monotone hazard rates. For that theorem, the stronger assumption on values is needed in order to uniquely pin down the revenue-maximizing parameters.

the half-normal distribution (recall that negative valuations are not possible), and any convex combination of them (as long as they have the same initial value $\underline{v}$).

Analogues of these assumptions in a discrete setting are employed by Pai and Vohra (2014). Unlike Pai and Vohra (2014) and Che, Gale and Kim (2013a), there will be no further assumptions on the budget distribution aside from requiring a continuous distribution of budgets.

2.4 Interpretation and Relation to Other Models

A fundamental difference between the current paper and others is that the current paper does not feature a unit-demand assumption.

Now, before proceeding to the main results, I discuss (i) a single-agent interpretation of the model and (ii) the relationship between the model studied here and those of Pai and Vohra (2014) and Che et al. (2013a). A natural single-agent interpretation would be one where a single agent independently draws a valuation from F and a budget from G. Accordingly, (1) expresses the ex-ante welfare objective $\mathcal{W}(x,t) = \int_W \int_V x(v,w) v f(v)\ dv\ g(w)\ dw$. Additionally, the conditions (NN), (BC), (IC), (IR) naturally translate to the single-agent setting since they don't involve any aggregation. The aggregate conditions (BB) and (LS) now represent *ex-ante* conditions, i.e. transfers and supply must be balanced on average and transfers are permitted across type realizations. Therefore, the mechanisms found in the main theorem solve the optimal mechanism design problems for the single-agent ex-ante formulation (see Alaei et al. (2012)).

For this single-agent ex-ante formulation, the found optimal mechanism is not prior-free. That is because the mechanisms found depend upon both distributions F and G. While envyfreeness is not a meaningful concept in a single agent setting, several papers have successfully analyzed envy-free mechanisms when buyers have budgets, especially those discussed in the related literature section: Kempe et al. (2009), Feldman et al. (2012), and Colini-Baldeschi et al. (2014).

The setting of the current model differs from that of Pai and Vohra (2014) in a number of ways. They study optimality in a single-good auction setting with either discrete types or approximate optimality in a continuous model. This enables them to follow a duality approach due to Border (1991). More fundamentally, in their model, there is a single unit of the good with an ex-post supply constraint. Formally, given any declaration of types $(v_1, w_1), \dots, (v_N, w_N)$, the allocation x is required to satisfy $\sum_{i=1}^{N} x(v_i, w_i) \leq 1$. The limited supply constraint (LS) formulated here can be thought of as a per-person limitation. As shown in the online supplement, the finite-agent model which approaches the model studied here, has the per-person supply constraint $\sum_{i=1}^{N} \frac{x(v_i, w_i)}{N} \leq S$.

The models of Che, Gale, and Kim (2013a, 2013b) are more closely related to the current one, though they differ in a few key aspects. First, their analysis focuses on a welfare objective and generally takes place in a 2x2 (valuation,budget) model, whereas the current paper also considers revenue and focuses on a continuum of types. In addition to an aggregate limited supply constraint, their models feature a unit demand requirement that $\forall v, w,\ x(v, w) \leq 1$. There is no such restriction in the current model, although this condition is additionally satisfied by the optimal mechanisms found here when the supply S is not too large, as demonstrated in Corollary 4. Finally, while the budget constraint of Che, Gale, and Kim (2013b) is the same as that used here, the budget constraint of Che, Gale and Kim (2013a) is not. There, an ex-post budget constraint is used, $t(v, w) \leq x(v, w)w \Leftrightarrow \frac{t(v,w)}{x(v,w)} \leq w$, whereas in the current paper, the budget constraint is $t(v, w) \leq w$. As discussed earlier, the difference is between a per-unit constraint and a restriction on the total amount that an agent can pay. Notice that the optimal mechanisms derived here will not satisfy this per-unit budget constraint.

Finally, Devanur et al. (2013) solves an optimal mechanism design problem with a finite set of agents and a common budget and without the assumptions made here which are necessary for the present analysis. The limit of their mechanism approaches the one found in this paper as the number of agents and with a supply constraint proportional to market size increase to infinity. The common budget literature of Laffont and Robert (1996), Maskin (2000) can be used to show Step 1 of this paper and likewise, Devanur et al. (2013) can be used when the assumptions do not hold.

3 The Main Result

3.1 A Preview

In this subsection, I provide a brief preview of the main result and an example. The main result shows that the optimal mechanisms, from either a welfare or revenue point of view, are linear mechanisms, perhaps with uniform lump-sum transfers.

Definition: A *linear mechanism with uniform transfers* is characterized by two parameters, (p, T), both of which are non-negative. In these linear mechanisms, agents receive a uniform lump-sum transfer T and can purchase as much of the good as they wish at the unit price p. Agents with "low" valuations, specifically $v < p$, will therefore purchase none of the good and will simply receive the transfer T. On the other hand, agents with "high" valuations, specifically $v > p$, will have induced budget $w + T$ and therefore will purchase $\frac{w+T}{p}$ of the good. This mechanism is clearly incentive-compatible since all agent choose their preferred bundle from the options they can afford. The mechanism is depicted in Figure 1.

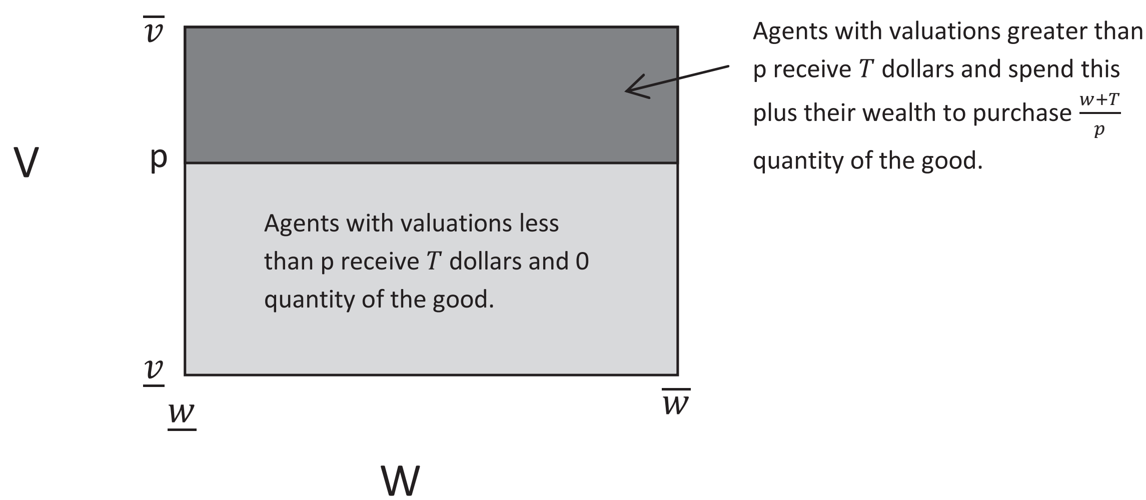

Figure 1: A Linear Mechanism with Uniform Transfers

Now, I present the main theorem, followed by three lemmas necessary for its proof:

Main Theorem: *Under the regularity and decreasing density assumptions, the following three statements hold:*

1. *A welfare-optimal mechanism is a linear mechanism with lump sum transfers,* (p_W, T_W)*, such that* $T_W = S \cdot p_W$ *and* $(1 - F(p_W))\mathbb{E}[w] = S \cdot p_W \cdot F(p_W)$*.*

2. *The revenue-optimal mechanism is a linear mechanism (with no transfers),* (p_R, T_R)*, such that* $T_R = 0$ *and* $(1 - F(p_R))\mathbb{E}[w] = S \cdot p_R$*.*

3. $p_W > p_R$ *.*

In the above theorem, the expectation $\mathbb{E}[w]$ is the average budget in the society, calculated as $\mathbb{E}[w] = \int_{\underline{w}}^{\bar{w}} wg(w)dw$.

Note that the theorem demonstrates that linear mechanisms with uniform lump-sum transfers are welfare- and revenue-optimal in the class of all admissible mechanisms, not just the class of linear mechanisms. Additionally, p_W is uniquely well-defined since the defining equation $(1 - F(p_W))\mathbb{E}[w] = S \cdot p_W \cdot F(p_W)$ has a strictly decreasing left-hand side and strictly increasing right-hand side in p_W. The price p_R is similarly uniquely well-defined.

An Example: The principal has a unit supply of the good and agents' valuations and budgets are both uniformly distributed on $[0, 1]$. That is, $S = 1$ and $F = G = U(0, 1)$.

Welfare Maximization: According to the main theorem above, the welfare-maximizing price satisfies $(1 - p_W)\frac{1}{2} = p_W^2$. The unique positive solution to this equation is $p_W = \frac{1}{2}$ and each agent receives the transfer $T_W = \frac{1}{2}$. Agents self-select into two regimes based upon their valuations: those with valuations below 1/2 simply receive the transfer, while agents with valuations above 1/2 receive the transfer and use it along with their budget to purchase the good. Specifically, high-value agents with original budget 0 will have budget 1/2 after the lump-sum transfer and will purchase one unit at the unit price of 1/2. The richest high-value agents, those with original budget 1, will have budget 3/2 after the uniform lump-sum transfer and therefore will purchase 3 units at the unit price of 1/2.

Revenue Maximization: Under the revenue-maximizing mechanism, there are no uniform lump-sum transfers. The market clearing price, given in the theorem is

$(1-p_R)\frac{1}{2} = p_R$ and therefore is uniquely determined as $p_R = \frac{1}{3}$. As noted in the theorem, this market-clearing price is lower than in the welfare-maximizing setting because agents receive no transfers. Since agents in the revenue-maximizing setting are "poorer", they have a lower aggregate demand function and a lower market-clearing price. Agents with valuations below 1/3 purchase none of the good. Agents with valuations above 1/3 purchase as much of the good as they can afford. Therefore, among high-value agents, the amount purchased ranges from 0 units for agents with budget 0 to 3 units for agents with budget 1. The principal's total revenue is 1/3, as opposed to 0 under the welfare-maximizing mechanism. On the other hand, the welfare of the revenue-maximizing mechanism is 2/3 as opposed to 3/4 under the welfare-maximizing mechanism.

Notice that the maximum amount purchased in the revenue-maximizing mechanism is the same as that of welfare-maximizing mechanism. This need not be the case in general. For these two mechanisms, agents fall under three possible comparisons:

1. Agents with valuations of less than 1/3 purchase none of the good in either mechanism, but receive a uniform lump-sum transfer of 1/2 in the welfare-maximizing mechanism. Therefore, they are clearly better off in the welfare-maximizing setting.

2. Agents with valuations between 1/3 and 1/2 purchase the good in the revenue-maximizing setting and receive lump-sum transfers in the welfare-maximizing setting. Of these agents, the agents whose utility increases the most in the revenue-maximizing setting are those with $v = 1/2$ and $w = 1$. They purchase 3 units for one dollar, which yields utility 3/2. On the other hand, in the welfare-maximizing setting, they have one dollar of budget and receive another 1/2 dollar from the uniform lump-sum transfer and thus have a total utility of 3/2. Therefore, the best-off agents in the revenue-maximizing mechanism in the $[1/3, 1/2]$ valuation range are indifferent between the welfare- and revenue-maximizing mechanisms while all other agents in this valuation range strictly prefer the welfare-maximizing mechanism.

3. Agents with valuations $v \geq 1/2$ purchase as much as they can afford in either mechanism. However, the amount they can purchase in the revenue-maximizing

mechanism, i.e. $\frac{w}{1/3} = 3w$, is weakly less than the amount they can purchase in the welfare-maximizing mechanism, i.e. $\frac{w+1/2}{1/2} = 2(w + 1/2) = 2w + 1$. Therefore, they are better off in the welfare-maximizing mechanism.

The above analysis demonstrates that the welfare-maximizing mechanism is a *Pareto improvement* over the revenue-maximizing mechanism. However, this is not a feature that needs to hold generally. More specifically, if one added a zero measure of agents with $w = 2$ and values uniformly distributed on $[0, 1]$, then agents with $w = 2$ and $v = 1$ would be worse off under the welfare-maximizing mechanism. This is because they value the good highly, but can purchase only $\frac{2+1/2}{1/2} = 5$ units in the welfare-maximizing mechanism as compared to $\frac{2}{1/3} = 6$ units in the revenue-maximizing mechanism. Therefore, while the welfare-maximizing mechanism improves utilitarian welfare relative to the revenue-maximizing mechanism, it may or may not be Pareto-improving as well.

Finally, the main result does not necessarily hold if the regularity and decreasing density assumptions are not satisfied. An example is provided in the appendix. The following is a brief outline of the argument behind the main theorem of this paper:

Step 1: I show that any admissible mechanism is simultaneously welfare- and revenue-dominated by a value-incentive compatible mechanism where every agent receives a take-it-or-leave-it offer based upon his budget type. These mechanisms have the budget-IC fully relaxed and I call such mechanisms *take-it-or-leave-it mechanisms.*

Step 2: I prove that any take-it-or-leave-it mechanism is welfare- and revenue-dominated by a mechanism that charges a linear price, that is, the per-unit price is the same for every budget level. This is the central innovation of the current paper.

Step 3: I show that one can consider an equivalent linear mechanism with *uniform lump-sum transfers*, that is, the transfers to agents no longer depend upon their budget-type.

Step 4: I find the welfare- and revenue-optimal linear mechanisms with uniform lump-sum transfers.

There are a couple aspects of the derivation that are potentially troublesome and warrant further discussion. First, *take-it-or-leave-it mechanisms* are value-incentive compatible, but may not be budget-incentive compatible. In fact, the only take-it-or-leave-it mechanisms that are budget-incentive compatible are those where rich agents are weakly favored. Fortunately, this does not pose a problem because in Steps 2 and 3 above, I show that an optimal take-it-or-leave-it mechanism is a linear pricing mechanism with uniform transfers and hence it is incentive compatible (with respect to both (IC) and (IC')). In more detail: while I temporarily focus on take-it-or-leave-it mechanisms that are value-incentive compatible and not necessarily budget-incentive compatible, the optimal take-it-or-leave-it mechanism turns out to be budget-incentive compatible. Therefore, the optimal take-it-or-leave-it mechanism is incentive compatible with respect to values and budgets and therefore is the solution to the optimal mechanism design problem. These optimal mechanisms are characterized in the statement of the theorem.

3.2 Three Lemmas

The first lemma shows that if all agents have the same known budget w, then any admissible mechanism is simultaneously welfare- and revenue-dominated by a take-it-or-leave-it offer (P, Q). I use the notation P here because it is an aggregate price rather than a per-unit price. The per-unit price will be the threshold $\hat{v}$ defined in the first lemma. Moreover, the optimal take-it-or-leave-it offer has the feature that the price P equals w, the known budget of each agent, which can be exchanged for a fixed quantity Q of the good.

I show this dominance via a weight-shifting argument as outlined in Figure 3.2 where two allocation functions are drawn. The solid dark allocation function consists of a take-it-or-leave-it offer. All agents with a valuation below $\hat{v}$ receive none of the good and all agents with valuations above $\hat{v}$ receive the same amount. The light dotted allocation function is another admissible allocation function. The weight-shifting argument relies upon finding $\hat{v}$ and shifting the allocation function from the dotted allocation function to the solid one. The choice of $\hat{v}$ is uniquely determined so that the shift that takes place allocates the same supply. Thus, the good is being shifted from agents with valuation less than $\hat{v}$ to agents with valuation above $\hat{v}$.

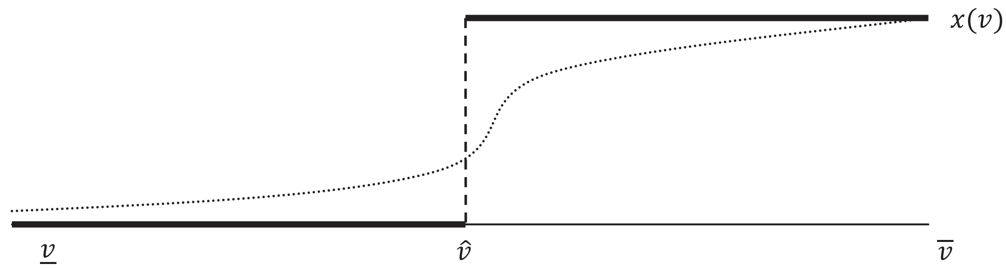

Figure 2: Agents below $\hat{v}$ no longer receive the good whereas all agents above $\hat{v}$ receive the same share as type $\bar{v}$.

Note: In the dark one-step allocation function above, it may be that the implied transfer paid by the highest type, i.e. $t(\bar{v})$ is strictly less than w. In this case, $\hat{v}$ (the threshold valuation/price) as well as the transferred quantity Q can both be increased in tandem so that the aggregate supply is maintained and welfare/revenue are both simultaneously improved. This step is also performed in the following lemma:

Lemma 1 *Under the regularity and decreasing density assumptions, for a fixed level of budget w, an admissible*[5] *allocation rule $x(v,w)$, and a transfer rule $t(v,w)$, there is a unique admissible welfare-optimal take-it-or-leave-it offer $\hat{x}$ with a transfer $\hat{t}$ that maintains the same reservation utility $U(\underline{v},w)$ and supply $\hat{S} = \int_{\underline{v}}^{\bar{v}} x(v,w)f(v)dv$. Moreover, this take-it-or-leave-it offer has the following properties:*

1. *This offer is welfare-improving.*

2. *This offer is revenue-improving.*

3. *$\hat{t} = w$.*

The lemma essentially states that any admissible mechanism is dominated by one in which an agent receives a take-it-or-leave-it offer with price equal to his budget. Such a mechanism is value-incentive compatible, but need not be wealth-incentive compatible. This will be demonstrated by Lemmas 2 and 3 which will show that the

[5]The fact that we are starting off with an admissible pair of rules x, t is fundamental for the proof here.

optimal take-it-or-leave-it mechanism features a linear price and is budget-incentive compatible with respect to both (IC) and (IC') after a smoothing of transfers.

While the formal proof of the above lemma is relegated to the appendix, the general idea is as follows: Shifting the allocation of the good from lower- to higher-value agents is feasible as the payments induced by the envelope theorem for the highest value type weakly decrease. As agents were previously respecting their budget constraints and the highest value agent's payments weakly decrease, then he must respect his budget. This argument relies on the decreasing density assumption. As all other agents purchase the same or less as the highest value agent, they are also asked for an affordable transfer. Shifting the good from lower-value types to higher value ones clearly improves welfare. The revenue improvement is due to the standard regularity assumption, so that the good is also being shifted to agents with higher virtual valuations.

Before turning to the next and central step, it is now a good time to compare the problem studied here with those with a common budget, most notably, Laffont and Robert (1996), Maskin (2000) and Devanur et al. (2013). In their problems, there is a common budget constraint and they solve for the optimal mechanisms under both revenue and welfare objectives. Their optimal mechanisms are auctions with pooling at the top rather than take-it-or-leave-it offers. The reason that the optimal mechanism here is much simpler is because the mechanism designer faces ex-ante supply constraints rather than interim constraints. This simpler solution in the common budget problem is what enables sewing the solution across different budget levels.

I now turn to formally defining a take-it-or-leave-it mechanism. A take-it-or-leave it offer gives the agents a quantity of the good in return for *their full budget*. However, instead of phrasing it in terms of quantities $\hat{x}(w)$, it turns out that for subsequent analysis, it is more tractable to phrase them in terms of cutoffs, $\hat{v}(w)$. These cutoffs specify value-thresholds so that agents with values above the relevant threshold participate and "take" the offer and agents below the threshold "leave it" instead. Either definition is equivalent via the indifference equation $\hat{x}\hat{v}-w = U(\underline{v}, w)$. Due to value-admissibility, these cutoffs serve as *both* the per-unit price that agents with budget w pay and the value-threshold for agents who buy the good.

Definition: Take-It-Or-Leave-It Mechanisms
A *take-it-or-leave-it mechanism* is a pair of functions (x, t) where
$x : V \times W \to \mathbb{R}_+,\ t : V \times W \to \mathbb{R}$ satisfying the following conditions:

$$\exists \hat{v} : W \to [\underline{v}, \bar{v}] \tag{2}$$

$$\int_W \int_V t(v, w) f(v)\ dv\ g(w)\ dw \geq 0 \tag{3}$$

$$\int_W \int_V x(v, w) f(v)\ dv\ g(w)\ dw \leq S \tag{4}$$

$$\forall v, w,\ v \geq \hat{v}(w) \Rightarrow x(v, w) = \frac{w + U(\underline{v}, w)}{\hat{v}(w)},\ t(v, w) = w \tag{5}$$

$$\forall v, w,\ v < \hat{v}(w) \Rightarrow x(v, w) = 0,\ t(v, w) = -U(\underline{v}, w) \leq 0 \tag{6}$$

To understand the above conditions, (2) stipulates that value-thresholds are well-defined and fall within the possible valuation range. Conditions (3) and (4) are simply budget-balance and limited supply. Condition (5) is more subtle. It states that higher-value agents "take it" and pay their entire budget. In return, value-incentive compatibility requires that the threshold type must be indifferent to such a payment, so $x(v, w)\hat{v}(w) - w = U(\underline{v}, w)$, which is just rewritten in (5). Finally, condition (6) says that those who "leave it" receive none of the good and they all receive the same monetary transfer, which gives them all the same reservation utility.

It is sufficient to consider take-it-or-leave-it mechanisms since Lemma 1 demonstrates that any admissible mechanism is simultaneously welfare- and revenue-dominated by such a mechanism. One can think of $\hat{v}(w)$ as being the per-unit price that an agent with budget w faces. This price then serves as a cutoff where agents with valuations higher than the per-unit price will fully expend their budgets buying the good and agents with valuations lower than the per-unit price will not purchase any of the good. Therefore, agents' IC conditions are built into equations (5) and (6) and it is assumed that agents cannot lie about their budget. In addition, notice that $U(\underline{v}, w) = -t(\underline{v}, w)$ and therefore equation (6) is also the individual rationality condition because agents who do not receive the good cannot be expected to pay anything. Take-it-or-leave-it mechanisms are a special subclass of all value-incentive compatible mechanisms. They are not necessarily budget-incentive compatible because agents may wish to lie downwards about their budget in order to obtain a cheaper per-unit price $\hat{v}(w)$.

The next lemma shows that any take-it-or-leave-it mechanism is welfare- and revenue-dominated by a linear price mechanism, or in other words, one where the cutoff values $\hat{v}(w)$ are constant in w. A *linear price mechanism*, or one with constant cutoffs, means that all agents face the same per-unit price, as in a competitive market.

This step is the central innovation of the current paper and so its proof is presented in full. I would also like to emphasize that both the continuum population assumption and most importantly, the unit-demand assumption are essential. If there was a unit-demand assumption, then the following lemma could fail to hold because when smoothing prices across different budget levels, it may give higher budget levels too much of the good, thereby violating the unit-demand constraint.

Lemma 2 *Under the regularity and decreasing density assumptions, the welfare-optimal and the revenue-optimal mechanisms both feature linear prices.*

Proof of Lemma 2:

Notice that a take-it-or-leave-it mechanism is defined by the reserve utility that the lowest value types receive, $U(\underline{v}, w)$, and the threshold $\hat{v}(w)$, which is also the linear price which the agents pay. Induced by these parameters is $\hat{x}(w) = \frac{w+U(\underline{v},w)}{\hat{v}(w)}$, the quantity offered in the take-it-or-leave-it offer and $S(w) = (1 - F(\hat{v}(w))\hat{x}(w)$, the per-capita consumption of the good by agents with budget w.

Focusing on one budget strip and suppressing the function arguments, one obtains the following set of defining equations.

$$S = (1 - F(\hat{v}))\hat{x} \tag{7}$$

$$\hat{x}\hat{v} = U + w \tag{8}$$

$$\mathcal{W} = \left(\int_{v}^{\bar{v}} zf(z)dz\right)\hat{x} \tag{9}$$

$$\mathcal{R} = (1 - F(\hat{v}))(U + w) - U \tag{10}$$

I will first show that $\frac{d\mathcal{W}}{dS}$ and $\frac{d\mathcal{R}}{dS}$ only depend on w via v. Then, I will show that $\mathcal{W}$ and $\mathcal{R}$ are in fact concave in S. These two facts together will imply that if $\hat{v}(w) < \hat{v}(w')$, then there is a joint welfare and revenue improvement by removing some supply from agents with budget w and giving it to agents with budget w'.

Replacing x in the equations above and taking derivatives, one has

$$\frac{\partial S}{\partial \hat{v}} = \frac{-\hat{v}f(\hat{v}) - (1 - F(\hat{v}))}{\hat{v}^2}(U + w) \tag{11}$$

$$\frac{\partial \mathcal{W}}{\partial \hat{v}} = \frac{-\hat{v}^2 f(\hat{v}) - \int_{\hat{v}}^{\bar{v}} z f(z)dz}{\hat{v}^2}(U + w) \tag{12}$$

$$\frac{\partial \mathcal{R}}{\partial \hat{v}} = -f(\hat{v})(U + w) \tag{13}$$

Therefore,

$$\frac{d\mathcal{W}}{dS} = \frac{\hat{v}^2 f(\hat{v}) + \int_{\hat{v}}^{\bar{v}} z f(z)dz}{\hat{v}f(\hat{v}) + (1 - F(\hat{v}))} \tag{14}$$

$$\frac{d\mathcal{R}}{dS} = \frac{\hat{v}^2 f(\hat{v})}{\hat{v}f(\hat{v}) + (1 - F(\hat{v}))} \tag{15}$$

Above, both derivatives are independent of agents' actual budget levels, and all that is relevant is the critical threshold $\hat{v}$. This suggests that a uniform $\hat{v}$ is optimal and this will in fact be proven once it is shown that the second derivatives are negative. However, notice that $\frac{dS}{d\hat{v}} < 0$, so it will in fact suffice to show that $\frac{\partial}{\partial \hat{v}} \frac{d\mathcal{W}}{dS} > 0$ and similarly $\frac{\partial}{\partial \hat{v}} \frac{d\mathcal{R}}{dS} > 0$.

The second derivatives are

$$\frac{\partial}{\partial \hat{v}} \frac{d\mathcal{W}}{dS} = \frac{\hat{v}f(\hat{v})(\hat{v}f(\hat{v}) + (1 - F(\hat{v}))) + \hat{v}f'(\hat{v})(\hat{v}(1 - F(\hat{v})) - \int_{\hat{v}}^{\bar{v}} z f(z)dz)}{(\hat{v}f(\hat{v}) + (1 - F(\hat{v})))^2} \tag{16}$$

$$\frac{\partial}{\partial \hat{v}} \frac{d\mathcal{R}}{dS} = \frac{2\hat{v}f(\hat{v})(\hat{v}f(\hat{v}) + 1 - F(\hat{v})) + \hat{v}^2 f'(\hat{v})(1 - F(\hat{v}))}{(\hat{v}f(\hat{v}) + (1 - F(\hat{v})))^2} \tag{17}$$

The first term of the numerator of the welfare equation (16) is positive. The second term is positive as well because $f'(z) \leq 0$ and $\hat{v}(1 - F(\hat{v})) - \int_{\hat{v}}^{\bar{v}} z f(z)dz \leq \hat{v}(1 - F(\hat{v})) - \int_{\hat{v}}^{\bar{v}} \hat{v} f(z)dz = 0$. The denominator is of course positive.

As for the revenue equation (17), by differentiating the regularity condition, $f'(v)(1 - F(v)) > -2f(v)^2$. Therefore, the numerator of the revenue equation is bounded below by $2\hat{v}f(\hat{v})(1 - F(\hat{v})) \geq 0$. ■

Lemmas 1 and 2 show that any admissible mechanism is simultaneously welfare- and revenue-dominated by a mechanism with linear prices (that is, every agent pays the same per-unit price). Moreover, as in any take-it-or-leave-it mechanism, when they purchase the good, they fully expend their budget in order to do so. These mechanisms are admissible according to (IC), but not necessarily for (IC') because richer agents may be receiving larger subsidies. The next lemma shows that for any such linear mechanism, there is a linear mechanism with lump-sum transfers that generates the same welfare and revenue. Lump-sum transfers and linear prices are (IC').

Lemma 3 *Any linear price mechanism that satisfies (BB), (LS), (NN), (BC), (IR), and (Value-IC) is simultaneously welfare- and revenue-equivalent to a linear price mechanism with uniform lump-sum transfers.*

3.3 The Main Theorem

The three lemmas show that any admissible mechanism is simultaneously welfare- and revenue-dominated by a linear price mechanism with lump-sum transfers. So, it remains to find the optimal such linear mechanisms. The theorem shows that the welfare optimum is characterized by the supply and budget balance conditions while the revenue-maximizing mechanism is characterized by the supply condition and $T = 0$. The theorem is restated below and the proof can be found in the Appendix.

Main Theorem: *Under the regularity and decreasing density assumptions, the following three statements hold:*

1. *A welfare-optimal mechanism is a linear mechanism with lump-sum transfers, (p_W, T_W), such that $T_W = S \cdot p_W$ and $(1 - F(p_W))\mathbb{E}[w] = S \cdot p_W \cdot F(p_W)$.*

2. *The revenue-optimal mechanism is a linear mechanism (with no transfers), (p_R, T_R), such that $T_R = 0$ and $(1 - F(p_R))\mathbb{E}[w] = S \cdot p_R$.*

3. $p_W > p_R$.

The above theorem shows that the optimal constrained mechanism from an efficiency point of view is one where the mechanism designer charges a linear price with lump-sum transfers. Agents with high valuations will purchase as much as they can afford and agents with low valuations do not purchase the good.

In fact, the welfare-maximizing mechanism is the linear mechanism with the *highest* feasible per-unit price while the revenue-maximizing mechanism is the linear mechanism with the lowest feasible per-unit price. This is due to the inverse relationship between transfers and the market-clearing price. The welfare-maximizing principal carries out transfers that are as large as possible in order to maximally relax the agents' budget constraints, thus increasing the market-clearing price. He does so until the budget balance constraint binds him. On the other hand, the revenue-maximizing principal makes no transfers, resulting in the lowest possible market-clearing price. Consequently, the per-unit prices have the perhaps counterintuitive relationship that prices are higher in the welfare-maximizing mechanism than in the revenue-maximizing one.

We now move on from discussing the main result of the paper to a few extensions.

Joint Objectives: The proof of the above theorem demonstrates that the optimal mechanism for a principal with a joint objective that depends both on welfare and revenue is a linear price mechanism. Depending upon the specific objective function, the optimal linear price can be any price between p_W and p_R and the balanced budget condition makes this a one-dimensional optimization problem.

Production: In some settings, the good can be produced, perhaps at a linear cost to the principal, rather than available in finite supply. In this case, the revenue-maximizing principal sets a price above this linear cost and produces according to demand. In contrast, the welfare-maximizing principal sets a price equal to the linear cost and therefore generates no revenue with which to make transfers. Therefore, the price relationship is reversed, the welfare-maximizing principal now sets a lower price than the revenue-maximizing principal. Intuitively, if a principal can produce the good, he prefers to do so rather than make lump-sum transfers because production is a more targeted subsidy. For more details, see the online supplement.

Aggregate Uncertainty: In the online supplement it is shown that the case of aggregate uncertainty with a finite state space and ex-ante constraints is a straightforward extension of the main theorem.

Finite Markets: While the model can be interpreted as either a continuum population problem or a single-agent setting, one naturally wonders what can be said for the finite agent setting with ex-post constraints. Fortunately, there is a significant strand of literature that focuses on computational approaches to optimal or approximately optimal mechanism design (see Yan (2011), Alaei et al. (2012) and Alaei et al. (2013); for a comprehensive survey, see Hartline (2013)). Of particular interest is, the sequential posted-price mechanism of Yan (2011). According to that mechanism, the seller orders the agents and then sells to them at the continuum optimal price p_R until either the supply of the good or the agents are exhausted. Given the current symmetric setting, this is equivalent to a posted price with rationing in case of surplus demand. When there are k goods to be sold, Yan (2011) demonstrates that this sequential posted-price mechanism is approximately optimal with a factor of $1/(1-1/\sqrt{2\pi k})$. So, the approximation becomes arbitrarily good as $k \to \infty$. Because of the setting difference (multi-unit demand), in the online appendix, I repeat a slightly modified version of his argument for completeness. Yan (2011) also shows that this approach can be used for other objectives, such as welfare. But, in the setting studied here, approximating the welfare optimal mechanism is trickier because it is not composed only of a price, there are also subsidies which must be balanced out via the sale of the good. So, instead, in the online appendix, a similar approach to Yan (2011) is taken where the good is freely and equally distributed and then a rationed market operates (rather than a rationed posted price mechanism). This mechanism is approximately optimal (with factor $1-\epsilon_N$) where ϵ_N vanishes as $N \to \infty$.

The approaches of Alaei et al. (2012) and Alaei et al. (2013) are intriguing, especially that of Alaei et al. (2012) who demonstrates how a multi-agent problem may be reduced to that of the single-agent problem because the main theorem of the current paper finds an optimal mechanism only in single-agent settings. But, in those papers, agents have unit-demand for each good (although there may be many types of goods) and therefore agents are either served or unserved. It remains interesting open work to see if the results therein can be extended to the setting studied here where agents may purchase any quantity of the good.

Unit Demand: While there is no unit demand assumption, if the supply of the good and budgets are both not too large, then the characterized optimal mechanism will allocate less than one unit to every agent. In that case, allocations can be understood as probabilities and the main theorem derives the optimal mechanism for a setting with a unit demand condition as well. Formally, the unit demand condition is $x(v, w) \leq 1, \ \forall v, w$ and the following corollary holds:

Corollary 4 *If $x(\bar{v}, \bar{w}) \leq 1$, then the optimal mechanisms from the main theorem are optimal for the unit demand problem as well. In terms of primitives:*

1. *For welfare optimality, the necessary and sufficient condition is* $S \leq \left(1 - F\left(\frac{\bar{w}}{1-S}\right)\right) \int_{\underline{w}}^{\bar{w}} \left(\frac{w(1-S)}{\bar{w}} + S\right) g(w)dw.$

2. *For revenue optimality, the necessary and sufficient condition is* $S \leq (1 - F(\bar{w})) \int_{\underline{w}}^{\bar{w}} \frac{w}{\bar{w}} g(w)dw.$

In the case where the above conditions are not satisfied, then the above mechanism is not optimal because it is not admissible. One may naturally wonder if a similar approach to the main theorem could be used. The first stage of the argument goes through. For each fixed budget, the optimal mechanism is still a take-it-or-leave-it offer, although the offer may be non budget-exhaustive. The difficulty lies in the second stage when sewing the take-it-or-leave-it offers together across different budget types. When there is no unit demand restriction and the planner gives some supply to agents with budget type w, he affects welfare by a concave function that depends solely upon the marginal type $\hat{v}(w)$. Because of this, when the planner optimizes across different budget types, he equalizes the thresholds, $\hat{v}(w)$. But, in the unit demand setting, the derivative of welfare with respect to supply, $\frac{d\mathcal{W}}{dS}$ depends upon the threshold $\hat{v}(w)$ and whether the agents are receiving 1 or less.[6] So, in the class of take-it-or-leave-it mechanisms, the optimizing planner would like to employ two different threshold regimes for when $x = 1$ and when $x < 1$. It turns out that these different thresholds favor poorer agents and therefore will not be incentive compatible across budget types as richer agents will imitate poorer types. That the mechanism design problem with unit demand cannot be solved in this fashion may also be guessed from the budget-constrained auction setting where: the common budget mechanism

[6] Formally, Equation (12) shows when $x < 1$, the derivative of welfare with respect to supply is $d\mathcal{W}/dS = \left(\hat{v}^2 f(\hat{v}) + \int_{\hat{v}}^{\bar{v}} z f(z)dz\right) / (\hat{v} f(\hat{v}) + (1 - F(\hat{v}))(U + w))$. On the other hand, when $x = 1$, it can be shown that $d\mathcal{W}/d\hat{v} = -\hat{v} f(\hat{v})$, $d\mathcal{S}/d\hat{v} = -f(\hat{v})$, and so $d\mathcal{W}/dS = \hat{v}$.

of Laffont and Robert (1996) differs from the mechanism of Pai and Vohra (2014) for a fixed budget level because the latter features pooling in the middle (Pai and Vohra (2014), Proposition 4) and the former does not.

4 Correlation

In this subsection, I solve for the revenue-maximizing mechanism in a setting where the revenue-maximizing mechanism will be generally nonlinear due to correlation between agents' valuations and budgets. While, in the provision of healthcare, budget and health may be independent, in other cases, such as the distribution of food or housing allowances, agents' budgets and values may be negatively correlated. On the other hand, in some cases, such as the "license raj" in India where the government allocated production quotas to family firms, some firms may be more efficient than others (see Esteban and Ray (2006)). In those cases, rich firms may have a higher valuation of production because they are more efficient, which is how they became rich in the first place.

Specifically, the assumption of the independence of budgets and valuations is replaced by the following weaker assumption.

Assumption 3: $\frac{f(v|w)}{1-F(v|w)}$ is weakly increasing in w.

The above assumption requires that virtual valuations are weakly increasing in w. Under this assumption, I obtain the following theorem.

Theorem 5 *Under the regularity, decreasing density, and the between-budgets monotone hazard rate assumptions, the revenue-maximizing mechanism is a take-it-or-leave-it mechanism with zero transfers. The linear prices that agents face are weakly decreasing in w and if the between-budgets monotone hazard rate assumption holds strictly, then the linear prices face are strictly decreasing in w whenever some agents purchase the good and some do not.*[7]

[7] In fact, a weaker condition would imply strictly decreasing linear prices, specifically that $\forall v$, $\frac{f(v,\underline{w})}{1-F(v,\underline{w})} < \frac{f(v,\bar{w})}{1-F(v,\bar{w})}$. This condition only requires that the hazard rate strictly increases in w at one point along every horizontal value strip rather than strictly increasing everywhere.

The proof is relegated to the appendix, but I now present the intuition behind it. As before, it is sufficient to restrict one's attention to take-it-or-leave-it mechanisms. First, it is not in the seller's interest to make any transfers to the agents since for that budget level, he could just offer a smaller supply of the good at the same linear price. This is revenue-improving because the seller is still taking in the same revenue, selling a smaller supply which he could monetize elsewhere, and saving on the transfers to the non-purchasing agents.

The other step of the proof is to show that the optimal cutoff levels $\hat{v}(w)$ are decreasing in budget. If this is the case, and there are no transfers to the agents, then this take-it-or-leave-it mechanism is admissible. This is because any take-it-or-leave-it mechanism is value-incentive compatible and for budget-incentive compatibility, it is sufficient that a mechanism favors the rich. Budget admissibility then follows as rich types do not wish to imitate poorer types (since they are favorably treated) while poor types would like to imitate rich types who benefit from a cheaper per-unit price, but they are unable to do so.

Example: Suppose that $V = [0, 1]$, $W = [1, 100]$ and that valuations are distributed according to a truncated exponential distribution for each w. Notice that a truncated exponential distribution, $f_\lambda(v) = \lambda e^{-\lambda v}/(1 - e^{-\lambda})$ has a Hazard Rate $\mathrm{HR}_\lambda(\mathrm{v}) = \frac{\mathrm{f}_\lambda(\mathrm{v})}{1-\mathrm{F}_\lambda(\mathrm{v})} = \frac{\lambda \mathrm{e}^{-\lambda \mathrm{v}}}{\mathrm{e}^{-\lambda \mathrm{v}} - \mathrm{e}^{-\lambda}} = \frac{\lambda}{1-\mathrm{e}^{\lambda(\mathrm{v}-1)}}$. This distribution satisfies both the regularity condition and the decreasing density condition. To see that for any fixed v the hazard rate is increasing in λ, consider the following derivative:

$$\frac{d\mathrm{HR}_\lambda(\mathrm{v})}{d\lambda} = \frac{\lambda^2 e^{\lambda(v-1)}}{(1 - e^{\lambda(v-1)})^2} > 0.$$

Therefore, the correlated distribution $f(v, w) = f_w(v)$ satisfies Assumptions 1-3.

If the supply of the good is such that cutoffs are chosen so that the derivative of revenue with respect to supply, $\frac{d\mathcal{R}}{dS} = 0.25$, then the value cutoffs for different types will range from about about 0.47 when $w = 1$ to about 0.25 when $w = 100$. As these cutoffs are also the per-unit prices, high-budget agents will face per-unit prices that are about half of what low-budget agents face.

The above take-it-or-leave-it mechanism has the interesting interpretation that a decreasing cutoff function corresponds to a concave pricing function. In other words, as the quantity purchased increases, the per-unit price is decreasing. So, this is a situation where agents have linear utility in the good, yet quantity discounts are profit-maximizing as a way to favor the rich agents. This is a form of price discrimination to capture richer agents who have virtual valuations. No such price discrimination would exist in a linear utility model without budget constraints.

Remark 1: Other similar examples may be constructed by taking the exponential parameter $\lambda(w)$ to be any increasing function of w. Further examples may also be constructed by initially taking a hazard rate that satisfies the above condition, such as $(1-v)\cdot(1-w)$ on $[0,1]^2$ and deriving the underlying distribution that generates it (for details, see Thomas (1971)). For other examples of nonlinear pricing, see Wilson (1993).

Remark 2: It may be that the between-budgets monotone hazard rate assumption is strict for some v, but constant for the critical threshold $\hat{v}$. In this case, the optimal mechanism will be linear, despite a lack of independence.

Remark 3: The above analysis was only performed for the revenue maximizing principal because he only needs to keep track of revenue. The same analysis could be done for the welfare-maximizing principal, but it adds complication with no additional insights as the welfare-maximizing principal must keep track of both welfare and revenue (to maintain aggregate budget balance).

5 Conclusion

A pair of optimal mechanism design problems were solved for in a setting with a finite supply of a single divisible good and a continuum of agents with private budget constraints. It was found that the optimal mechanisms for either objective take the form of linear prices. The welfare-maximizing mechanism additionally features uniform lump-sum transfers that partially alleviate agents' budget constraints. Agents with a high valuation for the good use these transfers along with their budget to purchase the good from the principal.

The revenue-maximizing principal offers no transfers to agents, implying that the gains in revenue from selling the good at a higher price due to the relaxation of budget constraints are more than outweighed by the cost of these transfers. An intuition of this result is that incentive compatibility requires transfers to be untargeted and thus they end up in in the hands of both agents who would use them to purchase the good and those who would not.

Additionally, an extension was considered that introduced correlation between valuations and budgets. I found that the revenue-maximizing mechanism when facing correlated types may feature nonlinear prices. This mechanism turns out to be a concave pricing rule and therefore offers a justification for quantity discounts, even when agents have linear utility. That is, quantity discounts enable profitable price discrimination along the budget dimension by taking advantage of the fact that agents with different budgets agents have different valuation distributions.

To conclude, a key motivation for studying the current problem is that budget constraints are prevalent and they may have a real impact on the structure of the optimal mechanism. Several interesting questions remain, including: when agents have concave utility in the good, when multiple goods are available, solving for the optimal mechanism in the binding unit-demand setting, and exploring whether multi-agent to single-agent reduction techniques can apply in the current multi-unit demand setting.

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Appendix

Lemma 1 *Under the regularity and decreasing density assumptions, for a fixed level of budget w, an admissible*[8] *allocation rule $x(v,w)$, and a transfer rule $t(v,w)$, there is a unique admissible welfare-optimal take-it-or-leave-it offer $\hat{x}$ with a transfer $\hat{t}$ that maintains the same reservation utility $U(\underline{v},w)$ and supply $\hat{S} = \int_{\underline{v}}^{\bar{v}} x(v,w)f(v)dv$. Moreover, this take-it-or-leave-it offer has the following properties:*

1. *This offer is welfare-improving.*

2. *This offer is revenue-improving.*

3. *$\hat{t} = w$.*

Proof of Lemma 1:

Throughout this proof, we fix a budget level w. Without loss of generality, consider the case where $x(\underline{v},w) = 0$. If this is not the case, then x can be changed to make that true and $t(\underline{v},w)$ is decreased by $x(\underline{v},w)\underline{v}$. This preserves $U(\underline{v},w)$ the utility of an agent with value $\underline{v}$, so he still does not wish to deviate after this change. For any other agent (v,w) with $v > \underline{v}$, she is currently receiving utility $U(v,w) \geq U(\underline{v},w)$ and would receive $U(\underline{v},w)$ if she lied to be the lowest type. Therefore, she also has no incentive to deviate to pretend to be $\underline{v}$ and incentive compatibility is preserved.

For simplicity, denote $\hat{x} := x(\bar{v},w)$. There is some $\hat{v}$ such that $(1-F(\hat{v}))\hat{x} = \hat{S}$. In addition, let $\hat{t} = \hat{x}\hat{v} - t(\underline{v},w)$. Now, consider the alternate transfer/allocation mechanism where

$$\hat{x}(v,w) = \begin{cases} 0 & v < \hat{v} \\ \hat{x} & v \geq \hat{v} \end{cases} \qquad\qquad \hat{t}(v,w) = \begin{cases} t(\underline{v},w) & v < \hat{v} \\ \hat{t} & v \geq \hat{v} \end{cases}$$

It needs to be shown that this alternate mechanism is feasible, improves welfare, and improves revenue for the principal. Notice that x is weakly increasing, so $\hat{x}$ is an improvement over x because some of the good that was going to lower-valuation types under the allocation function x is being shifted to higher-valuation agents under the allocation function $\hat{x}$.

[8] The fact that we are starting off with an admissible pair of rules x, t is fundamental for the proof here.

As for feasibility, it needs to be shown that $\hat{t} < w$. From the value-IC condition, transfers are given as: $t(v, w) = t(\underline{v}, w) + x(v, w)v - \int_{\underline{v}}^{v} x(z, w)dz$.

Then $t(\bar{v}, w) = t(\underline{v}, w) + x(\bar{v}, w)\bar{v} - \int_{\underline{v}}^{\bar{v}} x(z, w)dz = t(\underline{v}, w) + x(\bar{v}, w)\bar{v} - \int_{\underline{v}}^{\bar{v}} x(z, w)f(z)\frac{1}{f(z)}dz$ $\geq t(\underline{v}, w) + x(\bar{v}, w)\bar{v} - \int_{\underline{v}}^{\bar{v}} \hat{x}(z, w)f(z)\frac{1}{f(z)}dz = \hat{t}(\bar{v}, w)$. In the previous calculation, notice that the inequality follows because $\frac{1}{f(z)}$ is increasing, $\int_{\underline{v}}^{\bar{v}} x(z, w)f(z)dz = \int_{\underline{v}}^{\bar{v}} \hat{x}(z, w)f(z)dz$, and because $\hat{x}(v, w) - x(v, w)$ is a weakly negative, then weakly positive, function of v.

The above argument is important, because it yields the following inequalities: $w \geq t(\bar{v}, w) \geq \hat{t}(\bar{v}, w)$ and therefore $\hat{t}$ is feasible.

Finally, it needs to be shown that

$$\mathcal{R}(\hat{t}) := \int_{\underline{v}}^{\bar{v}} \hat{t}(v, w)f(v)dv \geq \int_{\underline{v}}^{\bar{v}} t(v, w)f(v)dv$$

Using the transfer equation from above, $\mathcal{R}(t) = \int_{\underline{v}}^{\bar{v}} \left(v - \frac{1-F(v)}{f(v)}\right) x(v, w)f(v)dv$. As above, we can see that $\int_{\underline{v}}^{\bar{v}} \left(v - \frac{1-F(v)}{f(v)}\right) \hat{x}(v, w)f(v)dv > \int_{\underline{v}}^{\bar{v}} \left(v - \frac{1-F(v)}{f(v)}\right) x(v, w)f(v)dv$ because $\frac{1-F(v)}{f(v)}$ is decreasing which implies $v - \frac{1-F(v)}{f(v)}$ is increasing, $\int_{\underline{v}}^{\bar{v}} x(z, w)f(z)dz$ $= \int_{\underline{v}}^{\bar{v}} \hat{x}(z, w)f(z)dz$, and because $\hat{x}(v, w) - x(v, w)$ is a negative, then positive, function of v. Therefore $\mathcal{R}(\hat{t}) \geq \mathcal{R}(t)$.

Therefore the optimal transfer/allocation mechanism for the principal is a take-it-or-leave-it offer. If it is not the case that $t(\bar{v}, w) = w$, then increase $\hat{x}$ by ϵ. As before, define $\hat{v}$ s.t. $(1 - F(\hat{v}))\hat{x} = \hat{S}$ and $\hat{t} := \hat{x}\hat{v} - t(\underline{v}, w)$. Notice that $\hat{v}$ is the threshold type who accepts the take-it-or-leave-it offer, i.e. this is the amount being spent per unit. Since this has just increased, we know that revenue has increased. Moreover, we know that welfare has increased because there is now a higher threshold type. Finally, we can find an ϵ small enough s.t. $\hat{t}$ is still less than w, hence we still have feasibility. ■

Proof of Lemma 3:

From the previous argument, one sees that the optimal mechanism is a linear-pricing system p. However, if non-uniform transfers are being made to the agents with lowest valuations $\underline{v}$, then this mechanism is not admissible with respect to (IC'). This is because, agents who do not wish to purchase the good may wish to misreport their types in order to secure a more favorable transfer. Formally, admissibility with respect to (IC') will fail if for $w \neq w'$, it is the case that $t(w, v) \neq t(w', v)$.

This problem is easily solved by distributing a uniform transfer equal to $\int_{\underline{w}}^{\bar{w}} t(w, \underline{v}) g(w) dw$ to agents who do not purchase the good. This may change the allocation for many agents (not just the ones who do not purchase the good), but notice that this change is welfare and revenue equivalent.[9]

Formally, the quantity demanded at the price p does not change:

$$
\begin{aligned}
&(1 - F(p)) \int_{\underline{w}}^{\bar{w}} \frac{w + t(\underline{v}, w)}{p} g(w) dw \\
=&\frac{1 - F(p)}{p} \left(\int_{\underline{w}}^{\bar{w}} w g(w) dw + \int_{\underline{w}}^{\bar{w}} t(\underline{v}, w) g(w) dw \right) \\
=&\frac{1 - F(p)}{f(p)} \int_{\underline{w}}^{\bar{w}} \left(w + \int_{\underline{w}}^{\bar{w}} t(\underline{v}, z) g(z) dz \right) g(w) dw
\end{aligned}
$$

where the first line is the amount demanded with non-uniform transfers and the last line is the quantity demanded with uniform transfers. This implies that the market clearing price p does not change. Since the quantity supplied does not change, and the market clearing price does not change, neither does welfare or revenue. This is because for any v above p, the same quantity is being bought and the same transfers are being made for that horizontal slice of agents. ■

Proof of Main Theorem:

Suppose that the linear mechanism with uniform transfers (p, T) does not supply the entire supply S of the good. Then the price p can be slightly reduced and this increases both the welfare of the agents (because every agent is receiving weakly more) and increases revenue (because a larger set of agents is paying their entire budget). Therefore, for either optimal mechanism, it must be that the supply constraint binds: $(1 - F(p)) \frac{\mathbb{E}[w]+T}{p} = S$.

Welfare-Maximization:

For the case of welfare-maximization, suppose that there is some leftover revenue. Then the principal could increase the transfers T slightly and increase the

[9]Recall that agents were using their budget w and transfer $t(w, \underline{v})$ to buy the good at price p, so agents who were receiving favorable transfers could purchase more of the good. Smoothing the transfers therefore affects purchasers because their effective budget changes.

price p slightly and improve the overall utilitarian welfare. Therefore, the welfare-maximizing mechanism needs to satisfy the budget balance constraint, specifically: $(1 - F(p))\mathbb{E}[w] = F(p)T$.

Solving for p and T yields the conditions provided in the theorem.

Revenue-Maximization:

For the case of revenue-maximization, suppose that $T > 0$. Then, an alternate mechanism where p is unchanged, and T is changed to 0 generates strictly more welfare. Therefore, the revenue-maximizing mechanism (p, T) is such that $T = 0$ and the supply constraint above binds.

Substituting in for T and multiplying the supply equation by p yields the conditions provided in the theorem. ■

Proof of Theorem 5:

Notice that the cutoff functions are determined by Equation (15), restated here in the case of non-independence of values and budgets.

$$\frac{d\mathcal{R}}{dS} = \frac{v^2 f(v|w)}{vf(v|w) + (1 - F(v|w))}$$

Now, because values and budgets are not independent, there may be differential returns to transferring supply from some budget levels to other budget levels. It needs to be checked that these returns are increasing in budget. Technically, it is necessary and sufficient that the above condition is increasing in budget. Dividing the top and bottom of the equation through by $f(v|w)$ yields:

$$\frac{d\mathcal{R}}{dS} = \frac{v^2}{v + \frac{1-F(v|w)}{f(v|w)}} \tag{18}$$

By the between-budgets monotone hazard rate assumption, the last term of the denominator is weakly decreasing in w, and hence, the whole expression is weakly increasing in w. In addition, the derivative is always positive. Therefore, the revenue-maximizing mechanism is where all S of the good is being sold, and where $\frac{d\mathcal{R}}{dS}$ is constant for every w being sold the good. This is precisely what is desired, and completes the proof. ■

A Example of Necessity

Here I demonstrate that the assumptions of the paper are necessary by giving an example where the assumptions do not hold and the optimal mechanism is not a linear mechanism (as the Main Theorem proves must be the case when the assumptions do hold).

Recall that, by a linear mechanism, there is a per-unit price p and to purchase a quantity of the good x, the price is px. In addition, my main result of the optimality of linear mechanisms applies to a either a single uniform budget constraint or individual unobservable budget constraints.

Therefore, for simplicity, I will demonstrate an example where agents have a single uniform budget and where the optimal mechanism is not linear. With a common budget, a linear mechanism has the additional feature that it is a take-it-or-leave-it mechanism. Specifically, any linear mechanism has a threshold value. Above this threshold, every agent receives the same quantity for the same transfer, and below this threshold, no agent purchases the good.

I prove the non-optimality of a take-it-or-leave-it mechanism in this setting by demonstrating a mechanism that dominates it.

Consider the following value distribution:

$$f(x) = \begin{cases} 1/12 & \text{if } 1 \leq x \leq 2 \\ 1/3 & \text{if } 2 < x \leq 3 \\ 7/12 & \text{if } 3 < x \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

The above distribution has three steps from 1 to 4. Suppose that $S = 11/12$ and $w = 2$ and consider a mechanism that sells one unit of the good at the price 2. This is a linear price mechanism for a single budget level and will generate welfare:[10]

$$\mathcal{W}_1 = \frac{1}{3} \cdot \frac{5}{2} + \frac{7}{12} \cdot \frac{7}{2} = \frac{23}{12} = 1.9333$$

On the other hand, consider a mechanism where the principal sells the good

[10]The welfare calculation is performed by looking at each of the populations and multiplying quantity (=1) times density (= 1/3 or 7/12) times average value (= 5/2 or 7/2).

according to:

$$x(v) = \begin{cases} 19/29 & \text{if } 1 \leq x \leq 3 \\ 32/29 & \text{if } 3 < x \leq 4 \end{cases} \qquad t(v) = \begin{cases} 19/29 & \text{if } 1 \leq x \leq 3 \\ 2 & \text{if } 3 < x \leq 4 \end{cases}$$

The above is a "two-step mechanism" since there are two different possible allocations and it generates welfare

$$\mathcal{W}_2 = \frac{19}{29} \cdot \frac{1}{12} \cdot \frac{3}{2} + \frac{19}{29} \cdot \frac{1}{3} \cdot \frac{5}{2} + \frac{32}{29} \cdot \frac{7}{12} \cdot \frac{7}{2} = 2.881$$

The two-step mechanism generates less revenue and more welfare than the one-step mechanism. The additional revenue from the one-step mechanism can be redistributed to the population, so that its welfare can be improved by finding a higher linear price at which trade can take place. Doing so would then make the welfare comparison between the two above mechanisms unclear. However, such a redistributional improvement can be prevented by adding a large population of agents with valuations between 0 and 1. These agents will then absorb most of the cash distributions. I show how this is done in the subsequent paragraphs.

Define a measure g as:

$$g(x) = \begin{cases} m & \text{if } 0 \leq x < 1 \\ f(x) & \text{otherwise} \end{cases}$$

The above is a measure and not a density because the integral of g is equal to $m + 1$ and not 1. Now, I will compare two mechanisms based upon the previously defined ones. The first is the one-step mechanism as defined before, with all of the money taken in, redistributed to the population so that a higher per-unit price can be established. This is the welfare-maximizing one-step mechanism in the class of all one-step mechanisms. The other is the two-step mechanism from before with no cash distributions. Therefore, it is clear that $0 = \mathcal{R}_1(m) < \mathcal{R}_2(m)$.

The important point is that as $m \to \infty$ the additional welfare value of the extra revenue that the one-step function generates converges to 0. Specifically, if all money that is received is redistributed, then, the one-step mechanism implies a cutoff price p where $(1 - F(p))w = Sp\frac{F(p)+m}{1+m}$. Moreover, recall that $(1 - F(2))w = S \cdot 2$. So, as $m \to \infty$, it is the case that $\frac{F(p)+m}{1+m} \to 1$ and therefore the threshold price $p \to 2$. This implies $\mathcal{W}_1(m) \to \mathcal{W}_1 < \mathcal{W}_2 = \mathcal{W}_2(m)$.

So, for large enough m, it is the case that $\mathcal{W}_1(m) < \mathcal{W}_2(m)$. While g is not a distribution, one can consider a rescaling that is, specifically, let $h(x) = \frac{g(x)}{m+1}$ and supply be $\frac{S}{m+1}$. Then the above allocation mechanisms are still applicable, but generate welfare $\frac{\mathcal{W}_1(m)}{m+1}$ and $\frac{\mathcal{W}_2(m)}{m+1}$ respectively. Therefore, it is the case that the 2-step mechanism generates strictly higher revenue and welfare then the 1-step mechanism for the distribution h and supply $\frac{S}{m+1}$. ■