ANALYSIS OF CRYPTOCURRENCIES ADOPTION USING FRACTIONAL GREY LOTKA-VOLTERRA MODELS

by

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Dedication

This dissertation is dedicated to my beloved late father Gratien Gatabazi, my mother Flora Uwera, my brothers and sisters, my aunt Marie-Goretti Mukakamali, my wife Josiane and my son Louis-Marie.

Declaration

The research work described in this dissertation was carried out in the Department of Mathematics and Applied Mathematics, University of Johannesburg, under the supervision of Dr. Jules Clément Mba and Dr. Edson Pindza.

The dissertation presents original work by the author and has not been submitted in any form for any degree or diploma to any university. Where use has been made of the work of others it has been duly acknowledged.

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Abstract

Solving analytically nonlinear dynamical system in continuous time scale is often problematic. The accumulation generating operations provide a tool of formulating a discrete dynamical form whose properties are relatively close to that of corresponding nonlinear systems. The present study discusses threes versions of 2- and 3- dimensional discrete Lotka-Volterra dynamical system with application to cryptocurrencies adoption. The application is interested on 3 cryptocurrencies namely Bitcoin, Litecoin and Ripple. The 2-dimensional application is on Bitcoin and Litecoin while the 3-dimensional application is on Bitcoin, Litecoin and Ripple. The dataset include records from 28-April-2013 to 10-February-2018 which provide forecasting values for Bitcoin and Litecoin through 2-dimensional study, while records from 7-August-2013 to 10-February-2018 provide forecasting values of Bitcoin, Litecoin and Ripple through 3-dimensional study. The thesis has produced four papers that have been published and presented in international conferences.

In Paper 1, Grey Lotka-Volterra Model (GLVM) of two and three dimensions is used for assessing the interaction between cryptocurrencies. The last 100 forecasting values, for n-dimensional GLVM, $n = \{2,3\}$ are presented. Lyapunov exponents of the 2 and 3-dimensional Lotka-Volterra models reveals that it is a chaotic dynamical system. Plots of 2 and 3-dimensional Lotka-Volterra models for filtered datasets suggest also a chaotic behavior. Using the Mean Absolute Percentage Error criterion, it was found that the accuracy of the GLVM is better than that of the classical Grey Model (GM(1,1)). By analysing the 2-dimensional GLVM, Bitcoin and Litecoin are found in *mutualism* or equivalently a win-win situation. The 3-dimensional GLVM

analysis evokes an increasing trend in transacting both Bitcoin, Litecoin and Ripple where Bitcoin keep relatively higher transaction counts. Paper 1 was published in the 122^{nd} volume of *Chaos, Solitons and Fractals*.

In Paper 2, Fractional Grey Lotka-Volterra Model (FGLVM) is introduced. Forecasting values of cryptocurrencies for n-dimensional FGLVM study, $n = \{2,3\}$ along 100 days of study time are displayed. The graph and Lyapunov exponents of the 2-dimensional Lotka-Volterra system using the results of FGLVM reveals that the system is a chaotic dynamical system. The chaos in 3-dimensional Lotka-Volterra is suggested by the values of Lyapunov exponents. The Mean Absolute Percentage Error indicates that FGLVM is better than GM(1,1) and GLVM. Both 2 and 3-dimensional FGLVMs analysis evokes a future constant trend in transacting Bitcoin and a future decreasing trend in transacting Litecoin and Ripple with Bitcoin at a relatively higher transaction while Litecoin transaction counts are everywhere superior to that of Ripple. Paper 2 was published in the 29^{th} volume of *Chaos*.

In Paper 3, Fractional Grey Lotka-Volterra Model with variable order is introduced. The actual values and the model values of n-dimensional model $n = \{2,3\}$ are displayed. The Mean Absolute Percentage Error (MAPE) suggests a high accuracy of the 3-dimensional Variable-order Fractional Lotka-Volterra model (VFGLVM) for the overall model values of Bitcoin and a reasonable accuracy for both model values of Litecoin and Ripple. The 2-dimensional VFGLVM has a good accuracy for the overall forecasting values of Bitcoin and a reasonable accuracy for the forecasting values of Litecoin. By analysing values of Lyapunov exponents and patterns of the corresponding Lotka-Volterra models, the 2 and 3-dimensional models show a chaotic behavior. Forecasting values indicate a future slight linear increase in transacting Bitcoin and a future decreasing transaction of Litecoin and Ripple. Bitcoin will keep relatively higher transaction counts and Litecoin transaction counts will be everywhere higher than that of Ripple. Paper 3 was published in the 127^{th} volume of *Chaos, Solitons and Fractals*.

In Paper 4, the error assessment is made on the classical Grey Model (GM(1,1)) and the variants of the Grey Lotka-Volterra dynamical system namely the GLVM, the FGLVM and the VFGLVM.

The error sequence patterns and the Mean Absolute Percentage Error (MAPE) suggest a relatively higher accuracy of the VFLVM in 2- and 3-dimensional framework. The results show that in most of the cases, the VFGLVM is relatively the best model followed by the FGLVM, the GLVM and then the GM(1,1). Paper 4 was submitted in *Chaos*.

List of papers

The following papers have been published from this thesis.

1. P. Gatabazi, J.C. Mba, E. Pindza, C. Labuschagne (2019). Grey Lotka-Volterra models with application to cryptocurrencies adoption. *Chaos, Solitons and Fractals*, **122**, 47-57.

2. P. Gatabazi, J.C. Mba, E. Pindza (2019). Fractional grey Lotka-Volterra model with application to cryptocurrencies adoption. *Chaos*, 29 (7), 073116 .

3. P. Gatabazi, J.C. Mba, E. Pindza (2019). Modeling cryptocurrencies transaction counts using variable-order Fractional Grey Lotka-Volterra dynamical system. *Chaos, Solitons and Fractals*, **127**, 283-290.

4. P. Gatabazi, J.C. Mba, E. Pindza (2019). Error assessment in forecasting cryptocurrencies transaction counts using variants of the Grey Lotka-Volterra dynamical system. *Chaos* , No: **CHA19-AR-01330** (Under review).

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PART 1
GENERAL INTRODUCTION AND PRELIMINARIES

CHAPTER 1

GENERAL INTRODUCTION

1.1 Background

The traditional Grey Model GM(1,1) proposed by Deng [9] has been widely applied in various studies on forecasting such as for example in wafer fabrication, computation in opto-electronics industry output value, estimating electricity costs, forecasting the integrated circuit industry, and fatality risk estimation [22]. However, GM(1,1) presents some disadvantages: As a single variable forecasting model, it cannot analyse the long-term relationship between the two variables and predict the values of two variables in social system or economic system. Also, it cannot reflect the new information priority principle, and, if it is necessary to obtain the ideal effect of modeling, the original data must meet the class ratio test. To overcome these restrictions, Fractional Grey Model FGM(q,1) was proposed by Mao et al. [13] and presents higher modeling precision and wider adaptability.

Beside the GM(1,1) and the FGM(q,1), the Lotka-Volterra Model (LVM) has been a tool of analysing competition in continuous time scale. The LVM was used for example in a study of competition among 200 mm and 300 mm wafers by using historical data [8]. Applying the GM(1,1) to the Lotka-Volterra competition models were established in [10] for testing the trade relationships between China and Singapore, Malaysia and Thailand, respectively, based on the data of import and export from 2003 to 2014.

The present study introduces a new approach to test the competitivity based on applying the

FGM(q,1) to the Lotka-Volterra model. This approach will then be used to study the competition between Bitcoin and its peer cryptocurrencies and also its social adoption. This study can be regarded as an important reference for investors in cryptocurrencies, and to assist governments regarding their monetary policy on cryptocurrencies.

1.2 Cryptocurrencies

The idea on digital currencies starts with Chaum [7] in early 1980's. The first virtual currency was called Digicash and was launched in 1990. In 2008, Satoshi [17] launched Bitcoin which consists of the online third-party free trade without centralized control [5]. The other modern cryptocurrencies that followed the foundation of Bitcoin include Litecoin launched in 2011, Ripple launched in 2012, Dogecoin launched in 2013 and Dash launched in 2014 [2]. Many other cryptocurrencies which followed include Ethereum, Peercoin, Primecoin, Chinacoin and Ven. Cryptocurrencies appear among the innovations that allow transfers of digital currencies without the intervention of banks. Numerous advantages of Bitcoin include anonymous online transaction, non-taxable purchases, mobile payments and relatively low transaction fees [3, 2]. Furthermore, Wayner [21] evokes that digital cash cannot have multiple copies, thanks to its strong cryptographic algorithm and network consensus on its blockchain. Hence, a cryptocurrency cannot be used more than once. Multiple transfers are therefore not easy for digital currencies. Following this, cryptocurrency has been viewed as a more secure and reliable mode of payment in recent years. Urquhart [20] tested the efficiency of Bitcoin by using the dataset on the exchange of Bitcoin for six years. This analysis raises the problem of a long-term return of Bitcoin. The rapid spread of Bitcoin trade urged some governments to ban or discourage the use of Bitcoin due to uncontrolled transactions that could affect the administration of monetary policy [8]. The efficient studies on cryptocurrencies could assist some governments which see cryptocurrencies as an economic threat, to tailor they monetary policy regarding the digital currencies. In fact, some governments note that cryptocurrency could facilitate illegal transactions, disrupting the government activities [8]. The comparison of the Lotka-Volterra models with other popular techniques, such as linear regression modeling on Bitcoin will provide another way of understanding the accuracy of Lotka-Volterra model for predicting the behaviour of Bitcoin.

This study integrates Fractional order Lotka-Volterra dynamical system which presents the advantage for better modeling and understanding the behaviour of more complicated predator-prey systems due to the LVM long memory principle. A combination of this model with grey modeling to obtain the fractional grey Lotka-Volterra models will be beneficial with regard to long-term behaviour and forecasting, as well as a better understanding of the social adoption of digital currencies.

The present study analyses mathematically a competition between Bitcoin, Litecoin and Ripple daily transaction counts. Litecoin differs from Bitcoin in three important points. Firstly, Litecoin performs the processing of a block every 2.5 minutes instead of every 10 minutes of Bitcoin, allowing faster confirmation of transactions [6]. Secondly Litecoin produces approximately 4 times more units than Bitcoin and thirdly, Litecoin uses the function Scrypt in its working test algorithm which is hard memory sequential function that facilitates mining and Litecoin does not need sophisticated equipment as Bitcoin does [6]. This effect enables Litecoin network to accommodate up to 84 million coins while Bitcoin network cannot exceed 21 million coins. This study includes Litecoin which was, by Bhosale and Mavale [4] report of 6^{th} March 2018, the second largest cryptocurrency by the market capitalization. Ripple is based on the honour and trust of the people in the network [6]. Ripple adopts the development of a credit system. Each Ripple node functions as a local exchange system, in such a way that the entire system forms a decentralized mutual bank based on the needs of the users and everything is for a common good among them. They can in such a way, exchange everything up to skills.

1.3 Study methodology

Innovation brought on by Bitcoin needs a mathematical understanding, especially using existing models from differential equations. This study prefers a use of Lotka-Volterra differential equations, which is a popular model for competing phenomena.

Lotka-Volterra equations have shown satisfactory results in various modeling. Marasco et al. [14] used Lotka-Volterra differential equations while studying economic competitions for forecasting market evolution of N firms in a dynamic oligopoly market. Their study was supported by two sets of historical data, namely the market shares of three Japanese beer companies with the inclusion of an outside good or option and the market shares of three mobile phone companies in Greece [14]. This study uses the mean square error for evaluating the fitting and forecasting performance of the fractional grey Lotka-Volterra model. Another case where the Lotka-Volterra model outperforms nicely is the case of competing technologies [5]: Through the Lotka-Volterra model and the real dataset, the markets of two different types of silicon wafer were compared. The Runge-Kutta numerical method was used to solve the model. A linear regression model was also conducted for the same dataset and by using the mean square error test, it was shown that Lotka-Volterra is much more accurate than linear regression for fitting the model [5]. The Lotka-Volterra model has been analysed and applied by several authors, some recent papers include Morris and Pratt [15], Wu et al. [22] and Hsi-Tse Wang and Ta-Chu Wang [10].

This study will firstly define the Lotka-Volterra dynamical system known also as the predator-prey model. The Lotka-Volterra system and related modifications can be found in [3] and [20]. The Lotka-Volterra dynamical system in discrete framework will be constructed by using the traditional Grey Modeling (GM(1,1)) as described in [22] and the fractional calculus theory. The basic concepts on fractional calculus in continuous time scale are presented in the next chapter. The performance of the models will be measured by the Mean Absolute Percentage Error criterion found for example in [10], [16, 22], [23] or [24].

CHAPTER 2

PRELIMINARIES

The present study will apply Grey model and fractional calculus theory to the Lotka-Volterra dynamical system. This chapter stats by discussing the basic concepts in fractional integral and derivative in continuous time scale, then presents the general Lotka-Volterra and close by presenting the overview on the classical Grey Modeling (GM(1,1)).

2.1 Basic concepts of fractional calculus in continuous time scale

The fractional calculus consists of defining real or complex powers of the integration linear operator $\mathscr{I}$ and differentiation linear operator $\mathscr{D}$. Several ways of defining fractional integral and derivative include Riemann-Liouville, Caputo, Grünwald-Letnikov and Hadamard approaches [1]. The fractional integrals in this chapter are limited on Riemman-Liouvile and Hadamard integrals while Riemann-Liouvile, Caputo, Grünwald-Letnikov and Hadamard derivatives are presented.

2.1.1 Fractional operators and fractional integral

Let $y : [a,b] \to \mathbb{R}$ be an integrable function and α a real positive number. Given the operator ${}_a\mathscr{I}_t^n$, the Cauchy formula for n-fold iterated integral is given by

$$\begin{aligned} {}_a\mathscr{I}_t^n y(t) &= \int_a^t d\zeta_1 \int_a^{\zeta_1} d\zeta_2 \int_a^{\zeta_2} d\zeta_3 \cdots \int_a^{\zeta_{n-1}} d\zeta_n \\ &= \frac{1}{(n-1)!} \int_a^t (t-\zeta)^{n-1} y(\zeta) d\zeta, n \in \mathbb{N}. \end{aligned} \tag{2.1}$$

The generalization of Equation (2.1) for non-integer values of n is given by the Riemann-Liouville fractional integral of order α as

$$ {}_a^{RL}\mathscr{I}_t^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\zeta)^{\alpha-1} y(\zeta) d\zeta, t > a \tag{2.2}$$

and

$$ {}_t^{RL}\mathscr{I}_b^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (t-\zeta)^{\alpha-1} y(\zeta) d\zeta, t < b. \tag{2.3}$$

Equations (2.2) and (2.3) are respectively left and right Riemann-Liouville fractional integrals of order α over the domain $[a,b]$. Assuming that $y(t)$ is continuous and $\alpha \to 0$, then

$$ {}_a\mathscr{I}_t^\alpha = {}_t\mathscr{I}_b^\alpha = \mathscr{I}$$

where $\mathscr{I}$ is the identity operator. Therefore,

$$ {}_a\mathscr{I}_t^\alpha y(t) = {}_t\mathscr{I}_b^\alpha y(t) = \mathscr{I} y(t) = y(t).$$

The left and right Hadamard fractional integrals of order α are given respectively by

$$ {}_a^{H}\mathscr{I}_t^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{\zeta} \right)^{\alpha-1} \frac{y(\zeta)}{\zeta} d\zeta, t > a \tag{2.4}$$

and

$$ {}_{t}^{H}\mathscr{I}_{b}^{\alpha}y(t) = \frac{1}{\Gamma(\alpha)}\int_{t}^{b}\left(\ln\frac{t}{\zeta}\right)^{\alpha-1}\frac{y(\zeta)}{\zeta}d\zeta, t < b. \quad (2.5) $$

2.1.2 Fractional derivative

Several definitions of fractional derivative in continuous time scale include the Riemann-Liouville, Caputo, Grünwald-Letnikov and Hadamard approaches [12]. Assume that $y : [a,b] \to \mathbb{R}$ is absolutely continuous on [a,b]. Let α be a positive real number and let n be the nearest integer greater than α. Below are three most popular definitions of the fractional derivative:

Definition 2.1.1. The left and right Riemann-Liouville fractional derivatives of order α are respectively

$$ \begin{aligned} {}_{a}^{RL}\mathscr{D}_{t}^{\alpha}y(t) &= \frac{d^n}{dt^n}{}_{a}\mathscr{I}_{t}^{n-\alpha}y(t) \\ &= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_{a}^{t}(t-\zeta)^{n-\alpha-1}y(\zeta)d\zeta, t > a \end{aligned} $$

and

$$ \begin{aligned} {}_{t}^{RL}\mathscr{D}_{b}^{\alpha}y(t) &= \frac{d^n}{dt^n}{}_{t}\mathscr{I}_{b}^{n-\alpha}y(t) \\ &= \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_{t}^{b}(\zeta-t)^{n-\alpha-1}y(\zeta)d\zeta, t < b \end{aligned} $$

where $n-1 < \alpha < n$.

Definition 2.1.2. The left and right Caputo fractional derivatives of order α are respectively

$$ \begin{aligned} {}_{a}^{C}\mathscr{D}_{t}^{\alpha}y(t) &= \frac{d^n}{dt^n}{}_{a}\mathscr{I}_{t}^{n-\alpha}y(t) \\ &= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_{a}^{t}(t-\zeta)^{n-\alpha-1}y^{(n)}(\zeta)d\zeta, t > a \end{aligned} $$

and

$$\begin{aligned}
{}_{t}^{C}\mathscr{D}_{b}^{\alpha}y(t) &= \frac{d^n}{dt^n}{}_{t}\mathscr{I}_{b}^{n-\alpha}y(t) \\
&= \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_t^b(\zeta-t)^{n-\alpha-1}y^{(n)}(\zeta)d\zeta, t<b
\end{aligned}$$

where $n-1<\alpha<n$.

Definition 2.1.3. The α^{th} order Grünwald-Letnikov fractional derivative of function y is given by

$$ {}_{a}^{GL}\mathscr{D}_{t}^{\alpha}y(t) = \lim_{h\to 0}\frac{1}{h^\alpha}\sum_{k=0}^{n}(-1)^k\frac{\Gamma(\alpha+1)}{k!\,\Gamma(\alpha-k+1)}y(t-kh)$$

where $nh=t-a$.

Definition 2.1.4. The left and right Hadamard fractional derivatives of order α are respectively

$$ {}_{a}^{H}\mathscr{D}_{t}^{\alpha}y(t) = \frac{t^n}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t\left(\ln\frac{t}{\zeta}\right)^{n-\alpha-1}\frac{y(\zeta)}{\zeta}d\zeta$$

and

$$ {}_{t}^{H}\mathscr{D}_{b}^{\alpha}y(t) = \frac{(-t)^n}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_t^b\left(\ln\frac{t}{\zeta}\right)^{n-\alpha-1}\frac{y(\zeta)}{\zeta}d\zeta$$

where $n-1<\alpha<n,\ t\in]a,b[$.

Remark 2.1.5. Consider the Caputo and Riemann-Liouvile fractional derivative and $y(t)=K$ where K is a constant. It follows from the Caputo derivative that

$$ {}_{a}^{C}\mathscr{D}_{t}^{\alpha}y(t) = {}_{t}^{C}\mathscr{D}_{b}^{\alpha}y(t) = 0$$

while

$$\begin{cases}
{}_{a}^{RL}\mathscr{D}_{t}^{\alpha}y(t) = \frac{K(t-a)^{-\alpha}}{\Gamma(1-\alpha)} \\
{}_{t}^{RL}\mathscr{D}_{b}^{\alpha}y(t) = \frac{K(b-t)^{-\alpha}}{\Gamma(1-\alpha)}.
\end{cases}$$

Caputo fractional derivatives seem to be more natural that the Riemann-Liouvile fractional derivatives.

Remark 2.1.6. For $\alpha \longrightarrow n^-$, $n \in \mathbb{N}$,

$$\begin{cases} {}_a^{RL}\mathscr{D}_t^\alpha ={}_a^C \mathscr{D}_t^\alpha = \frac{d^n}{dt^n} \\ {}_t^{RL}\mathscr{D}_b^\alpha ={}_t^C \mathscr{D}_b^\alpha = -\frac{d^n}{dt^n}. \end{cases}$$

2.1.3 Some properties of the fractional derivatives

This section presents some common properties in fractional differentiation and relationships between different types of fractional derivatives. The section is buckled by a characterization of integral and derivative Caputo and Riemann-Liouvile operators.

Lemma 2.1.7. Assume that $y : [a,b] \longrightarrow \mathbb{R}$ is absolutely continuous on $[a,b]$ and has the form $y(t) = (t-a)^\psi$ where ψ is a real number. Then we have the following results:

1. ${}_a^{RL}\mathscr{D}_t^\alpha (t-a)^\psi = \dfrac{\Gamma(\psi+1)}{\Gamma(\psi-\alpha+1)}(t-a)^{\psi-\alpha}$; $\psi > -1$.

2. ${}_a^{GL}\mathscr{D}_t^\alpha (t-a)^\psi = \dfrac{\Gamma(\psi+1)}{\Gamma(\psi-\alpha+1)}(t-a)^{\psi-\alpha}$; $\psi > 0,\ 0 < \alpha < 1$.

3. $$\begin{cases} {}_a^{C}\mathscr{D}_t^\alpha (t-a)^\psi) = \dfrac{\Gamma(\psi+1)}{\Gamma(\psi-\alpha+1)}(t-a)^{\psi-\alpha} \\ {}_t^{C}\mathscr{D}_b^\alpha (b-t)^\psi = \dfrac{\Gamma(\psi+1)}{\Gamma(\psi-\alpha+1)}(b-t)^{\psi-\alpha};\ \psi > 0,\ n-1 < \alpha < n, \psi > n-1,\ n \in \mathbb{N}. \end{cases}$$

The proof of Lemma 2.1.7 is straightforward from Definitions 2.1.1, 2.1.2 and 2.1.3.

Lemma 2.1.8. (Kilbas et al. [11]). Assume that $y : [a,b] \longrightarrow \mathbb{R}$ is absolutely continuous on $[a,b]$. The Riemann-Liouville and Caputo derivatives are related by the following relationships:

$$ {}_a^C\mathscr{D}_t^\alpha y(t) = {}_a^{RL}\mathscr{D}_t^\alpha \left[y(t) - \sum_{k=0}^{n-1} \frac{y^{(k)}(a)(t-a)^k}{k!} \right] \tag{2.6} $$

and

$$ {}_t^C\mathscr{D}_b^\alpha y(t) = {}_t^{RL}\mathscr{D}_b^\alpha \left[y(t) - \sum_{k=0}^{n-1} \frac{y^{(k)}(b)(b-t)^k}{k!} \right], \; n-1 < \alpha < n, \; n \in \mathbb{N}. \tag{2.7} $$

In particular, when $y(a) = y(b) = 0$, the Riemann-Liouville and Caputo derivatives are equal. The following theorems characterizes the composition of the integral and derivative operators in the sens of Caputo and Riemann-Liouvile [11].

Theorem 2.1.9. Assume that $y : [a,b] \longrightarrow \mathbb{R}$ is absolutely continuous on $[a,b]$. Let $\alpha > 0$. The following rules hold:

$$ {}_a^C\mathscr{D}_t^\alpha \, {}_a^{RL}\mathscr{I}_t^\alpha y(t) = y(t) $$

and

$$ {}_t^C\mathscr{D}_b^\alpha \, {}_t^{RL}\mathscr{I}_b^\alpha y(t) = y(t) $$

Theorem 2.1.10. Assume that $y : [a,b] \longrightarrow \mathbb{R}$ is absolutely continuous on $[a,b]$. Let α such that $n-1 < \alpha < n$, $n \in \mathbb{N}$. The following rules hold:

$$ {}_a^{RL}\mathscr{I}_t^\alpha \, {}_a^C\mathscr{D}_t^\alpha y(t) = y(t) - \sum_{k=0}^{n-1} \frac{y^{(k)}(a)}{k!}(t-a)^k $$

and

$$ {}_t^{RL}\mathscr{I}_b^\alpha \, {}_t^C\mathscr{D}_b^\alpha y(t) = y(t) - \sum_{k=0}^{n-1} \frac{(-1)^k y^{(k)}(b)}{k!}(b-t)^k $$

In particular when $\alpha \in]0,1[$,

$$ {}_{a}^{RL}\mathscr{I}_{t}^{\alpha}{}_{a}^{C}\mathscr{D}_{t}^{\alpha}y(t) = y(t) - y(a) $$

and

$$ {}_{t}^{RL}\mathscr{I}_{b}^{\alpha}{}_{t}^{C}\mathscr{D}_{b}^{\alpha}y(t) = y(t) - y(b). $$

2.2 General Lotka-Volterra system

2.2.1 Definition

The general Lotka-Volterra System (LVS) of competing relationships between n species is given by

$$ \begin{cases} \frac{dX_1}{dt} = X_1 \left[a_1 - (\alpha_{11}X_1 + \alpha_{12}X_2 + \cdots + \alpha_{1n}X_n)\right] \\ \frac{dX_2}{dt} = X_2 \left[a_2 - (\alpha_{21}X_1 + \alpha_{22}X_2 + \cdots + \alpha_{2n}X_n)\right] \\ \vdots \\ \frac{dX_n}{dt} = X_n \left[a_n - (\alpha_{n1}X_1 + \alpha_{n2}X_2 + \cdots + \alpha_{nn}X_n)\right] \end{cases} \tag{2.8} $$

[18].

In System (2.8), parameters $a_{i,\ i\in[1,n]}$ represent the capacity of growing of populations $X_{i,\ i\in[1,n]}$, while parameters $\alpha_{ij,\ i\in[1,n]\ j\in[1,n]}$ represent the effect species j has on species i. The expressions X_i^2 are interactions within species, $X_i\ X_j$, $i \neq j$ are interactions of different species.

2.2.2 LVS equilibrium point

Assuming nonzero competing species $X_{i,\ i\in[1,n]}$, the equilibrium point of System (2.8) satisfies the following system:

$$
\begin{cases}
\alpha_{11}X_1 + \alpha_{12}X_2 + \cdots + \alpha_{1n}X_n = a_1 \\
\alpha_{21}X_1 + \alpha_{22}X_2 + \cdots + \alpha_{2n}X_n = a_2 \\
\vdots \\
\alpha_{n1}X_1 + \alpha_{n2}X_2 + \cdots + \alpha_{nn}X_n = a_n
\end{cases}
\tag{2.9}
$$

whose solution for $\begin{vmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \dots & \alpha_{nn} \end{vmatrix} \neq 0$ is the equilibrium point

$$
(X_1, X_2, \dots, X_n) =
\left(\frac{\begin{vmatrix} a_1 & \alpha_{12} & \dots & \alpha_{1n} \\ a_2 & \alpha_{22} & \dots & \alpha_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_n & \alpha_{n2} & \dots & \alpha_{nn} \end{vmatrix}}{\begin{vmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \dots & \alpha_{nn} \end{vmatrix}},
\frac{\begin{vmatrix} \alpha_{11} & a_1 & \dots & \alpha_{1n} \\ \alpha_{21} & a_2 & \dots & \alpha_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{n1} & a_n & \dots & \alpha_{nn} \end{vmatrix}}{\begin{vmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \dots & \alpha_{nn} \end{vmatrix}},
\dots,
\frac{\begin{vmatrix} \alpha_{11} & \alpha_{12} & \dots & a_1 \\ \alpha_{21} & \alpha_{22} & \dots & a_2 \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \dots & a_n \end{vmatrix}}{\begin{vmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \dots & \alpha_{nn} \end{vmatrix}} \right)
$$

Theorem 2.2.1. (Strobeck [18]). System (2.8) for nonzero competing species $X_{i,\ i\in[1,n]}$ has a stable equilibrium if and only if

$$
\begin{vmatrix} a_1 & \alpha_{12} & \dots & \alpha_{1n} \\ a_2 & \alpha_{22} & \dots & \alpha_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_n & \alpha_{n2} & \dots & \alpha_{nn} \end{vmatrix} > 0,
\begin{vmatrix} \alpha_{11} & a_1 & \dots & \alpha_{1n} \\ \alpha_{21} & a_2 & \dots & \alpha_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{n1} & a_n & \dots & \alpha_{nn} \end{vmatrix} > 0,\ \dots,
\begin{vmatrix} \alpha_{11} & \alpha_{12} & \dots & a_1 \\ \alpha_{21} & \alpha_{22} & \dots & a_2 \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \dots & a_n \end{vmatrix} > 0
$$

The proof of Theorem 2.2.1 is found in [18].

2.3 Grey Modeling

The grey modeling (GM(1,1)) consists of predicting uncertain or incomplete information systems for determining the future dynamic situation of a certain sequence of numbers [23]. Assume an original data sequence $X^{(0)} = \left(x^{(0)}(1), x^{(0)}(2), \ldots, x^{(0)}(n)\right)$. The first order accumulation generating operation (1-AGO) is given by:

$$X^{(1)} = \left(x^{(1)}(1), x^{(1)}(2), \ldots, x^{(1)}(n)\right), \text{ with } x^{(1)}(k) = \sum_{i=1}^{k} x^{(0)}(i),\ k = 1, 2, \ldots, n,$$

We define the mean sequence of $X^{(1)}$ as

$$Z^{(1)} = \left(z^{(1)}(2), z^{(1)}(3), \ldots, z^{(1)}(n)\right),$$

with

$$z^{(1)}(k) = \frac{x^{(1)}(k) + x^{(1)}(k-1)}{2},\ k = 2, 3, \ldots, n.$$

The GM(1,1) based on the series $X^{(1)} = \left(x^{(1)}(1), x^{(1)}(2), \ldots, x^{(1)}(n)\right)$ is defined by the following differential equation:

$$\frac{dx^{(1)}(t)}{dt} + ax^{(1)}(t) = b. \tag{2.10}$$

Parameters a and b of GM(1,1) are calculated by the least square method and the initial condition $X^{(1)}(1) = X^{(0)}(1)$ as proposed by Tien [19]. Hsi-Tse Wang and Ta-Chu Wang [10] propose the following least square estimation:

$$\begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix} = (B'B)^{-1}B'M$$

where,

$$B = \begin{pmatrix} -z^{(1)}(2) & 1 \\ -z^{(1)}(3) & 1 \\ \vdots & \vdots \\ -z^{(1)}(n) & 1 \end{pmatrix}; \quad M = \begin{pmatrix} x^{(0)}(2) \\ x^{(0)}(3) \\ \vdots \\ x^{(0)}(n) \end{pmatrix}.$$

Proposition 2.3.1. The difference equation corresponding to Equation (2.10) can be written as:

$$x^{(0)}(k+1) + az^{(1)}(k) = b. \tag{2.11}$$

Proof. The term $\dfrac{dx^{(1)}(t)}{dt}$ can be written as

$$\begin{aligned}
\frac{dx^{(1)}(t)}{dt} &= \lim_{\Delta t \to 0} \frac{x^{(1)}(t+\Delta t) - x^{(1)}(t)}{\Delta t} \\
&\approx \frac{x^{(1)}(k+\Delta k) - x^{(1)}(k)}{\Delta k}|_{\Delta k=1},\ k = 1,2,\ldots,n \\
&= x^{(1)}(k+1) - x^{(1)}(k) \\
&= x^{(0)}(k+1).
\end{aligned} \tag{2.12}$$

The term $x^{(1)}(t)$ in continuous case is approximated by the mean of $x^{(1)}(k)$ and $x^{(1)}(k-1)$, $k = 2,3,\ldots,n$, that is

$$\begin{aligned}
x^{(1)}(t) &= \frac{x^{(1)}(k) + x^{(1)}(k-1)}{2} \\
&= z^{(1)}(k),\ k = 2,3,\ldots,n.
\end{aligned} \tag{2.13}$$

Replacing (2.12) and (2.13) into (2.10) yields the difference equation (2.11). □

2.4 Conclusion

This chapter reviewed some elements of fractional calculus, defined the Lotka-Volterra dynamical system and then presented the elements of grey modeling. The following part consist of applying successively grey modeling and fractional derivative to the Lotka-Volterra dynamical system for forecasting adoption of cryptocurrencies.

References

[1] Almeida, R., Tavares, D., and Torres, D. F. M. (2018). *The variable-order fractional calculus of variations*. SpringerBriefs in Applied Sciences and Technology, Springer, Cham.

[2] Angela, S. B. (2016). Ten types of digital currencies and how they work. *Online trading: Free introductory eBook, September 24, 2016*.

[3] Anisiu, M. C. (2014). Lotka-Volterra and their model. *Didactica Mathematica* **32**, *9-17*.

[4] Bhosale, J. and Mavale, S. (2018). Volatility of select crypto-currencies: A comparison of Bitcoin, Ethereum and Litecoin. *Annual Research Journal of SCMS, Pune* **6**.

[5] Blundell-Wignall, A. (2014). The Bitcoin question: Currency versus trust-less transfer technology. *OECD Working Papers on Finance, Insurance and Private Pensions, No 37, OECD Publishing*.

[6] Chan, S., Chu, J., Nadarajah, S., and Osterrieder, J. (2017). A statistical analysis of cryptocurrencies. *Journal of Risk and Financial Management* **2017**.

[7] Chaum, D. L. (1981). Untraceable electronic mail, return addresses, and digital pseudonyms. *Commun. ACM,* **24** *(2), 84-90*.

[8] Chiang, S. W. (2012). An application of Lotka-Volterra model to Taiwan's transition from 200 mm t0 330 mm silicon wafers. *Technological Forecasting & Social Change* **78**, *526-535*.

[9] Deng, J. (1989). Introduction grey system theory. *J. Grey Syst.* **1** *(1), 191-243*.

[10] Hsi-Tse Wang and Ta-Chu Wang (2016). Application of grey Lotka-Volterra model to forecast the diffusion and competition analysis of the TV and smart-phone industries. *Technological Forecasting Social Change* **106**, *37-44*.

[11] Kilbas, A. A., Srivastava, H. M., and Trujillo, J. J. (2006). *Theory and Applications of Fractional Differential Equations*. Elsevier Science.

[12] Koçak, H. and Yildirim, A. (2011). An efficient new iterative method for finding exact solutions of non-linear time-fractional partial differential equations. *Modelling and Control* **16** *(4), 403-414*.

[13] Mao, S. H., Gao, M. Y., Xiao, X. P., and Zhu, M. (2016). A novel fractional grey system model and its application. *Applied Mathematical Modelling,* **40**, *5063-5076*.

[14] Marasco, A., Picucci, A., and Romano, A. (2016). Market share dynamics using Lotka-Volterra models. *Technological Forecasting & Social Change* **105**, *49-62*.

[15] Morris, S. A. and Pratt, D. (2003). Analysis of the Lotka-Volterra competition equations as a technological substitution model. *Technological Forecasting Social Change,* **70**, *103-133*.

[16] Nai-Ming Xie and Si-Feng Liu (2009). Discrete grey forecasting model and its optimization. *Applied Mathematical Modelling* **33**, *1173-1186*.

[17] Satoshi, N. (2009). Bitcoin: A peer-to-peer electronic cash system.

[18] Strobeck, C. (1973). N species competition. *Ecology,* **54** *(3), 650-654*.

[19] Tien, T. L. (2009). A new grey prediction model FGM(1,1). *Mathematical and Computer Modelling* **49**, *1416-1426*.

[20] Urquhart, A. (2016). The inefficiency of Bitcoin. *Economics Letters,* **148**, *80-82*.

[21] Wayner, P. (1997). *Digital Cash*. AP Professional, London, 2^{nd} edition.

[22] Wu, L., Liu, S., and Wang, Y. (2012). Grey Lotka-Volterra model and its applications. *Technological Forecasting & Social Change* **79**, *1720-1730*.

[23] Xie, N. M., Liu, S. F., Yang, Y. J., and Yuan, C. Q. (2013). On a novel grey forecasting model based on no-homogeneous index sequence. *Applied Mathematical Modelling* **37** , *5059-5068*.

[24] Zhou, W. and Jian-Min He (2013). Generalised GM(1,1) model and its application in forecasting of fuel production. *Applied Mathematical Modelling* **37**, *6234-6243*.

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GREY LOTKA-VOLTERRA MODELS WITH APPLICATION TO CRYPTOCURRENCIES ADOPTION

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Abstract: The study uses Grey Lotka-Volterra Model (GLVM) of two and three dimensions for assessing the interaction between cryptocurrencies. The 2-dimensional study is on Bitcoin and Litecoin while the 3-dimensional study is on Bitcoin, Litecoin and Ripple. Records from 28-April-2013 to 10-February-2018 provide forecasting values for Bitcoin and Litecoin through 2-dimensional GLVM study, while records from 7-August-2013 to 10-February-2018 provide forecasting values of Bitcoin, Litecoin and Ripple through 3-dimensional GLVM study. The behavior of Bitcoin and Litecoin or both Bitcoin, Litecoin and Ripple in future is proposed by looking at the 100 last forecasting values of n-dimensional GLVM study, $n = \{2,3\}$. Lyapunov exponents of the 2 and 3-dimensional Lotka-Volterra models reveals that it is a chaotic dynamical system. Plots of 2 and 3-dimensional Lotka-Volterra models for filtered datasets suggest also a chaos. Using the Mean Absolute Percentage Error criterion, it was found that the accuracy of the GLVM is better than that of the grey model (GM(1,1)). By analysing the 2-dimensional GLVM, Bitcoin and Litecoin are found in the competition known as *mutualism* or equivalently a win-win situation where Bitcoin transaction is constant while Litecoin transaction has the increasing trend. The 3-dimensional GLVM analysis evokes however, an increasing trend in transacting both Bitcoin, Litecoin and Ripple where Bitcoin keep relatively higher transaction counts.

Keywords: Grey Lotka-Volterra Model, Mean Absolute Percentage Error, competition, interactions, continuous time model, differential equations, difference equations.

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1. Introduction

The grey model GM(1,1) was proposed by Deng [1, 2]. It has been applied in various studies on forecasting such as electricity costs, integrated circuit industry, wafer fabrication, opto-electronics industry output value and fatality risk estimation measure [3]. However, GM(1,1) can work only as a single variable forecasting model. It cannot analyse the long-term relationship between the two variables and predict the values of two variables in social system or economic system. To overcome the case of competition of several variables, Grey Lotka-Volterra Model (GLVM) was proposed by Wu et al. [3]. GLVM is one of the discrete time LVM which presents high modeling precision and wide adaptability. Czyzowicz et al. [4] suggest and prove that any discrete Lotka-Volterra model may converge to some absorbing state in time when any pair of agents is allowed to interact, and so is the GLVM.

The performance of GLVM has been observed for example in a study of testing the trade relationships between China and Singapore, Malaysia and Thailand, respectively, based on the data of import and export from 2003 to 2014 [5]; GLVM outperformed also in measuring the competition between TV and Smartphone industries [6] as compared to the GM(1,1). In spite of good performance of the GLVM, the high variability of the dataset may require an appropriate fractional differentiation rather than the total differentiation applied in GLVM. Further on fractional calculus and discrete fractional differentiation can be found in [7], [8], [9], [10], [11], [12] and [13].

The interest of this study is brought on the GLVM applied to three cryptocurrencies: Bitcoin, Litecoin and Ripple. As all other known cryptocurrencies, Bitcoin is also the online currency initiated in 2008 [14]. Bitcoin consists of direct trade which is not tracked by a third-party [15]. Bitcoin appears among the innovations that make transfers of digital currencies without the intervention of banks. Numerous advantages of Bitcoin include a discrete online transaction, third-party free transactions, non-taxable purchases, mobile payments and relatively low transaction fees [16]. Furthermore, Wayner [17] evokes that digital cash cannot have multiple copies. Hence, Bitcoin cannot be used more than once. Multiple transfers are therefore not easy for digital currencies. Following this, Bitcoin has been viewed as a more secure and reliable mode of payment in recent years. Urquhart [18] tested the efficiency of Bitcoin by using the dataset on the exchange of Bitcoin for six years. This analysis leaves the problem of a long-term adoption of Bitcoin. This problem will be addressed in this study using variants of Lotka-Volterra models.

This study assesses the type of interaction between competition of Bitcoin and Litecoin by considering the signs of interaction terms parameters. Various type of interactions include *pure competition* elaborated in [19], *predator-prey* elaborated in [20, 21, 22, 23, 24], *mutualism* found in [25, 26], *commensalism* found in [27] and neutralism elaborated in [26]. The calculation and interpretation of the Lyapunov exponents of the Lotka-Volterra model allows an easy understanding on the predictability of the model. The accuracy of the GLVM in this study is checked by the Mean Absolute Percentage Error (MAPE) criterion encountered in various related research such as [6, 28, 3, 29, 30].

The study is subdivided as follows: Section 2 presents the methodology of the study, that is a review on LV and GLV models. Section 3 presents the main results of the study with interpretation and Section 4 gives a conclusion.

2. Methodology

2.1. Concept of LV model

The LV system of differential equations also known as predator-prey model gives the competing relationships between the two species. Assuming that X and Y are two populations of competing species at time t, and given constant parameters a , b, c, p, q and r, the LV model is expressed by the following system [6].

$$\begin{cases} \frac{dX}{dt} = aX - bX^2 - cXY \\ \frac{dY}{dt} = pY - qY^2 - rYX. \end{cases} \tag{1}$$

The expressions X^2 and Y^2 are interactions within species, XY and YX are interactions of different species. Parameters a and p represents the capacity of growing of population X and Y respectively. Parameters b and q are the limiting parameter of decrease in size of populations, while parameters c and r represent the competition rate between the two species. The signs of parameters c and r reveals the type of relationship between species as indicates Table 1 suggested by Marasco et al. [31].

Table 1: Type of interaction between species according to the sign of parameters c and r.

Sign of c	Sign of r	Type of interaction
+	+	Pure competition
-	+	Predator-prey
-	-	Mutualism
-	0	Commensalism
+	0	Amensalism
0	0	Neutralism

Proposition 1. (Leslie, 1958)The continuous time model (1) can be converted to the following LV difference equations:

$$\begin{cases} X(t+1) = \frac{\alpha X(t)}{1+\beta X(t)+\gamma Y(t)},\ t = 1,2,\ldots,n-1 \\ Y(t+1) = \frac{\phi Y(t)}{1+\psi Y(t)+\omega X(t)},\ t = 1,2,\ldots,n-1, \end{cases} \tag{2}$$

where α, ϕ, β and ψ are the parameters of the individual signal species, while γ and ω indicate the magnitude of the effect that each species has on the rate of increase of the other, with relationships of parameters in Equation (1) and (2) given by:

$$a = \ln\alpha;\ b = \frac{\beta \ln\alpha}{\alpha - 1};\ c = \frac{\gamma \ln\alpha}{\alpha - 1}$$
$$p = \ln\phi;\ q = \frac{\psi \ln\phi}{\phi - 1};\ r = \frac{\omega \ln\phi}{\phi - 1}$$

The proof of Proposition 1 is found in [32].

2.2. Grey modeling

The grey forecast modeling also known as one order and one variable grey forecasting model (GM(1,1)) consists of predicting uncertain or incomplete information systems for determining the future dynamic situation of a certain sequence of numbers [29]. Given an original data sequence $X^{(0)} = \left(x^{(0)}(1), x^{(0)}(2), \ldots, x^{(0)}(n)\right)$ and the first order accumulation generating operation (1-AGO) given by:

$$X^{(1)} = \left(x^{(1)}(1), x^{(1)}(2), \ldots, x^{(1)}(n)\right),\ \text{with}\ x^{(1)}(k) = \sum_{i=1}^{k} x^{(0)}(i),\ k = 1,2,\ldots,n,$$

the mean sequence of $X^{(1)}$ given by

$$Z^{(1)} = \left(z^{(1)}(2), z^{(1)}(3), \ldots, z^{(1)}(n)\right),$$

with

$$z^{(1)}(k) = \frac{x^{(1)}(k) + x^{(1)}(k-1)}{2},\ k = 2,3,\ldots,n;$$

the GM(1,1) based on the series $X^{(1)} = \left(x^{(1)}(1), x^{(1)}(2), \ldots, x^{(1)}(n)\right)$ is defined by the following differential equation:

$$\frac{dx^{(1)}(t)}{dt} + ax^{(1)}(t) = b; \tag{3}$$

where parameters a and b of GM(1,1) are calculated by the least square method and the initial condition $X^{(1)}(1) = X^{(0)}(1)$ [33]. . Hsi-Tse Wang and Ta-Chu Wang [6] propose the following least square estimation:

$$\begin{pmatrix}\hat{a}\\ \hat{b}\end{pmatrix} = (B'B)^{-1}B'M$$

where,

$$B = \begin{pmatrix} -z^{(1)}(2) & 1 \\ -z^{(1)}(3) & 1 \\ \vdots & \vdots \\ -z^{(1)}(n) & 1 \end{pmatrix}; \quad M = \begin{pmatrix} x^{(0)}(2) \\ x^{(0)}(3) \\ \vdots \\ x^{(0)}(n) \end{pmatrix}.$$

Proposition 2. The difference equation corresponding to Equation (3) can be written as:

$$x^{(0)}(k+1) + az^{(1)}(k) = b. \tag{4}$$

Proof. The term $\frac{dx^{(1)}(t)}{dt}$ can be written as

$$\begin{aligned} \frac{dx^{(1)}(t)}{dt} &= \lim_{\Delta t \to 0} \frac{x^{(1)}(t+\Delta t) - x^{(1)}(t)}{\Delta t} \\ &\approx \frac{x^{(1)}(k+\Delta k) - x^{(1)}(k)}{\Delta k}\Big|_{\Delta k=1},\ k = 1,2,\ldots,n \\ &= x^{(1)}(k+1) - x^{(1)}(k) \\ &= x^{(0)}(k+1). \end{aligned} \tag{5}$$

The term $x^{(1)}(t)$ in continuous case is approximated by the mean of $x^{(1)}(k)$ and $x^{(1)}(k-1)$, $k = 2,3,\ldots,n$, that is

$$\begin{aligned} x^{(1)}(t) &= \frac{x^{(1)}(k) + x^{(1)}(k-1)}{2} \\ &= z^{(1)}(k),\ k = 2,3,\ldots,n. \end{aligned} \tag{6}$$

Replacing (5) and (6) in (3) yields the difference equation (4). □

Further concepts on difference equations can be found for example in [34].

2.3. Grey LV equations

Assume two sets of original series $X^{(0)}$ and $Y^{(0)}$

$$\begin{aligned} X^{(0)} &= \left(x^{(0)}(1), x^{(0)}(2), \ldots, x^{(0)}(n)\right) \\ Y^{(0)} &= \left(y^{(0)}(1), y^{(0)}(2), \ldots, y^{(0)}(n)\right). \end{aligned}$$

The 1-AGO of $X^{(0)}$ and $Y^{(0)}$ are given by:

$$X^{(1)} = \left(x^{(1)}(1), x^{(1)}(2), \ldots, x^{(1)}(n)\right)$$

and

$$Y^{(1)} = \left(y^{(1)}(1), y^{(1)}(2), \ldots, y^{(1)}(n)\right)$$

with

$$x^{(1)}(k) = \sum_{i=1}^{k} x^{(0)}(i),\ k = 1, 2, \ldots, n$$

and

$$y^{(1)}(k) = \sum_{i=1}^{k} y^{(0)}(i),\ k = 1, 2, \ldots, n.$$

Applying the grey model to the system (1) yields the following approximations:

$$\begin{cases} x^{(0)}(k+1) \approx a z_x^{(1)}(k) - b\left(z_x^{(1)}(k)\right)^2 - c z_x^{(1)}(k) z_y^{(1)}(k) \\ y^{(0)}(k+1) \approx p z_y^{(1)}(k) - q\left(z_y^{(1)}(k)\right)^2 - r z_y^{(1)}(k) z_x^{(1)}(k) \end{cases} \tag{7}$$

with error sequences expressed by

$$\begin{cases} \varepsilon_{xk} = x^{(0)}(k+1) - \left(a z_x^{(1)}(k) - b\left(z_x^{(1)}(k)\right)^2 - c z_x^{(1)}(k) z_y^{(1)}(k)\right) \\ \varepsilon_{yk} = y^{(0)}(k+1) - \left(p z_y^{(1)}(k) - q\left(z_y^{(1)}(k)\right)^2 - r z_y^{(1)}(k) z_x^{(1)}(k)\right). \end{cases} \tag{8}$$

The least square estimates of parameters in (7) are found as follows:

$$\begin{pmatrix} \hat{a} \\ \hat{b} \\ \hat{c} \end{pmatrix} = (B_x' B_x)^{-1} B_x' M_x, \quad \begin{pmatrix} \hat{p} \\ \hat{q} \\ \hat{r} \end{pmatrix} = (B_y' B_y)^{-1} B_y' M_y \tag{9}$$

where,

$$B_x = \begin{pmatrix} z_x^{(1)}(2) & -\left(z_x^{(1)}(2)\right)^2 & -z_x^{(1)}(2) z_y^{(1)}(2) \\ z_x^{(1)}(3) & -\left(z_x^{(1)}(3)\right)^2 & -z_x^{(1)}(3) z_y^{(1)}(3) \\ \vdots & \vdots & \vdots \\ z_x^{(1)}(n) & -\left(z_x^{(1)}(n)\right)^2 & -z_x^{(1)}(n) z_y^{(1)}(n) \end{pmatrix}; \quad M_x = \begin{pmatrix} x^{(0)}(2) \\ x^{(0)}(3) \\ \vdots \\ x^{(0)}(n) \end{pmatrix}.$$

$$B_y = \begin{pmatrix} z_y^{(1)}(2) & -\left(z_y^{(1)}(2)\right)^2 & -z_y^{(1)}(2)z_x^{(1)}(2) \\ z_y^{(1)}(3) & -\left(z_y^{(1)}(3)\right)^2 & -z_y^{(1)}(3)z_x^{(1)}(3) \\ \vdots & \vdots & \vdots \\ z_y^{(1)}(n) & -\left(z_y^{(1)}(n)\right)^2 & -z_y^{(1)}(n)z_x^{(1)}(n) \end{pmatrix}; \quad M_y = \begin{pmatrix} y^{(0)}(2) \\ y^{(0)}(3) \\ \vdots \\ y^{(0)}(n) \end{pmatrix}.$$

Note that the usual LV estimation based on datasets $X^{(0)}$ and $Y^{(0)}$ follows from the LV difference equations (2), that is:

$$\begin{cases} \hat{x}^{(1)}(k+1) = \frac{\hat{\alpha} x^{(1)}(k)}{1+\hat{\beta} x^{(1)}(k)+\hat{\gamma} y^{(1)}(k)}, \\ \hat{y}^{(1)}(k+1) = \frac{\hat{\phi} y^{(1)}(k)}{1+\hat{\psi} y^{(1)}(k)+\hat{\omega} x^{(1)}(k)}. \end{cases} \tag{10}$$

2.4. Extension to 3-dimensional system

Strobeck [35] propose the LV model for n competing species as

$$\frac{dX_i}{dt} = X_i\left(a_i - \sum_{j=1}^{n} \alpha_{ij} X_j\right). \tag{11}$$

Parameters a_i represent the capacity of growing of populations X_i, while parameters α_{ij} represent the effect species j has on species i. Assuming three competitive species X, Y and W, the system (11) yields the 3-dimensional system as

$$\begin{cases} \frac{dX}{dt} = a_1 X - b_1 X^2 - c_1 XY - d_1 XW \\ \frac{dY}{dt} = a_2 Y - b_2 Y^2 - c_2 YX - d_2 YW \\ \frac{dW}{dt} = a_3 W - b_3 W^2 - c_3 WX - d_3 WY. \end{cases} \tag{12}$$

Applying the grey model to the system (12) yields the following GLVM:

$$\begin{cases} x^{(0)}(k+1) \approx & a_1 z_x^{(1)}(k) - b_1\left(z_x^{(1)}(k)\right)^2 - c_1 z_x^{(1)}(k) z_y^{(1)}(k) \\ & -d_1 z_x^{(1)}(k) z_w^{(1)}(k) \\ y^{(0)}(k+1) \approx & a_2 z_y^{(1)}(k) - b_2\left(z_y^{(1)}(k)\right)^2 - c_2 z_y^{(1)}(k) z_x^{(1)}(k) \\ & -d_2 z_y^{(1)}(k) z_w^{(1)}(k) \\ w^{(0)}(k+1) \approx & a_3 z_w^{(1)}(k) - b_3\left(z_w^{(1)}(k)\right)^2 - c_3 z_w^{(1)}(k) z_x^{(1)}(k) \\ & -d_3 z_w^{(1)}(k) z_y^{(1)}(k) \end{cases} \tag{13}$$

with error sequences expressed by

$$\begin{cases} \varepsilon_{xk} = x^{(0)}(k+1) - \Big(a_1 z_x^{(1)}(k) - b_1\left(z_x^{(1)}(k)\right)^2 - c_1 z_x^{(1)}(k) z_y^{(1)}(k) \\ \qquad -d_1 z_x^{(1)}(k) z_w^{(1)}(k)\Big) \\ \varepsilon_{yk} = y^{(0)}(k+1) - \Big(a_2 z_y^{(1)}(k) - b_2\left(z_y^{(1)}(k)\right)^2 - c_2 z_y^{(1)}(k) z_x^{(1)}(k) \\ \qquad -d_2 z_y^{(1)}(k) z_w^{(1)}(k)\Big) \\ \varepsilon_{wk} = z^{(0)}(k+1) - \Big(a_3 z_w^{(1)}(k) - b_3\left(z_w^{(1)}(k)\right)^2 - c_3 z_w^{(1)}(k) z_x^{(1)}(k) \\ \qquad -d_3 z_w^{(1)}(k) z_y^{(1)}(k)\Big). \end{cases} \tag{14}$$

The least square estimates of parameters in (13) are found as follows:

$$\begin{pmatrix}\hat{a}_1\\ \hat{b}_1\\ \hat{c}_1\\ \hat{d}_1\end{pmatrix} = (B'_xB_x)^{-1}B'_xM_x,\ \begin{pmatrix}\hat{a}_2\\ \hat{b}_2\\ \hat{c}_2\\ \hat{d}_2\end{pmatrix} = (B'_yB_y)^{-1}B'_yM_y,\ \begin{pmatrix}\hat{a}_3\\ \hat{b}_3\\ \hat{c}_3\\ \hat{d}_3\end{pmatrix} = (B'_wB_w)^{-1}B'_wM_w \tag{15}$$

where,

$$B_x = \begin{pmatrix} z_x^{(1)}(2) & -\left(z_x^{(1)}(2)\right)^2 & -z_x^{(1)}(2)z_y^{(1)}(2) & -z_x^{(1)}(2)z_w^{(1)}(2) \\ z_x^{(1)}(3) & -\left(z_x^{(1)}(3)\right)^2 & -z_x^{(1)}(3)z_y^{(1)}(3) & -z_x^{(1)}(3)z_w^{(1)}(3) \\ \vdots & \vdots & \vdots & \vdots \\ z_x^{(1)}(n) & -\left(z_x^{(1)}(n)\right)^2 & -z_x^{(1)}(n)z_y^{(1)}(n) & -z_x^{(1)}(n)z_w^{(1)}(n) \end{pmatrix};\ M_x = \begin{pmatrix} x^{(0)}(2) \\ x^{(0)}(3) \\ \vdots \\ x^{(0)}(n) \end{pmatrix}.$$

$$B_y = \begin{pmatrix} z_y^{(1)}(2) & -\left(z_y^{(1)}(2)\right)^2 & -z_y^{(1)}(2)z_x^{(1)}(2) & -z_y^{(1)}(2)z_w^{(1)}(2) \\ z_y^{(1)}(3) & -\left(z_y^{(1)}(3)\right)^2 & -z_y^{(1)}(3)z_x^{(1)}(3) & -z_y^{(1)}(3)z_w^{(1)}(3) \\ \vdots & \vdots & \vdots & \vdots \\ z_y^{(1)}(n) & -\left(z_y^{(1)}(n)\right)^2 & -z_y^{(1)}(n)z_x^{(1)}(n) & -z_y^{(1)}(n)z_w^{(1)}(n) \end{pmatrix};\ M_y = \begin{pmatrix} y^{(0)}(2) \\ y^{(0)}(3) \\ \vdots \\ y^{(0)}(n) \end{pmatrix}.$$

$$B_w = \begin{pmatrix} z_w^{(1)}(2) & -\left(z_w^{(1)}(2)\right)^2 & -z_w^{(1)}(2)z_x^{(1)}(2) & -z_w^{(1)}(2)z_y^{(1)}(2) \\ z_w^{(1)}(3) & -\left(z_w^{(1)}(3)\right)^2 & -z_w^{(1)}(3)z_x^{(1)}(3) & -z_w^{(1)}(3)z_y^{(1)}(3) \\ \vdots & \vdots & \vdots & \vdots \\ z_w^{(1)}(n) & -\left(z_w^{(1)}(n)\right)^2 & -z_w^{(1)}(n)z_x^{(1)}(n) & -z_w^{(1)}(n)z_y^{(1)}(n) \end{pmatrix};\ M_w = \begin{pmatrix} w^{(0)}(2) \\ w^{(0)}(3) \\ \vdots \\ w^{(0)}(n) \end{pmatrix}.$$

2.5. Adequacy checking of the Grey Lotka-Volterra Model

The present study uses the Mean Absolute Percentage Error (MAPE) for checking the accuracy of the model. Assuming that Y_i and $\widehat{Y_i}$ are the actual and predicted values respectively with $i = 1, 2, \ldots, n$,

$$\text{MAPE} = \frac{1}{n}\sum_{i=1}^{n}\left|\frac{Y_i - \widehat{Y_i}}{Y_i}\right| \times 100.$$

Table 2 gives the prediction capability levels of MAPE.

Table 2: Prediction capability levels of MAPE

MAPE	Prediction capability
Less than 10	High accuracy
10 and less than 20	Good accuracy
20 and less than 50	Reasonable accuracy
Above 50	Lack of accuracy

2.6. Predictability of the model

This study uses Lyapunov exponents for checking predictability of the model. Lyapunov exponent of a dynamical system is obtained by assuming two close trajectories $X(t)$ and $X_0(t)$ of a dynamical system. The separation of these trajectories is given by

$$\delta\mathbf{X}(t) = \mathbf{X}(t) - \mathbf{X}_0(t);\ \delta\mathbf{X}_0 = \mathbf{X}(0) - \mathbf{X}_0(0)$$

Lyapunov exponent is a quantity λ that satisfy the condition:

$$|\delta\mathbf{X}(t)| \approx e^{\lambda t}|\delta\mathbf{X}_0|.$$

If the trajectory $\mathbf{X}(t)$ is given by a n-dimensional linear dynamical system with constant coefficients, that is

$$\dot{\mathbf{X}} = \mathbf{M}\mathbf{X} + \mathbf{f}(t). \tag{16}$$

with $n \times n$ matrix $\mathbf{A}$, then the n real parts of the different eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_n$ are the Lyapunov exponents of the dynamical system (16).
The maximum Liapunov exponent is given by:

$$\lambda_{max} = \lim_{t\to\infty}\lim_{\delta\mathbf{X}_0\to 0}\frac{1}{t}\frac{|\delta\mathbf{X}(t)|}{|\delta X_0|} \tag{17}$$

More generaly, if dynamical system is a nonlinear system, Lyapunov exponents are approximated by that of the corresponding linearized dynamical system. The method of linearizing a nonlinear equation consists of using a Taylor series of nonlinear integrand around an equilibrium point [36]. The linear form of model (11) can now be written as

$$\frac{d\mathbf{X}}{dt} = \mathbf{J}\,(X - \mathbf{X}_0)$$

where $\mathbf{X} = (X_1, X_2, \ldots, X_n)'$ and $\mathbf{J}$ is the Jacobian matrix at the equilibrium point $\mathbf{X}_0 = (X_{01}, X_{02}, \ldots, X_{0n})$.

The equilibrium points are found by equating the integrand to zero. For models (1), the possible equilibrium points are

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} b & c \\ r & q \end{pmatrix}^{-1} \begin{pmatrix} a \\ p \end{pmatrix},$$

while for model (12), the possible equilibrium points are

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} b_1 & c_1 & d_1 \\ c_2 & b_2 & d_2 \\ c_3 & d_3 & b_3 \end{pmatrix}^{-1} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}.$$

Lyapunov [37] shows that if the dynamical system of the first approximation is regular with the negative maximal Lyapunov exponent, then the solution of the original system is asymptotically stable, while a strange attractor is generated by a chaotic dynamical system if at least one exponent is positive.

2.7. Dataset

Two datasets on daily transaction counts of Bitcoin and Litecoin from 28-April-2013 up to 10-February-2018 are of interest for 2-dimensional analysis, while three datasets on daily transaction counts of Bitcoin, Litecoin and Ripple from 7-August-2013 up to 10-February-2018 are of interest for 3-dimensional analysis. Tables 3 and Table 4 give respectively the portion of evolution in transaction counts of Bitcoin and Litecoin along the study time, and the portion of evolution in transaction counts of Bitcoin, Litecoin and Ripple with daily records.

Table 3: Transaction counts of Bitcoin and Litecoin from 28-April-2013 up to 10-February-2018

Date	Bitcoin	Litecoin
28-Apr-13	40035	9408
29-Apr-13	52266	9092
30-Apr-13	46802	9205
1-May-13	52443	8927
2-May-13	55169	8290
⋮	⋮	⋮
6-Feb-2018	243950	59946
7-Feb-2018	213578	50320
8-Feb-2018	173158	37148
9-Feb-2018	177725	44811
10-Feb-2018	181640	46594

Table 4: Transaction counts of Bitcoin, Litecoin and Ripple from 7-August-2013 to 10-February-2018

Date	Bitcoin	Litecoin	Ripple
7-Aug-13	56974	4385	3335
8-Aug-13	56992	3932	3477
9-Aug-13	52486	3649	2219
10-Aug-13	52316	3924	1887
11-Aug-13	47995	3585	2207
⋮	⋮	⋮	⋮
6-Feb-2018	243950	59946	37098
7-Feb-2018	213578	50320	27775
8-Feb-2018	173158	37148	16700
9-Feb-2018	177725	44811	30748
10-Feb-2018	181640	46594	36859

The entire transaction counts of Bitcoin, Litecoin and Ripple are plotted in Figure 1, Figure 2 and Figure 3.

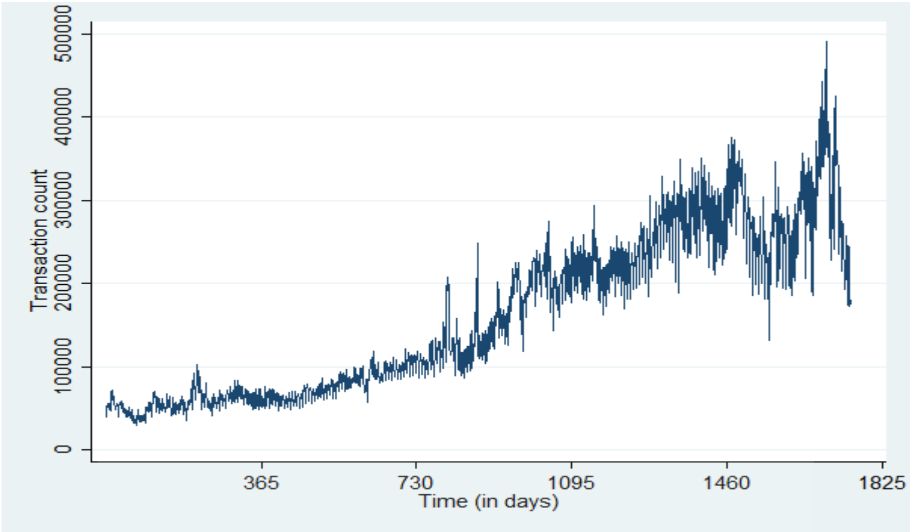


Figure 1: Transaction counts of Bitcoin from 28 April 2013 to 10 February 2018.

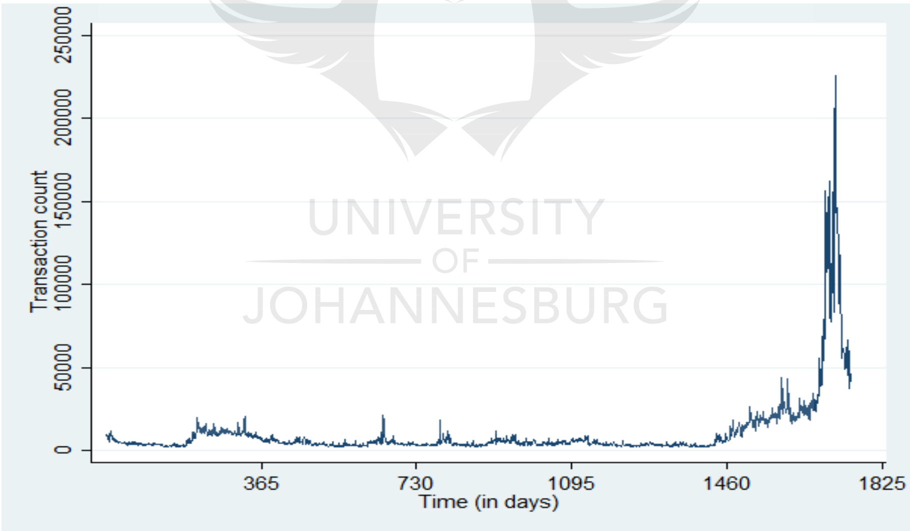


Figure 2: Transaction counts of Litecoin from 28-April-2013 to 10-February-2018.

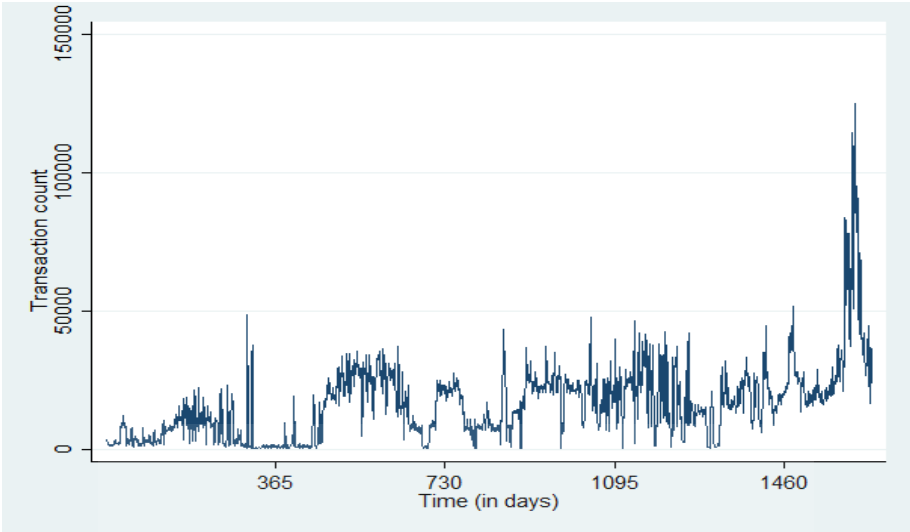


Figure 3: Transaction counts of Ripple from 7-August-2013 to 10-February-2018

From 28-April-2013, transaction counts of Bitcoin increased linearly along the subsequent 4 years, fluctuate with abrupt increase in the second half of the 5th year, and then starts to decrease slightly as shows Figure 1. Figure 2 shows that transaction counts of Litecoin are constant along the first 4 years, increase slightly up to the mid-5th year, make an abrupt increase in the second half of the 5th year and then start to decrease. Ripple transaction counts presented Figure 3 are subject of relatively high fluctuation along the study time, with an abrupt jump at the end of January 2018. Ripple transaction counts keep values less than that of Bitcoin and Litecoin of the same period.

3. Results and interpretation

3.1. 2-dimensional Grey Lotka-Volterra model for Bitcoin and Litecoin

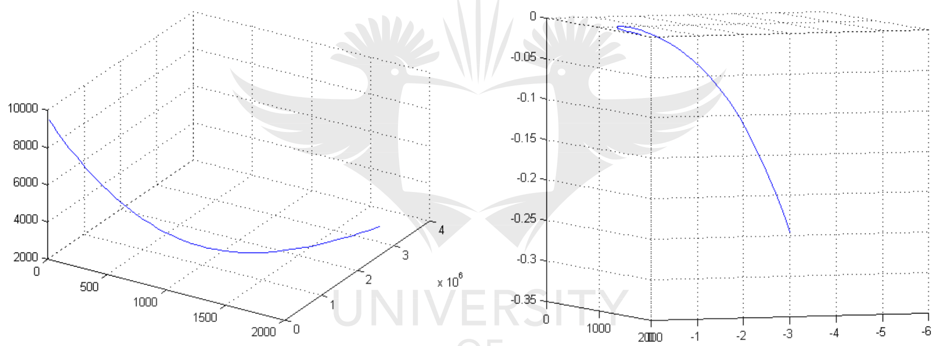
We apply the model (7) to the dataset summarised in Table 3. Equations (9) give the least square estimates of model parameters as

$$\begin{pmatrix} \hat{a} \\ \hat{b} \\ \hat{c} \end{pmatrix} = \begin{pmatrix} 2.548 \times 10^{-3} \\ 5.805 \times 10^{-12} \\ -4.198 \times 10^{-12} \end{pmatrix}, \quad \begin{pmatrix} \hat{p} \\ \hat{q} \\ \hat{r} \end{pmatrix} = \begin{pmatrix} -8.759 \times 10^{-4} \\ -3.460 \times 10^{-10} \\ -3.863 \times 10^{-13} \end{pmatrix}.$$

The Grey Lotka-Volterra model (7) can then be written as:

$$\begin{cases} x^{(0)}(k+1) \approx 2.548 \times 10^{-3} z_x^{(1)}(k) - 5.805 \times 10^{-12} \left(z_x^{(1)}(k) \right)^2 \\ \qquad +4.198 \times 10^{-12} z_x^{(1)}(k) z_y^{(1)}(k) \\ y^{(0)}(k+1) \approx -8.759 \times 10^{-4} z_y^{(1)}(k) + 3.460 \times 10^{-10} \left(z_y^{(1)}(k) \right)^2 \\ \qquad +3.863 \times 10^{-13} z_y^{(1)}(k) z_x^{(1)}(k) \end{cases} \tag{18}$$

$k = 1, 2, \ldots, n.$
The 2-dimensional Lyapunov exponents for two different equilibrium points present at least one positive exponent $\left(\boldsymbol{\lambda}_1 = \{2.548 \times 10^{-3};\ -8.759 \times 10^{-4}\}\right.$ and $\boldsymbol{\lambda}_2 = \{-4.209 \times 10^{-1};\ 2.429 \times 10^{-15}\}\big)$, this suggest that the system (1) is a chaotic dynamical system. The 2-dimensional LVM plot does not shows chaos (Figure 4 (a)) but the plot after filtration suggests chaos as shows Figure 4 (b).



(a) Original dataset (b) Dataset after filtration

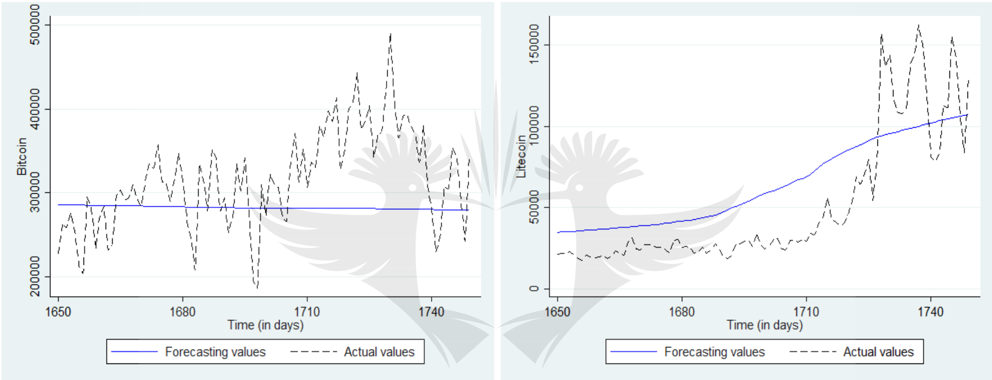
Figure 4: 2-dimensional LVM plots.

Under the mean absolute percentage error criterion, the GLVM (18) is reasonably accurate for the overall values of Bitcoin (MAPE=22) and reasonably accurate for the last 300 values of Litecoin (MAPE=35). The GLVM shows the better accuracy as compared to the GM(1,1) for which the MAPE is relatively greater, that is MAPE=49 for the overall values of Bitcoin and MAPE=44 for the last 300 values of Litecoin. In model (18), both estimates of parameters c and r of interactions are negative. Bitcoin and Litecoin are then in the mutualism system , or equivalently, there is a win-win situation.

Table 5 gives the last 100 GLVM forecasting values of Bitcoin (MAPE=20) and Litecoin (MAPE=37) and the last 100 GM(1,1) forecasting values of Bitcoin (MAPE=79) and Litecoin (MAPE=47). Clearly, the GLVM keeps the higher accuracy as compared to the

GM(1,1). The 100 last GLVM forecasting values show that in future, there will be a linear slight decrease in adopting Bitcoin and a slight linear increase for Litecoin adoption.

Figure 5 represents the 100 GLVM forecasting values of Bitcoin and Litecoin along the last 4 months of the study period. Figure 5 (a) shows that the actual values of Bitcoin fluctuate around the forecasting values, the same situation is observed for Litecoin especially along the last month of the study time (Figure 5 (b)). The graph in Figure 5 (a) is approximately constant along the last four months of the study time while in Figure 5 (b), the graph increase slightly along the first and last month and present an abrupt increase in the second and third month. This suggest a future increase in adopting Litecoin as found in Table 5.



(a) 100 last GLVM Bitcoin forecasting values (b) 100 last GLVM Litecoin forecasting values

Figure 5: 100 last GLVM forecasting values of Bitcoin and Litecoin.

Table 5: Last 100 GLVM and GM(1,1) forecasting values of Bitcoin and Litecoin transactions.

	Actual values		GLVM Values		GM(1,1) values			Actual values		GLVM Values		GM(1,1) values	
N0	BTC	LTC	BTC	LTC	BTC	LTC	N0	BTC	LTC	BTC	LTC	BTC	LTC
1	277479	24524	285705	34715	59982	32559	51	308072	117738	281883	58816	60015	43585
2	293991	19249	285637	34869	60000	32639	52	279371	81111	281846	59721	59984	43951
3	251587	18545	285568	35003	59955	32709	53	228791	77925	281806	60450	59931	44243
4	270896	20785	285504	35142	59976	32781	54	247298	82613	281775	61191	59951	44539
5	335480	27637	285429	35313	60044	32870	55	307486	112765	281742	62098	60014	44898
6	301202	27657	285352	35509	60007	32972	56	304904	111207	281710	63145	60011	45310
7	341128	29077	285274	35711	60050	33076	57	353659	155481	281689	64404	60063	45801
8	271625	29698	285201	35921	59976	33184	58	344260	141900	281675	65822	60053	46348
9	194554	25978	285151	36119	59895	33287	59	290259	105948	281648	67016	59996	46804
10	185886	34159	285117	36334	59886	33397	60	241601	83076	281614	67934	59945	47152
11	309159	26605	285063	36553	60016	33509	61	340809	127924	281579	68965	60049	47540
12	271867	24375	284988	36736	59977	33603	62	395806	186764	281560	70518	60107	48119
13	321636	26290	284910	36920	60029	33696	63	424840	225860	281573	72580	60138	48878
14	310244	30400	284827	37125	60017	33801	64	342564	197217	281606	74724	60051	49657
15	306450	31024	284748	37348	60013	33914	65	358679	173712	281626	76629	60068	50339
16	270738	24365	284673	37550	59975	34015	66	368025	143412	281607	78278	60078	50922
17	264695	23815	284600	37726	59969	34104	67	345506	146511	281575	79800	60054	51456
18	336029	29536	284517	37922	60044	34202	68	360101	145848	281546	81350	60070	51994
19	370918	29760	284417	38140	60081	34311	69	347227	140304	281512	82881	60056	52520
20	311885	28407	284318	38354	60019	34418	70	337766	120843	281469	84292	60046	53001
21	352050	30623	284223	38572	60061	34527	71	299913	106887	281418	85531	60006	53420
22	305586	28816	284128	38792	60012	34636	72	265586	93443	281372	86629	59970	53788
23	336533	33856	284037	39025	60045	34752	73	234890	88779	281334	87633	59938	54124
24	329524	33004	283942	39274	60037	34875	74	273473	90381	281291	88627	59978	54453
25	379086	40417	283841	39548	60090	35010	75	303566	117447	281244	89786	60010	54836
26	365821	43480	283737	39862	60076	35164	76	315604	113111	281197	91081	60023	55260
27	397917	55781	283636	40236	60109	35347	77	309322	95276	281134	92259	60016	55643
28	384219	42914	283529	40608	60095	35528	78	243454	70009	281068	93199	59947	55947
29	412725	40679	283410	40926	60125	35682	79	240433	66798	281004	93981	59943	56199
30	326193	38855	283298	41229	60034	35828	80	215435	55466	280940	94683	59917	56424
31	352868	39697	283198	41529	60062	35973	81	245395	61730	280870	95358	59949	56640
32	400505	45698	283084	41857	60112	36130	82	271759	59717	280786	96060	59976	56863
33	405531	53733	282965	42240	60117	36313	83	250247	59072	280698	96749	59954	57082
34	443399	68780	282847	42714	60157	36538	84	236422	61836	280622	97453	59939	57304
35	374765	64009	282740	43231	60085	36783	85	220304	57452	280553	98150	59922	57524
36	384936	70853	282645	43758	60096	37031	86	193421	49382	280489	98777	59894	57720
37	403225	79163	282551	44349	60115	37307	87	213288	51278	280424	99369	59915	57905
38	341256	53943	282456	44876	60050	37552	88	232028	50067	280346	99967	59935	58092
39	368427	79265	282368	45407	60078	37797	89	236442	55270	280264	100590	59939	58286
40	372821	156717	282328	46353	60083	38231	90	204159	54531	280192	101242	59905	58488
41	424393	136446	282305	47542	60137	38770	91	257504	57962	280114	101912	59961	58695
42	490459	143609	282246	48692	60207	39286	92	235750	66669	280034	102656	59938	58924
43	405507	116514	282179	49773	60117	39764	93	194733	49384	279967	103352	59895	59137
44	364051	108366	282120	50716	60074	40178	94	173509	45225	279907	103921	59873	59311
45	391725	107871	282059	51632	60103	40576	95	216178	51043	279841	104502	59918	59489
46	394057	110604	281990	52565	60105	40978	96	243950	59946	279761	105173	59947	59693
47	378482	138052	281940	53637	60089	41435	97	213578	50320	279681	105842	59915	59896
48	370141	143881	281913	54866	60080	41954	98	173158	37148	279609	106375	59872	60057
49	335350	162372	281910	56215	60043	42518	99	177725	44811	279546	106875	59877	60207
50	380493	149956	281906	57608	60091	43092	100	181640	46594	279485	107434	59881	60376

3.2. 3-dimensional Grey Lotka-Volterra model for Bitcoin, Litecoin and Ripple

Applying model (13) to the dataset summarised in Table 4, Equations (15) give the least square estimates of model parameters as

$$\begin{pmatrix}\hat{a}_1\\ \hat{b}_1\\ \hat{c}_1\\ \hat{d}_1\end{pmatrix}=\begin{pmatrix}2.709\times10^{-3}\\ 6.072\times10^{-12}\\ 2.323\times10^{-12}\\ -5.632\times10^{-9}\end{pmatrix},\quad \begin{pmatrix}\hat{a}_2\\ \hat{b}_2\\ \hat{c}_2\\ \hat{d}_2\end{pmatrix}=\begin{pmatrix}-1.104\times10^{-3}\\ -2.553\times10^{-12}\\ -5.543\times10^{-12}\\ -9.947\times10^{-8}\end{pmatrix},$$

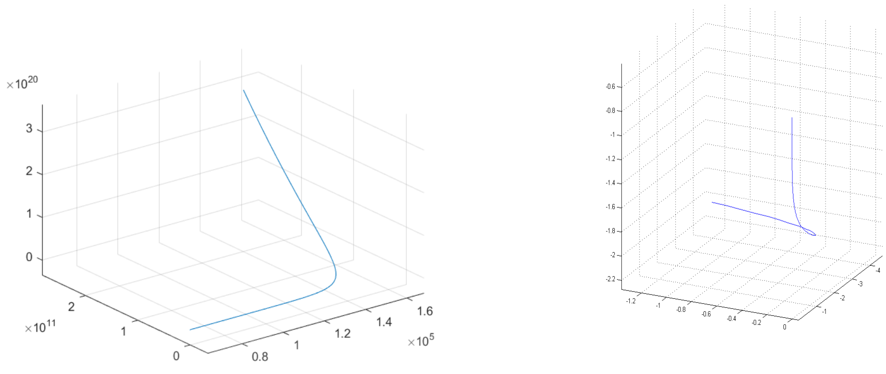
$$\begin{pmatrix}\hat{a}_3\\ \hat{b}_3\\ \hat{c}_3\\ \hat{d}_3\end{pmatrix}=\begin{pmatrix}1.002\\ -1.740\times10^{-7}\\ -1.354\times10^{-10}\\ 3.527\times10^{-9}\end{pmatrix}.$$

The Grey Lotka-Volterra model (7) can then be written as:

$$\begin{cases} x^{(0)}(k+1)\approx 2.709\times10^{-3}z_x^{(1)}(k)-6.072\times10^{-12}\left(z_x^{(1)}(k)\right)^2\\ \qquad -2.323\times10^{-12}z_x^{(1)}(k)z_y^{(1)}(k)+5.632\times10^{-9}z_x^{(1)}(k)z_w^{(1)}(k)\\ y^{(0)}(k+1)\approx -1.104\times10^{-3}z_y^{(1)}(k)+2.553\times10^{-12}\left(z_y^{(1)}(k)\right)^2\\ \qquad +5.543\times10^{-12}z_y^{(1)}(k)z_x^{(1)}(k)+9.947\times10^{-8}z_y^{(1)}(k)z_w^{(1)}(k)\\ w^{(0)}(k+1)\approx 1.002z_w^{(1)}(k)+1.740\times10^{-7}\left(z_w^{(1)}(k)\right)^2\\ \qquad +1.354\times10^{-10}z_w^{(1)}(k)z_x^{(1)}(k)-3.527\times10^{-9}z_w^{(1)}(k)z_y^{(1)}(k)\end{cases} \tag{19}$$

$k=1,2,\ldots,n.$

The Lyapunov exponents for two different equilibrium points present at least one positive exponent $\left(\boldsymbol{\lambda}_1=\{1.002;\ 0.002709;\ -0.001104\}\text{ and }\boldsymbol{\lambda}_2=\{-2.166\times10^{-1};\ 1.183\times10^{-1};\ -4.638\times10^{-4}\}\right)$. As for the 2-dimensional model, this suggest that the system (1) is a chaotic dynamical system. The 3-dimensional LVM plot does not shows chaos (Figure 6 (a)). However, the plot after filtration suggests a chaos (Figure 6 (b)).



(a) Original dataset

(b) Dataset after filtration

Figure 6: 3-dimensional LVM plots.

Under the MAPE criterion, model GLVM (19) is reasonably accurate for the overall values of Bitcoin (MAPE=24) and Ripple (MAPE=25) and reasonably accurate for the last 300 values of Litecoin where (MAPE=27). The GM(1,1) suggests no accuracy for both Bitcoin (MAPE=65), Ripple (MAPE=72) and the last 300 values of Litecoin (MAPE=60). Table 6 gives the last 100 GLVM forecasting values of Bitcoin (MAPE=24), Litecoin (MAPE=19) and Ripple (MAPE=9) and the last 100 GM(1,1) forecasting values of Bitcoin (MAPE=61), Litecoin (MAPE=37) and Ripple (MAPE=47) and still the GLVM is much more accurate that the GM(1,1) by the MAPE criterion. The 100 last GLVM forecasting values show that in future, there will be a trend of increase in transacting both Bitcoin, Litecoin and Ripple; with Bitcoin keeping relatively higher transaction counts.

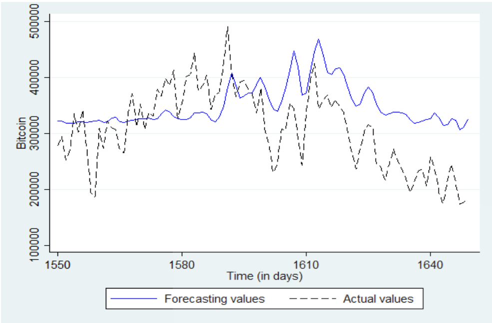
Table 6: Last 100 GLVM and GM(1,1) forecasting values of Bitcoin, Litecoin and Ripple transactions

	Actual values			GLVM values			GM(1,1) values		
N0	BTC	LTC	RPL	BTC	LTC	RPL	BTC	LTC	RPL
1	277479	24524	22368	322453	26156	22537	973094	33855	28546
2	293991	19249	18280	322066	24798	21105	974145	33943	28565
3	251587	18545	17602	318169	22246	18624	975149	34020	28581
4	270896	20785	19099	318125	22722	19051	976110	34099	28598
5	335480	27637	18915	317845	23549	19735	977226	34197	28616
6	301202	27657	21174	320781	24670	20817	978398	34309	28635
7	341128	29077	20553	321333	25689	21671	979579	34424	28654
8	271625	29698	20273	319533	25317	21203	980707	34543	28673
9	194554	25978	19687	321048	24844	20753	981564	34656	28692
10	185886	34159	23231	322502	26560	22295	982264	34778	28712
11	309159	26605	20843	322716	27281	22899	983175	34901	28733
12	271867	24375	19507	319495	25319	20958	984244	35004	28752
13	321636	26290	21017	321135	25547	21050	985336	35106	28771
14	310244	30400	29874	327826	31241	26458	986499	35221	28794
15	306450	31024	23694	328504	32857	27857	987633	35345	28819
16	270738	24365	19622	320698	27370	22508	988695	35458	28840
17	264695	23815	20202	319714	25453	20687	989680	35555	28858
18	336029	29536	24025	322238	27968	22984	990786	35663	28879
19	370918	29760	23637	323455	30038	24777	992086	35783	28901
20	311885	28407	24401	325264	30360	24975	993343	35901	28923
21	352050	30623	25098	325704	31281	25739	994564	36020	28947
22	305586	28816	26788	326213	32696	26987	995774	36141	28971
23	336533	33856	23877	326839	32123	26350	996956	36267	28994
24	329524	33004	24769	324857	31050	25297	998181	36403	29017
25	379086	40417	27720	326356	33410	27305	999485	36551	29042
26	365821	43480	35197	335318	39390	32759	1000855	36721	29071
27	397917	55781	37889	341435	45417	38089	1002261	36922	29105
28	384219	42914	28598	338232	41840	34632	1003700	37122	29136
29	412725	40679	28029	329988	36337	29474	1005166	37291	29163
30	326193	38855	24537	325878	34162	27352	1006525	37452	29187
31	352868	39697	24695	324961	32442	25611	1007775	37611	29210
32	400505	45698	26216	325019	33547	26490	1009161	37784	29234
33	405531	53733	29965	327588	36789	29246	1010644	37985	29260
34	443399	68780	36219	336357	42803	34484	1012206	38233	29291
35	374765	64009	32358	336899	44536	35740	1013711	38502	29323
36	384936	70853	35158	337648	44235	35186	1015109	38775	29355
37	403225	79163	29702	334529	42873	33796	1016559	39079	29385
38	341256	53943	21276	323243	34859	26533	1017929	39348	29409
39	368427	79265	24347	320805	31818	23736	1019235	39618	29430
40	372821	156717	34990	329579	40568	30908	1020598	40095	29458
41	424393	136446	48312	348496	55758	43479	1022065	40689	29497
42	490459	143609	83758	383234	86771	69224	1023748	41256	29558
43	405507	116514	82657	407270	109138	87478	1025397	41782	29636
44	364051	108366	52537	385581	90543	70879	1026813	42237	29699
45	391725	107871	52218	362332	71894	54790	1028203	42675	29748
46	394057	110604	56891	367136	75330	57092	1029649	43117	29799
47	378482	138052	61541	372588	82087	62016	1031070	43621	29854
48	370141	143881	58156	372407	83854	62689	1032448	44191	29910
49	335350	162372	78147	386176	95909	71488	1033746	44811	29974
50	380493	149956	77987	399511	110690	82023	1035063	45443	30047
51	308072	117738	58024	383561	97843	71338	1036330	45985	30110
52	279371	81111	42168	358516	74034	52397	1037410	46388	30157
53	228791	77925	37359	342633	60291	41521	1038345	46710	30194
54	247298	82613	37608	338600	57434	39126	1039221	47034	30229
55	307486	112765	63774	357028	76300	53030	1040242	47430	30277
56	304904	111207	67160	381001	97792	68656	1041369	47883	30338
57	353659	155481	108608	413486	131100	92518	1042581	48423	30420
58	344260	141900	114645	447660	165983	117974	1043865	49025	30524
59	290259	105948	64844	417567	136237	94520	1045032	49527	30608
60	241601	83076	50779	368411	90252	60567	1046011	49909	30662
61	340809	127924	71079	372240	95482	63862	1047082	50336	30719
62	395806	186764	98324	409663	131504	89140	1048437	50973	30798
63	424840	225860	121276	445436	171415	116034	1049947	51809	30900
64	342564	197217	125177	467630	194408	130526	1051359	52665	31016
65	358679	173712	92750	444875	174657	115150	1052649	53416	31117
66	368025	143412	78515	409256	139902	90154	1053987	54058	31197
67	345506	146511	84686	404440	135056	85850	1055299	54645	31274
68	360101	145848	95356	416086	150049	94846	1056598	55236	31358
69	347227	140304	86853	416445	153163	96006	1057899	55815	31443
70	337766	120843	78796	405457	140322	87165	1059159	56344	31520
71	299913	106887	55717	381082	116497	70611	1060332	56805	31583
72	265586	93443	52404	360107	95630	56629	1061373	57211	31633
73	234890	88779	41844	348145	84806	49302	1062294	57579	31677
74	273473	90381	53869	351867	86473	50089	1063229	57942	31722
75	303566	117447	70937	372524	110858	65465	1064291	58363	31780
76	315604	113111	66633	381985	122844	72246	1065430	58829	31845
77	309322	95276	58456	373812	113014	65623	1066580	59251	31903
78	243454	70009	39989	352884	91056	51534	1067597	59586	31949
79	240433	66798	40426	337520	76134	42026	1068487	59863	31987
80	215435	55466	34088	332361	71408	38923	1069326	60110	32021
81	245395	61730	40442	334979	71658	38945	1070174	60347	32056
82	271759	59717	39433	337661	76482	41752	1071125	60593	32093
83	250247	59072	42078	338134	78223	42610	1072086	60834	32131
84	236422	61836	37857	336354	77178	41783	1072981	61078	32169
85	220304	57452	36261	333296	72354	38722	1073821	61320	32203
86	193421	49382	26703	324308	62961	32870	1074583	61536	32233
87	213288	51278	28291	317595	56103	28692	1075331	61740	32258
88	232028	50067	30034	319532	59246	30438	1076150	61945	32286
89	236442	55270	33106	322652	63678	32966	1077012	62158	32315
90	204159	54531	30543	324494	64331	33233	1077823	62380	32345
91	257504	57962	34994	325361	66274	34226	1078672	62608	32376
92	235750	66669	44598	335915	78957	41622	1079580	62860	32413
93	194733	49384	27038	328841	72145	37431	1080372	63095	32446
94	173509	45225	22840	314057	53009	26016	1081049	63287	32469
95	216178	51043	31168	316086	56796	28181	1081766	63482	32495
96	243950	59946	37098	326527	69782	35664	1082613	63706	32527
97	213578	50320	27775	323283	66952	33883	1083455	63929	32557
98	173158	37148	16700	306803	48817	23190	1084166	64107	32578
99	177725	44811	30748	311191	51615	24747	1084812	64272	32600
100	181640	46594	36859	326307	70083	35322	1085473	64457	32631

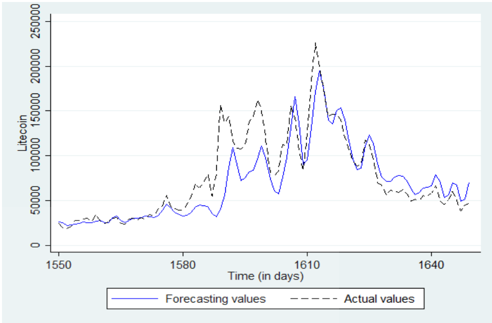
Figure 7 represents the 100 GLVM forecasting values of Bitcoin, Litecoin and Ripple along the last 4 months of the study period. Figure 7 (a), Figure 5 (b) and Figure 7 (c) emphasize a relatively high quality of forecasting values compared to that from the 2-dimensional GLVM. The evolution of actual values for both cryptocurrencies is similar to that of the forecasting values. The performance of the 3-dimensional GLVM is much better for Ripple and Litecoin.

The graph in Figure 7 (a) present an abrupt increase at the end of the second month and then decrease slightly with a slight increase trend at the end of the study time. This suggest a slight increase in adopting Bitcoin in future as suggested Table 6. In Figure 7 (b), the graph present a jump upward at the end of the second month and then decreases slightly with an increase trend at the end of the study time, suggesting the future slight increase in adopting Litecoin as found in Table 6.

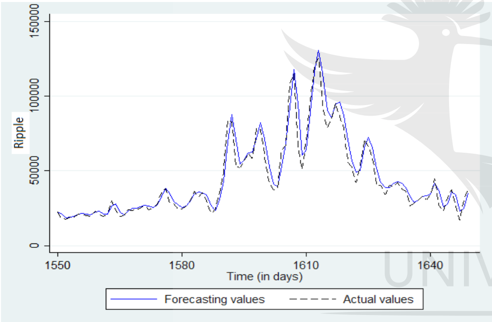
Ripple behaves similarly as the Litecoin as shows Figure 7 (c). Ripple keeps forecasting values less than that of Litecoin along the study time as shows Table 6.



(a) 100 last GLVM Bitcoin forecasting values



(b) 100 last GLVM Litecoin forecasting values



(c) 100 last GLVM Ripple forecasting values

Figure 7: 100 last GLVM forecasting values of Bitcoin, Litecoin and Ripple.

4. Conclusions

This paper reviewed models for competing species, namely the Grey Model (GM(1,1)), the Lotka-Volterra Model (LVM) and Grey Lotka-Volterra Model (GLVM). Predictability of the LVM is indicated by the estimated Lyapunov exponents of the model. It was found that the $n-$dimensional LVM, $n = 2;3$, is a chaotic dynamical system. GLVM is then used for assessing the competition and forecasting the transaction counts of Bitcoin and Litecoin from 28-April-2018 to 10-February-2018 as the 2-dimensional study and for assessing the competition and forecasting the transaction counts of Bitcoin, Litecoin and Ripple from 7-August-2018 to 10-February-2018 as the 3-dimensional study. The test of model accuracy done by the Mean Absolute Percentage Error (MAPE) show high accuracy of the GLVM as compared to the GM(1,1).

Under MAPE in 2-dimensional study, the overall forecasting values of Bitcoin are found to be reasonably accurate (MAPE=22) while reasonable accuracy for forecasting values of Litecoin occurred at 300 last forecasting values where MAPE=35. The last 100 forecasting values along the 4 last months of study period revealed a constant Bitcoin adoption and a slight increase for Litecoin adoption. The results of 3-dimensional study provide a relatively good performance compared to that of 2-dimensional study for Bitcoin and Litecoin and reveals the trend of a slight increase in trading both Bitcoin, Litecoin and Ripple. Ripple behaves similarly as the Litecoin with reasonable accuracy (MAPE=25) for the overall forecasting values.

The study shows that transaction counts of Bitcoin are relatively higher than that of Ripple and Litecoin along the study time and the trend in future is not significantly different according to the 3-dimensional GLVM. This confirms a long term strength in transacting Bitcoin relatively to Litecoin and Ripple.

The future work will consist of deriving and running a fractional grey Lotka-Volterra model for improving accuracy of forecasting values.

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Data Availability

The datasets analysed in the current study are available from anyone of the authors on request.

Conflicts of Interest

No conflicts of interest regarding the publication of this paper.

References

[1] J. Deng, "Introduction grey system theory," *J. Grey Syst.* **1** *(1), 191-243*, 1989.

[2] S. F. Liu and Y. Lin, *Grey systems: theory and practical applications*. Springer-Verlag London Ltd., London, 2010.

[3] L. Wu, S. Liu, and Y. Wang, "Grey Lotka-Volterra model and its applications," *Technological Forecasting & Social Change* **79**, *1720-1730*, 2012.

[4] J. Czyzowicz, L. Gasieniec, A. Kosowski, E. Kranakis, P. G. Spirakis, and P. Uznanski, "On convergence and threshold properties of discrete Lotka-Volterra population protocols," *International Colloquium on Automata, Languages, and Programming (ICALP 2015), Proceedings,* **9134** *393-405*, 2015.

[5] Zheng-Xin Wang and Hong-Tao Zhu, "Testing the trade relationships between China, Singapore, Malaysia and Thailand using grey Lotka-Volterra competition model," *Kybernetes* **45** *(6), 931-945*, 2016.

[6] Hsi-Tse Wang and Ta-Chu Wang, "Application of grey Lotka-Volterra model to forecast the diffusion and competition analysis of the TV and smart-phone industries," *Technological Forecasting Social Change* **106**, *37-44*, 2016.

[7] A. Atangana and J. F. Gómez-Aguilar, "Hyperchaotic behaviour obtained via a nonlocal operator with exponential decay and Mittag-Leffler laws," *Chaos, Solitons & Fractals,* **102** *doi: 10.1016/j.chaos.2017.03.022*, 2017.

[8] A. Atangana and J. F. Gómez-Aguilar, "Decolonisation of fractional calculus rules: Breaking commutativity and associativity to capture more natural phenomena," *The European Physical Journal Plus,* **133**, *1-23*, 2018.

[9] J. F. Gómez-Aguilar and A. Atangana, "New insight in fractional differentiation: power, exponential decay and Mittag-Leffler laws and applications," *The European Physical Journal Plus,* **133***, 1-23*, 2017.

[10] A. Atangana and J. F. Gómez-Aguilar, "A new derivative with normal distribution kernel: Theory, methods and applications," *Physica A: Statistical mechanics and its applications,* **476***, 1-14*, 2018.

[11] A. Atangana and J. F. Gómez-Aguilar, "Fractional derivatives with no-index law property: application to chaos and statistics," *Chaos, Solitons & Fractals,* **114***, 516-535*, 2018.

[12] H. Farid, "Discrete-time fractional differentiation from integer derivatives," *TR* **2004***, 528-536*, 2004.

[13] H. Koçak and A. Yildirim, "An efficient new iterative method for finding exact solutions of non-linear time-fractional partial differential equations," *Modelling and Control* **16** *(4), 403-414*, 2011.

[14] S. B. Angela, "Ten types of digital currencies and how they work," *Online trading: Free introductory eBook, September 24, 2016*, 2016.

[15] A. Blundell-Wignall, "The Bitcoin question: Currency versus trust-less transfer technology," *OECD Working Papers on Finance, Insurance and Private Pensions, No 37, OECD Publishing*, 2014.

[16] J. R. Hendrickson, T. L. Hogan, and W. J. Luther, "The political economy of Bitcoin," *Economics Inquiry,* **54** *(2), 925-939*, 2016.

[17] P. Wayner, *Digital Cash*. AP Professional, London, 2^{nd} ed., 1997.

[18] A. Urquhart, "The inefficiency of Bitcoin," *Economics Letters,* **148***, 80-82*, 2016.

[19] S. Lakka, C. Michalakelis, D. Varoutas, and D. Martakos, "Competitive dynamics in the operating systems market: modeling and policy implications," *Technological Forecasting & Social Change* **80***, 88-105*, 2013.

[20] R. Cerqueti, F. Tramontana, and M. Ventura, "On the coexistence of innovetors and imitators," *Technological Forecasting & Social Change* **90***, 487-496*, 2015.

[21] S. W. Chiang, "An application of Lotka-Volterra model to Taiwan's transition from 200 mm t0 330 mm silicon wafers," *Technological Forecasting & Social Change* **78***, 526-535*, 2012.

[22] H. B. Duan, L. Zhou, and Y. Fan, “A cross-country study on the relationship between diffusion of wind and photovoltaic solar technology,” *Technological Forecasting & Social Change* **83***, 156-169*, 2014.

[23] L. C. M. Miranda and C. A. S. Lima, “Technology substitution and innovation adoption: the cases of imaging and mobile communication markets,” *Technological Forecasting & Social Change* **80***, 1179-1193*, 2013.

[24] F. M. Tseng, Y. L. Liu, and H. S. Wu, “Market penetration among competitive innovation products: the case of the smartphone operating system,” *J. Eng. Technol. Manag.* **32***, 40-59*, 2014.

[25] B. H. Tsai and Y. Li, “Cluster evolution of IC industry from Taiwan to China,” *Technological Forecasting & Social Change* **76***, 1092-1104*, 2009.

[26] J. Wulf, M. Düser, and R. Zarnekow, “A methodology for analysing substitution effects between qos and best effort based services,” *In: Telecommunication, Media and Internet Techno-Economics (CTTE). 10th Conference of VDE, pp.1-5*, 2011.

[27] C. S. Lin, “Forecasting and analyzing the competitive diffusion of mobile cellular broadband and fixed broadband in Taiwan with limited historical data,” *Econ. Model.* **35***, 207-2013*, 2013.

[28] Nai-Ming Xie and Si-Feng Liu, “Discrete grey forecasting model and its optimization,” *Applied Mathematical Modelling* **33***, 1173-1186*, 2009.

[29] N. M. Xie, S. F. Liu, Y. J. Yang, and C. Q. Yuan, “On a novel grey forecasting model based on no-homogeneous index sequence,” *Applied Mathematical Modelling* **37** *, 5059-5068*, 2013.

[30] W. Zhou and Jian-Min He, “Generalised GM(1,1) model and its application in forecasting of fuel production,” *Applied Mathematical Modelling* **37***, 6234-6243*, 2013.

[31] A. Marasco, A. Picucci, and A. Romano, “Market share dynamics using Lotka-Volterra models,” *Technological Forecasting & Social Change* **105***, 49-62*, 2016.

[32] P. H. Leslie, “A stochastic model for studying the properties of certain biological systems by numerical methods,” *Biometrika* **45***, 16-31*, 1958.

[33] T. L. Tien, “A new grey prediction model FGM(1,1),” *Mathematical and Computer Modelling* **49***, 1416-1426*, 2009.

[34] F. Balibrea, “On problems of topological dynamics in non-autonomous discrete systems,” *Applied Mathematics and Nonlinear Sciences,* **1** *(2), 391-404*, 2016.

[35] C. Strobeck, "N species competition," *Ecology,* **54** *(3), 650-654*, 1973.

[36] J. Polking, A. Boggess, and D. Arnold, *Differential equations*. Prentice-Hall Inc., New Jersey, 2001.

[37] A. M. Lyapunov, *The general problem of the stability of motion*. Taylor & Francis, London, 1992.

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FRACTIONAL GREY LOTKA-VOLTERRA MODELS WITH APPLICATION TO CRYPTOCURRENCIES ADOPTION

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Keywords: Fractional model, Grey Lotka-Volterra Model, Mean Absolute Percentage Error, continuous time model, differential equations, difference equations.

Abstract: Fractional Grey Lotka-Volterra Model (FGLVM) is introduced and used for modeling the transaction counts of three cryptocurrencies namely Bitcoin, Litecoin and Ripple. The 2-dimensional study is on Bitcoin and Litecoin while the 3-dimensional study is on Bitcoin, Litecoin and Ripple. Dataset from 28-April-2013 to 10-February-2018 provides forecasting values for Bitcoin and Litecoin through 2-dimensional FGLVM study while dataset from 7-August-2013 to 10-February-2018 provides forecasting values of Bitcoin, Litecoin and Ripple through 3-dimensional FGLVM study. Forecasting values of cryptocurrencies for n-dimensional FGLVM study, $n = \{2,3\}$ along 100 days of study time are displayed. The graph and Lyapunov exponents of the 2-dimensional Lotka-Volterra system using the results of FGLVM reveals that the system is a chaotic dynamical system, while the 3-dimensional Lotka-Volterra system displays parabolic patterns in spite of the chaos indicated by the Lyapunov exponents. The Mean Absolute Percentage Error indicates that 2-dimensional FGLVM has a good accuracy for the overall forecasting values of Bitcoin and a reasonable accuracy for the last 300 forecasting values of Litecoin while the 3-dimensional FGLVM has a good accuracy for the overall forecasting values of Bitcoin and a reasonable accuracy for the last 300 forecasting values of both Litecoin and Ripple. Both 2 and 3-dimensional FGLVMs analysis evokes a future constant trend in transacting Bitcoin and a future decreasing trend in transacting Litecoin and Ripple. Bitcoin will keep relatively higher transaction counts, with Litecoin transaction counts everywhere superior to that of Ripple.

The Grey Lotka-Volterra model (GLVM) provides good estimates in several phenomenons. However the high variability of the dataset may require an appropriate fractional differentiation rather than the total differentiation applied in GLVM. Adequate Fractional Lotka-Volterra model may improve accuracy of the estimates, with relatively higher precision on different mathematical properties of the phenomenon of interest such as chaotic behavior and convergence.

1. Introduction

The concept fractional differentiation was found in a letter from Leibniz written to L'Hospital in 1695 [1]. Fractional differentiation consists of defining real or complex powers of the differentiation operator *D*. Fractional differentiation has been explored and applied in various subsequent studies such as the iterative methods in fractional calculus [2], the study on discrete time fractional calculus [3]; study on numerical approach of fractional differentiation [4], study on numerical discrete time fractional calculus [5] and many others recent studies such as for example [6], [7], [8], [9], [10], [11], [12], [13], [14] and [15]. The relationship between the two or more variables that uses Lotka-Volterra Model and related transformation such for example Grey Lotka-Volterra Model (GLVM) proposed by Wu et al. [16] presents modeling precision in social system or economic system.

Models for competing species, namely the Grey Model (GM(1,1)) and Grey Lotka-Volterra Model (GLVM) were reviewed and applied to cryptocurrencies adoption in [17]. The test of model accuracy done by the Mean Absolute Percentage Error (MAPE) in [17] showed high accuracy of the GLVM as compared to the GM(1,1). However due to the high variability in the dataset, the total differentiation can in some instance leave a challenge on the degree of the accuracy of the model and by using fractional differentiation, the accuracy may be improved. The Fractional Grey Lotka-Volterra Model (FGLVM) is therefore proposed by replacing total differentiation by fractional differentiation in GLVM. Considering that Grey Modeling is invariant with types of fractional differentiation and following the dynamic of cryptocurrencies, Caputo derivative is used since fluctuations are not showing critical jumps showing that there is no critical immediate change in transaction.

The application of FGLVM of this study is brought on three cryptocurrencies: Bitcoin, Litecoin and Ripple. Bitcoin as other cryptocurrencies, is the online currency initiated in 2008 [18] which consists of direct trade that is not tracked by a third-party [19] and without intermediary with the bank. Transactions of Bitcoin are mobile payments that are non-taxable

[20]. Wayner [21] evokes that digital cash cannot have multiple copies. Hence, Bitcoin cannot be used more than once. Multiple transfers are therefore not easy for digital currencies. Following this, Bitcoin has been viewed as a more secure and reliable mode of payment in recent years. The study on Bitcoin have been done for example by Urquhart [22] where the efficiency of Bitcoin is studied by using the dataset on the exchange of Bitcoin for six years. This analysis does not tackle however the problem of a long-term adoption of Bitcoin which will be addressed in this study.

Litecoin differs from Bitcoin in three important points. Firstly, Litecoin performs the processing of a block every 2.5 minutes instead of every 10 minutes of Bitcoin, allowing faster confirmation of transactions [23]. Secondly Litecoin produces approximately 4 times more units than Bitcoin and thirdly, Litecoin uses the function Scrypt in its working test algorithm which is hard memory sequential function that facilitates mining and Litecoin does not need sophisticated equipment as Bitcoin does [23]. This effect enables Litecoin network to accommodate up to 84 million coins while Bitcoin network cannot exceed 21 million coins. This study includes Litecoin as it is the second largest currency by the market capitalization [24]. Ripple for its part is based on the honour and trust of the people in the network [23]. Ripple adopts the development of a credit system. Each Ripple node functions as a local exchange system, in such a way that the entire system forms a decentralized mutual bank based on the needs of the users and everything is for a common good among them. They can in such a way, exchange everything up to skills.

The accuracy of the FGLVM in this study is checked by the Mean Absolute Percentage Error (MAPE) criterion encountered in various related research such as [25], [26, 16], [27] or [28]. The method of MAPE consists of checking accuracy of the model by using the quantity $\text{MAPE} = \frac{1}{n}\sum_{i=1}^{n}\left|\frac{Y_i-\widehat{Y}_i}{Y_i}\right| \times 100$, where Y_i and $\widehat{Y}_i$ are respectively the i^{th}observed and estimated quantities, $i = 1,2,\ldots,n$. The accuracy is high for MAPE less than 10, good when MAPE range from 10 to 20, the accuracy is reasonable if MAPE is between 20 and 50, and there is lack of accuracy if MAPE is 50 or above. The predictability of the FGLVM will be proposed by the pattern of the system of the model where Lyapunov exponents [29] will be also taken into account.

The study is subdivided as follows: Section 2 presents the methodology of the study, that is a review on Lotka-Volterra (LV) and GLV models, the introduction of the FGLVM and a description of the datasets. Section 3 presents the main results of the study with interpretation and Section 4 gives a conclusion.

2. Methodology

2.1. Fractional differentiation

Several definitions of fractional derivative in continuous time include the Caputo, Riemann-Liouville, Riesz and Hadamard approaches [2, 30]. The Caputo approach is of interest in this paper. Consider $f(t), t > 0$, the Caputo derivative of order α with α a real number such that $n-1 < \alpha < n$, is defined in [2] as

$$D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)ds}{(t-s)^{\alpha-n+1}}, \tag{1}$$

where $\Gamma(x)$ is a gamma function defined by

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt, \text{ with } x, \text{ a complex number.} \tag{2}$$

Diaz and Osier [31] generalised the discrete fractional derivative of order α for any sequence of complex or real numbers $f(n)$, by the following difference equation:

$$D_t^{\alpha} f(n) = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(n+\alpha-k), \tag{3}$$

with extended binomial coefficients defined by $\binom{\alpha}{k} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)\Gamma(k+1)}$ [32].

2.2. General Lotka-Volterra Model (GLVM)

The general Lotka-Volterra system or model of competing relationships between n species is given by

$$\frac{dX_i}{dt} = X_i \left(a_i - \sum_{j=1}^{n} \alpha_{ij} X_j \right) \tag{4}$$

[33, 34]. Parameters a_i represent the capacity of growing of populations X_i, while parameters α_{ij} represent the effect species j has on species i. The expressions X_i^2 are interactions within species, $X_i X_j$, $i \neq j$ are interactions of different species.
Assuming competitive species X, Y and W, the system (4) yields the 2-dimensional model for $n = 2$ as follows:

$$\begin{cases} \frac{dX}{dt} = a_1 X - b_1 X^2 - c_1 XY \\ \frac{dY}{dt} = a_2 Y - b_2 Y^2 - c_2 YX, \end{cases} \tag{5}$$

and the 3-dimensional model for $n = 3$ as

$$\begin{cases} \frac{dX}{dt} = a_1X - b_1X^2 - c_1XY - d_1XW \\ \frac{dY}{dt} = a_2Y - b_2Y^2 - c_2YX - d_2YW \\ \frac{dW}{dt} = a_3W - b_3W^2 - c_3WX - d_3WY. \end{cases} \tag{6}$$

2.3. Fractional Grey modeling

Assume original data sequences $X_i^{(0)} = \left(x_i^{(0)}(1), x_i^{(0)}(2), \ldots, x_i^{(0)}(n)\right)$ with the corresponding first order accumulation generating operations (1-AGO) given by:

$$X_i^{(1)} = \left(x_i^{(1)}(1), x_i^{(1)}(2), \ldots, x_i^{(1)}(n)\right)$$

with

$$x_i^{(1)}(k) = \sum_{j=1}^{k} x_i^{(0)}(j),\ k = 1, 2, \ldots, n, \tag{7}$$

and corresponding mean sequence of $X_i^{(1)}$ given by

$$Z_i^{(1)} = \left(z_i^{(1)}(2), z_i^{(1)}(3), \ldots, z_i^{(1)}(n)\right),$$

where,

$$z_i^{(1)}(k) = \frac{x_i^{(1)}(k) + x_i^{(1)}(k-1)}{2},\ k = 2, 3, \ldots, n;$$

Let GM(1,1) be the grey model based on the series $X_i^{(1)} = \left(x_i^{(1)}(1), x_i^{(1)}(2), \ldots, x_i^{(1)}(n)\right)$ and modeled by the differential equation

$$\frac{dx_i^{(1)}(t)}{dt} + a x_i^{(1)}(t) = b; \tag{8}$$

yielding the difference equation

$$x_i^{(0)}(k+1) + a z_i^{(1)}(k) = b. \tag{9}$$

where parameters a and b of GM(1,1) are calculated by least square method and the initial condition $X_i^{(1)}(1) = X_i^{(0)}(1)$ [35].
The expression of 1-AGO in (7) can be written in matrix form as

$$\mathbf{X}^{(1)} = \mathbf{X}^{(0)}\mathbf{U} \tag{10}$$

where the first order accumulated matrix $\mathbf{U}$ is given by

$$\mathbf{U} = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ 0 & 1 & \ldots & 1 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix}. \tag{11}$$

The second order accumulated sequence is then given by

$$\begin{aligned} \mathbf{X}^{(2)} &= \mathbf{X}^{(1)}\mathbf{U} \\ &= \mathbf{X}^{(0)}\mathbf{U}^{2}. \end{aligned}$$

More generally, the M^{th} order accumulated sequence is given by

$$\mathbf{X}^{(M)} = \mathbf{X}^{(0)}\mathbf{U}^{M} \tag{12}$$

Elements of $\mathbf{U}^{M}$ can be written as

$$u_{ik}^{M} = \begin{cases} 1 & \text{if } i = k \\ M(M+1)(M+2)\dots(M+k-i-1) & \text{if } i < k\,; \\ 0 & \text{if } i > k \end{cases} \tag{13}$$

and thus, the k^{th} accumulation in $X^{(M)}$ is given by

$$x^{(M)}(k) = \sum_{i=1}^{k} u_{ik}^{M} x^{(0)}(i) \tag{14}$$

Using Equation (12), Wu et al. [36] propose the fractional accumulation of order q, for the sequence $X^{(q)} = \{x^{(q)}(1), x^{(q)}(2), \dots, x^{(q)}(n)\}$ as

$$\mathbf{X}^{(q)} = \mathbf{X}^{(0)}\mathbf{U}^{q} \tag{15}$$

where q is a positive fractional number less than 1.
The elements of $\mathbf{U}^{q}$ can then be written as

$$u_{ik}^{q} = \begin{cases} 1 & \text{if } i = k \\ q(q+1)(q+2)\dots(q+k-i-1) & \text{if } i < k\,; \\ 0 & \text{if } i > k \end{cases} \tag{16}$$

or equivalently

$$u_{ik}^{q} = \begin{cases} 1 & \text{if } i = k \\ \frac{\Gamma(q+k-i)}{\Gamma(q)\Gamma(k-i+1)} & \text{if } i < k\,; \\ 0 & \text{if } i > k \end{cases} \tag{17}$$

Using Equation (14), the k^{th} fractional accumulation can be writen as

$$\begin{aligned} x^{(q)}(k) &= \sum_{i=1}^{k} u_{ik}^{q} x^{(0)}(i) \\ &= \sum_{i=1}^{k} \frac{\Gamma(q+k-i)}{\Gamma(q)\Gamma(k-i+1)} x^{(0)}(i) \end{aligned} \tag{18}$$

Proposition 1. The expression $x^{(q)}(k) = \sum_{i=1}^{k} \frac{\Gamma(q+k-i)}{\Gamma(q)\Gamma(k-i+1)} x^{(0)}(i)$ (Equation (18)) is equivalent to the expression

$$x^{(q)}(k) = \sum_{i=1}^{k} \frac{e^{\ln\Gamma(q+k-i) - \ln\Gamma(k-i+1)} x^{(0)}(i)}{\Gamma(q)}. \tag{19}$$

Proof. Consider Equation (18) and write the expression $\frac{\Gamma(q+k-i)}{\Gamma(q)\Gamma(k-i+1)}$ by using logarithm as

$$\begin{aligned} \frac{\Gamma(q+k-i)}{\Gamma(k-i+1)} &= e^{\ln\left[\frac{\Gamma(q+k-i)}{\Gamma(k-i+1)}\right]} \\ &= e^{\ln\Gamma(q+k-i) - \ln\Gamma(k-i+1)} \end{aligned} \tag{20}$$

Substituting (20) into Equation (18) yields Equation (19). □

2.4. Fractional grey LV equations

Assume the sets of original series $X_i^{(0)}$

$$X^{(0)} = \left(x^{(0)}(1), x^{(0)}(2), \ldots, x^{(0)}(n)\right)$$

The q-AGO of $X_i^{(0)}$ are given by:

$$X_i^{(q)} = \left(x_i^{(q)}(1), x_i^{(q)}(2), \ldots, x_i^{(q)}(n)\right)$$

with

$$x_i^{(q)}(k) = \sum_{j=1}^{k} u_{jk}^{q} x_i^{(0)}(j),\ k = 1, 2, \ldots, n;$$

the mean sequence of $X_i^{(q)}$ is given by

$$Z_i^{(q)} = \left(z_i^{(q)}(2), z_i^{(q)}(3), \ldots, z_i^{(q)}(n)\right) \tag{21}$$

with

$$z_i^{(q)}(k) = \frac{x^{(q)}(k) + x^{(q)}(k-1)}{2},\ k = 2, 3, \ldots, n \tag{22}$$

Applying the fractional grey model to the system (4) yields the following approximations:

$$x_i^{(0)}(k+1) \approx a_i z_i^{(q)}(k) - b_i \left(z_i^{(q)}(k)\right)^2 - \sum_{j \neq i}^{n} c_{ij} z_i^{(q)}(k) z_j^{(q)}(k); \tag{23}$$

with error sequences expressed by

$$\varepsilon_i = x_i^{(0)}(k+1) - \left(a_i z_i^{(q)}(k) - b_i \left(z_i^{(q)}(k) \right)^2 - \sum_{j \neq i}^{n} c_{ij} z_i^{(q)}(k) z_j^{(q)}(k) \right); \tag{24}$$

The least square estimates of parameters in (23) are found as follows:

$$\begin{pmatrix} \hat{a}_i \\ \hat{b}_i \\ \hat{c_{ij}} \end{pmatrix} = (B_i' B_i)^{-1} B_i' M_i \tag{25}$$

where,

$$B_i = \begin{pmatrix} z_i^{(q)}(2) & -\left(z_i^{(q)}(2)\right)^2 & -z_i^{(q)}(2) z_1^{(q)}(2) & \dots & -z_i^{(q)}(2) z_j^{(q)}(2) \\ z_i^{(q)}(3) & -\left(z_i^{(q)}(3)\right)^2 & -z_i^{(q)}(3) z_1^{(q)}(3) & \dots & -z_i^{(q)}(3) z_j^{(q)}(2) \\ \vdots & \vdots & \vdots & & \\ z_i^{(q)}(n) & -\left(z_i^{(q)}(n)\right)^2 & -z_i^{(q)}(n) z_1^{(q)}(n) & \dots & -z_i^{(q)}(n) z_j^{(q)}(2) \end{pmatrix}; \forall j \neq i;$$

$$M_i = \begin{pmatrix} x_i^{(0)}(2) \\ x_i^{(0)}(3) \\ \vdots \\ x_i^{(0)}(n) \end{pmatrix}$$

2.5. Data

The 2-dimensional analysis considers two datasets on daily transaction counts of Bitcoin and Litecoin from 28-April-2013 up to 10-February-2018, while 3-dimensional study takes three datasets on daily transaction counts of Bitcoin, Litecoin and Ripple from 7-August-2013 up to 10-February-2018. The evolution in transaction counts of Bitcoin and Litecoin along the study time and that of both Bitcoin, Litecoin and Ripple recorded daily with the related graphs are presented in [17] and can also be found via the authors of this paper. Bitcoin, Litecoin and Ripple (Table 1) represented in the same coordinate plane (Figure 1) indicate no critical difference between Litecoin and Ripple transactions and therefore, the dynamic of Bitcoin and Ripple does not differ critically from that of Bitcoin and Litecoin presented in this study.

Table 1: Transaction counts of Bitcoin, Litecoin and Ripple from 7-August-2013 to 10-February-2018

Date	Bitcoin	Litecoin	Ripple
7-Aug-13	56974	4385	3335
8-Aug-13	56992	3932	3477
9-Aug-13	52486	3649	2219
10-Aug-13	52316	3924	1887
11-Aug-13	47995	3585	2207
⋮	⋮	⋮	⋮
6-Feb-2018	243950	59946	37098
7-Feb-2018	213578	50320	27775
8-Feb-2018	173158	37148	16700
9-Feb-2018	177725	44811	30748
10-Feb-2018	181640	46594	36859

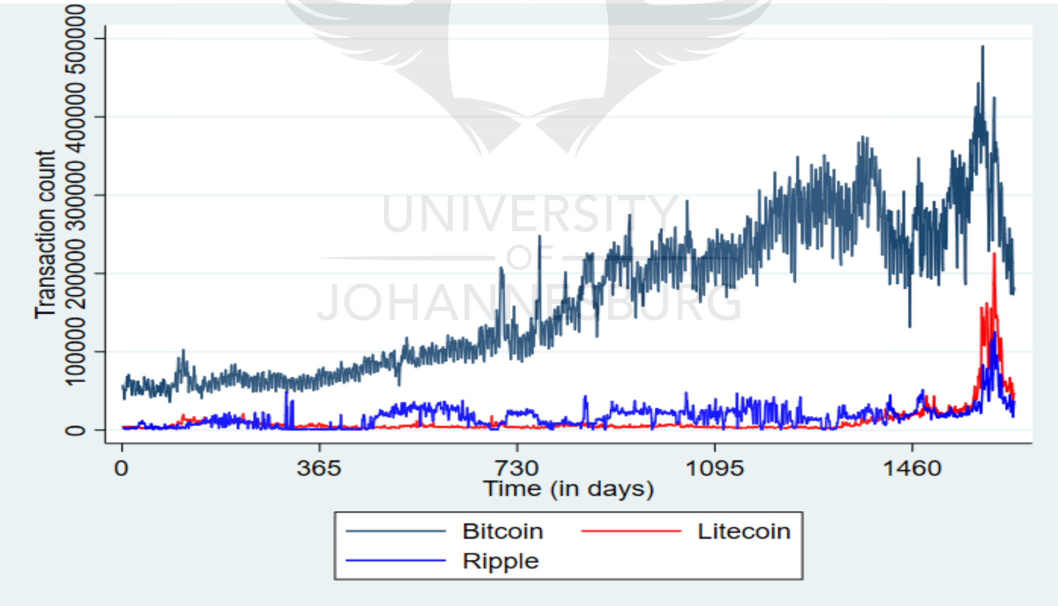


Figure 1: Transaction counts of Bitcoin, Litecoin and Ripple from 7-August-2013 to 10-February-2018.

3. Results

3.1. 2-dimensional Fractional Grey Lotka-Volterra model for Bitcoin and Litecoin

Model (23) is applied to the dataset. Using Equations (25) with $q = 0.5$, the least square estimates of model parameters are obtained as

$$\begin{pmatrix} \hat{a}_1 \\ \hat{b}_1 \\ \hat{c}_1 \end{pmatrix} = \begin{pmatrix} 5.293 \times 10^{-2} \\ 2.137 \times 10^{-9} \\ 2.403 \times 10^{-9} \end{pmatrix}, \quad \begin{pmatrix} \hat{a}_2 \\ \hat{b}_2 \\ \hat{c}_2 \end{pmatrix} = \begin{pmatrix} 1.641 \times 10^{-2} \\ -3.176 \times 10^{-8} \\ -3.319 \times 10^{-9} \end{pmatrix}.$$

The Fractional Grey Lotka-Volterra model (23) can then be written as:

$$\begin{cases} x^{(0)}(k+1) \approx 5.293 \times 10^{-2} z_x^{(q)}(k) - 2.137 \times 10^{-9} \left(z_x^{(q)}(k)\right)^2 \\ \qquad -2.403 \times 10^{-9} z_x^{(q)}(k) z_y^{(q)}(k) \\ y^{(0)}(k+1) \approx 1.641 \times 10^{-2} z_y^{(q)}(k) + 3.176 \times 10^{-8} \left(z_y^{(q)}(k)\right)^2 \\ \qquad +3.319 \times 10^{-9} z_y^{(q)}(k) z_x^{(q)}(k) \end{cases} \tag{26}$$

$k = 1, 2, \ldots, n$ and $q = 0.5$.

The pattern of the 2-dimensional FGLVM (Figure 2) is a connection between two figures that are not compatible, suggesting that the system is a chaotic dynamical system as suggest Thietart and Forgues [37]. The chaotic property of the 2-dimensional FLVM is also confirmed by the positive Lyapunov exponents found at the equilibrium points $(0,0)$ as $\boldsymbol{\lambda} = \{5.293 \times 10^{-2}, 1.641 \times 10^{-2}\}$.

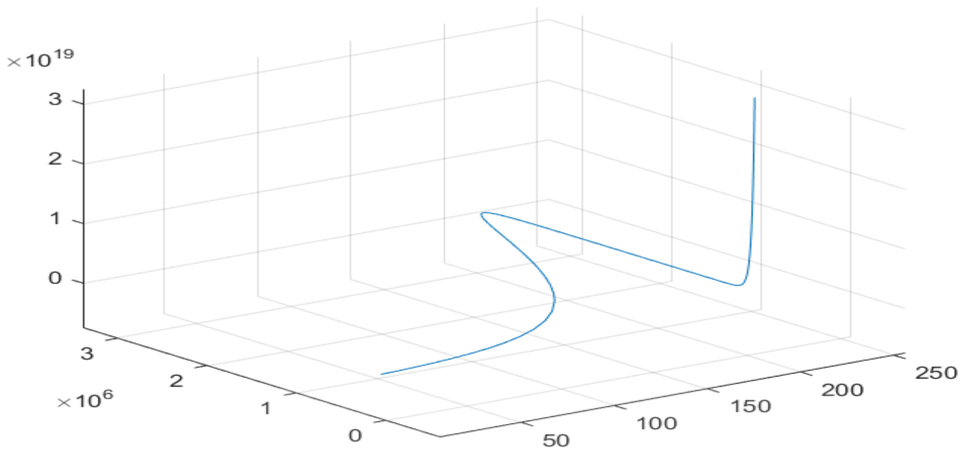


Figure 2: 2-dimensional LVM plot with initial conditions $X(0) = 40035$; $Y(0) = 9408$.

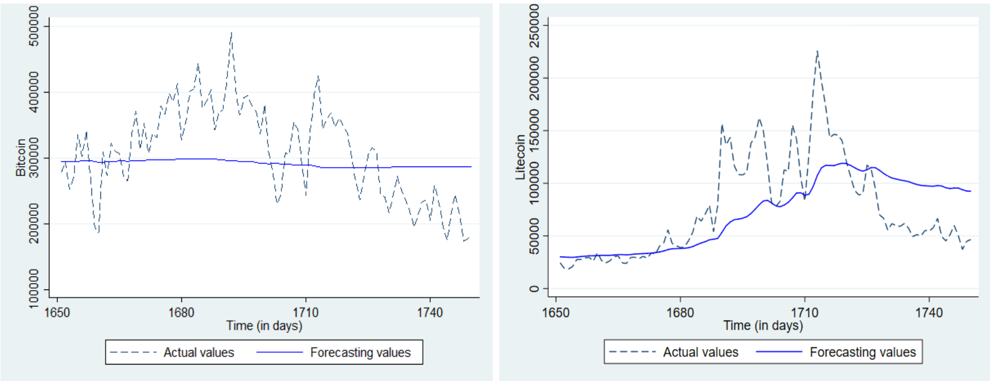
Under the MAPE, the accuracy of Model (26) is good for the overall values of Bitcoin (MAPE=16) and reasonably accurate for the last 300 values of Litecoin (MAPE=25). By considering the MAPE, the FGLVM suggests a better accuracy than that of the GLVM where reasonable accuracy is observed for the overall values of Bitcoin (MAPE = 22) and for the last 300 values of Litecoin (MAPE = 35) as reported in [17].

Table 2 gives the last 100 reasonably accurate forecasting values of the FGLVM of Bitcoin (MAPE=20) and Litecoin (MAPE=38). The 100 last forecasting values show that in future, there will be a linear slight decrease in adopting Bitcoin and a slight linear increase for Litecoin adoption.

Figure 3 represents the 100 forecasting values of Bitcoin (BTC) and Litecoin (LTC) along the last 4 months of the study period. Figure 3 (a) shows that the actual values of Bitcoin fluctuate around the forecasting values and the same situation is observed for Litecoin (Figure 3 (b)). The graph of forecasting values of Bitcoin in Figure 3 (a) is approximately constant along the last four months of the study time while in Figure 3 (b), the graph of forecasting values of Litecoin increases slightly up the beginning of the last month and then decreases towards the end of study time. This suggests a future decrease in adopting Litecoin as found in Table 2.

Table 2: Last 100 forecasting values of the Fractional Grey Lotka-Volterra Model for Bitcoin and Litecoin transactions.

	Actual values		FGLVM values			Actual values		FGLVM values	
N0	BTC	LTC	BTC	LTC	N0	BTC	LTC	BTC	LTC
1	277479	24524	294919	30155	51	308072	117738	291039	83781
2	293991	19249	294782	29948	52	279371	81111	291139	81279
3	251587	18545	294675	29659	53	228791	77925	291164	78595
4	270896	20785	294487	29575	54	247298	82613	290908	77655
5	335480	27637	294843	29937	55	307486	112765	290619	79292
6	301202	27657	295129	30404	56	304904	111207	290281	81881
7	341128	29077	295341	30749	57	353659	155481	289685	86079
8	271625	29698	295307	31068	58	344260	141900	289057	90707
9	194554	25978	294480	31083	59	290259	105948	288925	90967
10	185886	34159	293506	31295	60	241601	83076	288968	88634
11	309159	26605	293674	31588	61	340809	127924	288803	89616
12	271867	24375	294266	31497	62	395806	186764	288006	96864
13	321636	26290	294593	31540	63	424840	225860	286628	107323
14	310244	30400	294971	31880	64	342564	197217	285428	114660
15	306450	31024	295070	32273	65	358679	173712	284937	117045
16	270738	24365	295021	32277	66	368025	143412	285106	116989
17	264695	23815	294877	32061	67	345506	146511	285214	116824
18	336029	29536	295180	32280	68	360101	145848	285059	118028
19	370918	29760	295912	32741	69	347227	140304	284955	119080
20	311885	28407	296186	32963	70	337766	120843	285021	118748
21	352050	30623	296319	33191	71	299913	106887	285202	117063
22	305586	28816	296443	33408	72	265586	93443	285304	114847
23	336533	33856	296457	33724	73	234890	88779	285299	112714
24	329524	33004	296622	34164	74	273473	90381	285323	111522
25	379086	40417	296920	34794	75	303566	117447	285203	112833
26	365821	43480	297205	35701	76	315604	113111	284987	114988
27	397917	55781	297344	36979	77	309322	95276	285117	114860
28	384219	42914	297593	37841	78	243454	70009	285379	112304
29	412725	40679	298031	37949	79	240433	66798	285565	109364
30	326193	38855	298063	38065	80	215435	55466	285721	106862
31	352868	39697	297880	38221	81	245395	61730	285883	105015
32	400505	45698	298166	38818	82	271759	59717	286143	104087
33	405531	53733	298436	39910	83	250247	59072	286341	103090
34	443399	68780	298564	41694	84	236422	61836	286313	102372
35	374765	64009	298419	43291	85	220304	57452	286235	101598
36	384936	70853	298097	44532	86	193421	49382	286178	100148
37	403225	79163	297913	46327	87	213288	51278	286183	98836
38	341256	53943	297793	46855	88	232028	50067	286323	98020
39	368427	79265	297543	47600	89	236442	55270	286432	97598
40	372821	156717	296332	53239	90	204159	54531	286327	97346
41	424393	136446	295256	59436	91	257504	57962	286344	97228
42	490459	143609	295190	63181	92	235750	66669	286333	97823
43	405507	116514	295066	65468	93	194733	49384	286160	97381
44	364051	108366	294765	65935	94	173509	45225	286048	95762
45	391725	107871	294554	66820	95	216178	51043	286033	95045
46	394057	110604	294433	68234	96	243950	59946	286129	95480
47	378482	138052	293855	71245	97	213578	50320	286151	95451
48	370141	143881	292985	75266	98	173158	37148	286112	93887
49	335350	162372	291957	79523	99	177725	44811	285987	92676
50	380493	149956	291219	83211	100	181640	46594	285838	92432



(a) Forecasting values of Bitcoin along 100 last days of the study time.

(b) Forecasting values of Litecoin and along 100 last days of the study time.

Figure 3: Forecasting values of Bitcoin and Litecoin along 100 last days of the study time.

3.2. 3-dimensional Fractional Grey Lotka-Volterra model for Bitcoin, Litecoin and Ripple

We apply Model (23) with $q = 0.5$ to the dataset. Equations (25) give the least square estimates of model parameters as

$$\begin{pmatrix}\hat{a}_1\\ \hat{b}_1\\ \hat{c}_1\\ \hat{d}_1\end{pmatrix} = \begin{pmatrix}5.976\times 10^{-2}\\ 1.416\times 10^{-9}\\ -5.266\times 10^{-9}\\ 2.153\times 10^{-8}\end{pmatrix}, \begin{pmatrix}\hat{a}_2\\ \hat{b}_2\\ \hat{c}_2\\ \hat{d}_2\end{pmatrix} = \begin{pmatrix}5.359\times 10^{-2}\\ -1.314\times 10^{-7}\\ -1.996\times 10^{-8}\\ 2.947\times 10^{-7}\end{pmatrix}, \begin{pmatrix}\hat{a}_3\\ \hat{b}_3\\ \hat{c}_3\\ \hat{d}_3\end{pmatrix} = \begin{pmatrix}4.865\times 10^{-2}\\ -2.162\times 10^{-8}\\ 5.488\times 10^{-9}\\ -3.136\times 10^{-8}\end{pmatrix}.$$

The Fractional Grey Lotka-Volterra model (23) can then be written as:

$$\begin{cases} x^{(0)}(k+1) \approx 5.976\times 10^{-2} z_x^{(q)}(k) - 1.416\times 10^{-9}\left(z_x^{(q)}(k)\right)^2 \\ \qquad + 5.266\times 10^{-9} z_x^{(q)}(k) z_y^{(q)}(k) - 2.153\times 10^{-8} z_x^{(q)}(k) z_w^{(q)}(k) \\ y^{(0)}(k+1) \approx 5.359\times 10^{-2} z_y^{(q)}(k) + 1.314\times 10^{-7}\left(z_y^{(q)}(k)\right)^2 \\ \qquad + 1.996\times 10^{-8} z_y^{(q)}(k) z_x^{(q)}(k) - 2.947\times 10^{-7} z_y^{(q)}(k) z_w^{(q)}(k) \\ w^{(0)}(k+1) \approx 4.865\times 10^{-2} z_w^{(q)}(k) + 2.162\times 10^{-8}\left(z_w^{(q)}(k)\right)^2 \\ \qquad - 5.488\times 10^{-9} z_w^{(q)}(k) z_x^{(q)}(k) + 3.136\times 10^{-8} z_w^{(q)}(k) z_y^{(q)}(k) \end{cases} \tag{27}$$

$k = 1, 2, \ldots, n$, $q = 0.5$.

The 3-dimensional LVM is parabolic as shows (Figure 4). However, positive Lyapunov exponents are found at the equilibrium points $(0,0,0)$ as $\boldsymbol{\lambda} = \{5.976 \times 10^{-2}, 5.359 \times 10^{-2}, 4.865 \times 10^{-2}\}$. This suggest that the system is a chaotic dynamical system.

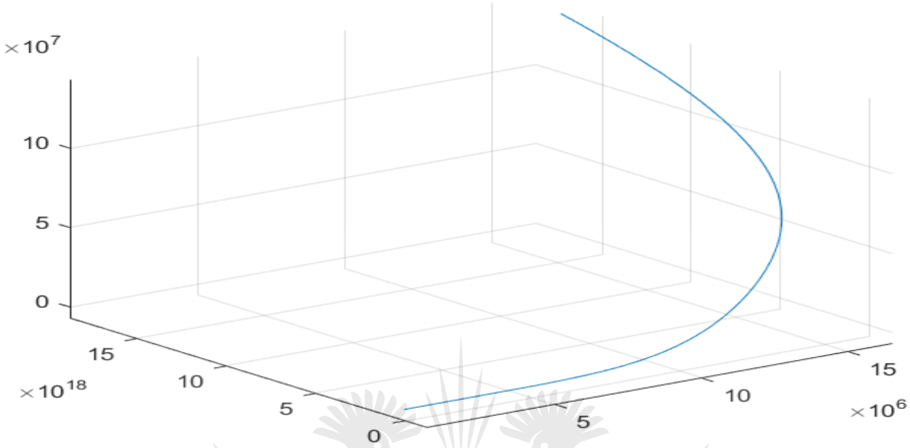


Figure 4: 3-dimensional LVM plot with initial conditions $X(0) = 56974$; $Y(0) = 4385$; $W(0) = 3335$.

Under the MAPE, the accuracy of Model (27) is good (MAPE=16). Model (27) is reasonably accurate for the last 300 values of Litecoin and Ripple with MAPE=28 and MAPE=29 respectively. Considering the MAPE and the Bitcoin forecasting values, the 3-dimensional FGLVM suggests a better accuracy than that of the 3-dimensional GLVM where reasonable accuracy is observed with MAPE = 22. The 3-dimensional GLVM accuracy takes over for the forecasting values of Ripple and Litecoin as found in [17].

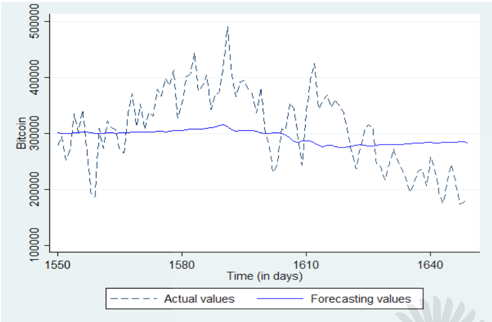
Table 3 gives the last 100 forecasting values of the FGLVM for Bitcoin with good accuracy (MAPE=19), Litecoin with reasonable accuracy (MAPE=39) and Ripple also with reasonable accuracy (MAPE=35). The 100 last forecasting values show that in future, there will be a slight decrease in transacting both Bitcoin, Litecoin and Ripple; with Bitcoin keeping relatively higher transaction counts.

Table 3: Last 100 forecasting values of the Fractional Grey Lotka-Volterra Model for Bitcoin and Litecoin and Ripple transactions

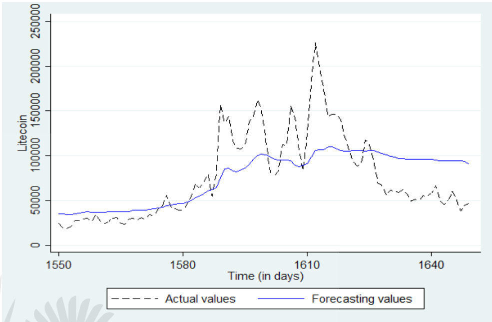
	Actual values			FGLVM values				Actual values			FGLVM values		
N0	BTC	LTC	RPL	BTC	LTC	RPL	N0	BTC	LTC	RPL	BTC	LTC	RPL
1	277479	24524	22368	300656	35357	22857	51	308072	117738	58024	299966	101178	45364
2	293991	19249	18280	300218	34877	22869	52	279371	81111	42168	300274	98468	44316
3	251587	18545	17602	300136	34541	22758	53	228791	77925	37359	300560	95568	43208
4	270896	20785	19099	299833	34330	22793	54	247298	82613	37608	300755	94913	42822
5	335480	27637	18915	300645	35138	22763	55	307486	112765	63774	299526	95480	43876
6	301202	27657	21174	301313	36007	22812	56	304904	111207	67160	296950	95306	45743
7	341128	29077	20553	301691	36571	22875	57	353659	155481	108608	292179	93931	49018
8	271625	29698	20273	301876	37101	22974	58	344260	141900	114645	285382	89660	53227
9	194554	25978	19687	300647	36678	23281	59	290259	105948	64844	283998	87962	53839
10	185886	34159	23231	299093	36320	23754	60	241601	83076	50779	286447	88978	52087
11	309159	26605	20843	299315	36728	23834	61	340809	127924	71079	287046	91082	52294
12	271867	24375	19507	300223	36967	23571	62	395806	186764	98324	285880	98041	55641
13	321636	26290	21017	300715	37199	23470	63	424840	225860	121276	282879	105998	60988
14	310244	30400	29874	300552	37263	23658	64	342564	197217	125177	278006	106726	65888
15	306450	31024	23694	300266	37449	23901	65	358679	173712	92750	276165	106431	67641
16	270738	24365	19622	300632	37756	23816	66	368025	143412	78515	278011	109535	66841
17	264695	23815	20202	300565	37536	23737	67	345506	146511	84686	278169	109573	66681
18	336029	29536	24025	300928	37894	23743	68	360101	145848	95356	276134	107440	67958
19	370918	29760	23637	301987	38822	23678	69	347227	140304	86853	274463	105721	69025
20	311885	28407	24401	302277	39148	23699	70	337766	120843	78796	274118	104716	69014
21	352050	30623	25098	302285	39350	23793	71	299913	106887	55717	275573	105307	67775
22	305586	28816	26788	302132	39421	23920	72	265586	93443	52404	277526	106166	66186
23	336533	33856	23877	302193	39832	24043	73	234890	88779	41844	278936	106205	64868
24	329524	33004	24769	302803	40725	24083	74	273473	90381	53869	279542	105894	64189
25	379086	40417	27720	303338	41718	24215	75	303566	117447	70937	278125	104991	65255
26	365821	43480	35197	303266	42580	24607	76	315604	113111	66633	277044	105640	66536
27	397917	55781	37889	302882	43713	25235	77	309322	95276	58456	277143	105606	66405
28	384219	42914	28598	303585	45161	25414	78	243454	70009	39989	278184	104407	64981
29	412725	40679	28029	304705	45951	25160	79	240433	66798	40426	279258	102852	63424
30	326193	38855	24537	305150	46433	25100	80	215435	55466	34088	279854	100980	62215
31	352868	39697	24695	305343	46885	25133	81	245395	61730	40442	280154	99332	61360
32	400505	45698	26216	306302	48230	25132	82	271759	59717	39433	280356	98478	60864
33	405531	53733	29965	307209	50167	25343	83	250247	59072	42078	280321	97194	60446
34	443399	68780	36219	307807	52808	25911	84	236422	61836	37857	280317	96412	60182
35	374765	64009	32358	308077	55122	26500	85	220304	57452	36261	280677	96198	59771
36	384936	70853	35158	308072	56873	27029	86	193421	49382	26703	281382	95790	58963
37	403225	79163	29702	309171	60250	27491	87	213288	51278	28291	282385	95967	58080
38	341256	53943	21276	310475	62209	27392	88	232028	50067	30034	282941	95861	57529
39	368427	79265	24347	311611	64406	27429	89	236442	55270	33106	283158	95664	57263
40	372821	156717	34990	313980	75057	29126	90	204159	54531	30543	283262	95630	57147
41	424393	136446	48312	314968	85283	31319	91	257504	57962	34994	283454	95795	57028
42	490459	143609	83758	311507	86391	33658	92	235750	66669	44598	282708	95265	57536
43	405507	116514	82657	305797	82817	36100	93	194733	49384	27038	282535	94628	57461
44	364051	108366	52537	304104	81642	36799	94	173509	45225	22840	283628	94684	56440
45	391725	107871	52218	305422	84458	36794	95	216178	51043	31168	283907	94353	56056
46	394057	110604	56891	305371	86258	37358	96	243950	59946	37098	283609	94282	56322
47	378482	138052	61541	304890	89889	38714	97	213578	50320	27775	283861	94634	56215
48	370141	143881	58156	305072	95867	40298	98	173158	37148	16700	284971	94700	55193
49	335350	162372	78147	303524	100026	42514	99	177725	44811	30748	284825	93239	54780
50	380493	149956	77987	300806	101500	44854	100	181640	46594	36859	283140	90443	55321

Figure 5 represents the 100 forecasting values of Bitcoin, Litecoin and Ripple along the last 100 days of the study period.

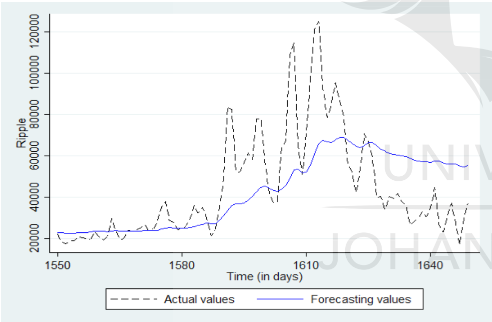
The graph in Figure 5 (a) shows that forecasting values of Bitcoin are approximately constant along last three month of the study time with a slight decrease tendency at the end. In Figure 5 (b), the graph representing Litecoin forecasting values increases slightly and then decrease slightly towards the end of the study time. The graph of Ripple forecasting values behave similarly as that of Litecoin as shows Figure 5 (c). Ripple and Litecoin keeps forecasting values less than that of Bitcoin and Ripple forecasting values are everywhere less than that of litecoin as shows Table 3.



(a) Forecasting values of Bitcoin along 100 last days of the study time.



(b) Forecasting values of Litecoin along 100 last days of the study time.



(c) Forecasting values of Ripple along 100 last days of the study time.

Figure 5: Forecasting values of Bitcoin, Litecoin and Ripple along 100 last days of the study time.

4. Conclusions

This paper introduced a fractional discrete differentiation on the Grey Lotka-Volterra Model (GLVM). The Fractional Grey Lotka-Volterra Model (FGLVM) formulated is applied for forecasting the adoption of cryptocurrencies namely Bitcoin, Litecoin and Ripple. Lyapunov exponents and graphs of the formulated model for dataset on cryptocurrencies were investigated for checking predictability. The pattern of the 2-dimensional FGLVM for Bitcoin and Litecoin suggested a chaotic dynamical system; that is also confirmed by the presence of positive Lyapunov exponents at the equilibrium point $(0,0)$. The 3-dimensional FGLVM displayed a parabolic pattern but also, positive Lyapunov exponents are found at the equilibrium point $(0,0,0)$. The later fact suggests that the model is a chaotic dynamical system. The adequacy checking of the FGLVM was checked by the Mean Absolute Percentage Error (MAPE) for forecasting values of cryptocurrencies.

The MAPE for 2-dimensional study suggested that the model accuracy is good for overall forecasting values of Bitcoin (MAPE=16). Reasonable accuracy for 2-dimensional FGLVM is observed at the last 300 forecasting values where MAPE=25. The 2-dimensional FGLVM suggests the better accuracy as compared to the 2-dimensional GLVM reported in [17] where reasonable accuracy is observed for the overall values of Bitcoin with MAPE = 22 and for the last 300 values of Litecoin with MAPE = 35. The last 100 forecasting values along the last 100 days of study period revealed a constant Bitcoin adoption and an earlier increase and later slight decrease in adopting Litecoin. The 3-dimensional FGLVM for Bitcoin, Litecoin and Ripple suggests good accuracy (MAPE=16) for all Bitcoin forecasting values while reasonable accuracy is suggested for the last 300 forecasting values of Litecoin and Ripple with MAPE=28 and MAPE=29 respectively. The 3-dimensional FGLVM is accurately better than the 3-dimensional GLVM for the forecasting values of Bitcoin but also the GLVM is accurately better than the FGLVM by considering the forecasting values of Litecoin and Ripple. The 3-dimensional FGLVM for last 100 days reveals a constant adoption of Bitcoin and a later decrease in adopting both Litecoin and Ripple.

The study shows that transaction counts of Bitcoin are relatively higher than that of Litecoin with Ripple transaction counts less than that of Litecoin along the study time.

The future work will consist of conducting a comparative study of the performance of classical Grey Model, Grey Lotka-Volterra Model and Fractional Grey Lotka-Volterra Model for Bitcoin, Litecoin and Ripple.

References

[1] G. Leibniz, *Mathematische Schriften*. Georg Olms Verlagsbuchhandlung, Hildesheim, 1962.

[2] H. Koçak and A. Yildirim, "An efficient new iterative method for finding exact solutions of non-linear time-fractional partial differential equations," *Modelling and Control* **16** *(4), 403-414*, 2011.

[3] H. Farid, "Discrete-time fractional differentiation from integer derivatives," *TR* **2004***, 528-536*, 2004.

[4] P. Ostalczyk and D. Brzezinński, "Numerical evaluation of Variable-Fractional-Order derivatives," *Automatyka* **15** *(3), 431-441*, 2011.

[5] F. Ikeda, "A numerical algorithm of discrete fractional calculus by using inhomogeneous sampling data," *Trans. of the Society of Instrument and Control Engineers* **6** *(1), 1-8*, 2007.

[6] V. E. Tarasov, "Differential equations with fractional derivative and universal map with memory," *J. Phys. A: Math. Theor.* **42***, doi: 10.1088/1751-8113/42/46/465102*, 2009.

[7] P. Kotha and B. T. Krishna, "Comparative study of fractional order derivative based on image enhancement techniques," *International Journal of Research in Computer and Communication Technology* **3** *(2), 231-235*, 2014.

[8] D. Y. Liu, O. Gibaru, W. Perruquetti, and T. M. Laleg-Kirati, "Fractional order differentiation by integration and error analysis in noisy environment," *IEEE Transactions on Automatic Control* **60** *(11), 2945-2960*, 2015.

[9] H. Bao, J. H. Park, and J. Cao, "Synchronization of fractional-order delayed neural networks with hybrid coupling," *Complexity,* **21** *(S1), 106-112*, 2016.

[10] Y. Shen, B. He, and P. Qin, "Fractional-order grey prediction method for non-equidistant sequences," *Entropy 2016,* **18** *(6), 227*, 2016.

[11] D. Baleanu and A. Fernandez, "On some new properties of fractional derivatives with Mittag-Leffler kernel," *Communications in Nonlinear Science and Numerical Simulation* **59***, 444-462*, 2017.

[12] M. Ciesielski and T. Blaszczyk, "The multiple composition of the left and right fractional Riemann-Liouville integrals-analytical and numerical calculations," *Filomat* **31** *(19), 6087-6099*, 2017.

[13] H. B. Bao, J. D. Cao, and J. Kurths, "State estimation of fractional-order delayed memristive neural networks," *Nonlinear Dynamics,* **94** *(2), 1215-1225*, 2018.

[14] J. Zhang, J. Wu, H. Bao, and J. Cao, "Synchronization analysis of fractional-order three-neuron BAM neural networks with multiple time delays," *Applied Mathematics and Computation,* **339** *(C), 441-450*, 2018.

[15] V. Zakharchenko and I. G. Kovalenko, "Best approximation of the fractional semi-derivative operator by exponential series," *Mathematics* **6** *(12), doi:10.3390/math6010012*, 2018.

[16] L. Wu, S. Liu, and Y. Wang, "Grey Lotka-Volterra model and its applications," *Technological Forecasting & Social Change* **79***, 1720-1730*, 2012.

[17] P. Gatabazi, J. C. Mba, E. Pindza, and C. Labuschagne, "Grey Lotka-Volterra model with application to cryptocurrencies adoption," *Chaos, Solitons & Fractals,* **122***, 47-57*, 2019.

[18] S. B. Angela, "Ten types of digital currencies and how they work," *Online trading: Free introductory eBook, September 24, 2016*, 2016.

[19] A. Blundell-Wignall, "The Bitcoin question: Currency versus trust-less transfer technology," *OECD Working Papers on Finance, Insurance and Private Pensions, No 37, OECD Publishing*, 2014.

[20] J. R. Hendrickson, T. L. Hogan, and W. J. Luther, "The political economy of Bitcoin," *Economics Inquiry,* **54** *(2), 925-939*, 2016.

[21] P. Wayner, *Digital Cash.* AP Professional, London, 2^{nd} ed., 1997.

[22] A. Urquhart, "The inefficiency of Bitcoin," *Economics Letters,* **148***, 80-82*, 2016.

[23] S. Chan, J. Chu, S. Nadarajah, and J. Osterrieder, "A statistical analysis of cryptocurrencies," *Journal of Risk and Financial Management* **2017**, 2017.

[24] J. Bhosale and S. Mavale, "Volatility of select crypto-currencies: A comparison of Bitcoin, Ethereum and Litecoin," *Annual Research Journal of SCMS, Pune* **6**, 2018.

[25] Hsi-Tse Wang and Ta-Chu Wang, "Application of grey Lotka-Volterra model to forecast the diffusion and competition analysis of the TV and smart-phone industries," *Technological Forecasting Social Change* **106***, 37-44*, 2016.

[26] Nai-Ming Xie and Si-Feng Liu, "Discrete grey forecasting model and its optimization," *Applied Mathematical Modelling* **33***, 1173-1186*, 2009.

[27] N. M. Xie, S. F. Liu, Y. J. Yang, and C. Q. Yuan, "On a novel grey forecasting model based on no-homogeneous index sequence," *Applied Mathematical Modelling* **37** *, 5059-5068*, 2013.

[28] W. Zhou and Jian-Min He, "Generalised GM(1,1) model and its application in forecasting of fuel production," *Applied Mathematical Modelling* **37***, 6234-6243*, 2013.

[29] A. M. Lyapunov, *The general problem of the stability of motion*. Taylor & Francis, London, 1992.

[30] Y. Povstenko, *Linear fractional diffusion-wave equation for scientists and engineers*. Bikhäuser, New York, 2015.

[31] J. B. Diaz and T. J. Osier, "Differences of fractional order," *Mathematics of computation* **28** *(125), 185-202*, 1974.

[32] J. J. Mohan and G. V. S. R. Deekshitulu, "Fractional order difference equations," *International journal of differential equations* **2012***, Article ID 780619*, 2012.

[33] C. Strobeck, "N species competition," *Ecology,* **54** *(3), 650-654*, 1973.

[34] M. C. Anisiu, "Lotka-Volterra and their model," *Didactica Mathematica* **32***, 9-17*, 2014.

[35] T. L. Tien, "A new grey prediction model FGM(1,1)," *Mathematical and Computer Modelling* **49***, 1416-1426*, 2009.

[36] L. F. Wu, S. F. Liu, W. Cui, D. L. Liu, and T. X. Yao, "Non-homogeneous discrete grey model with fractional-order accumulation," *Neural Comput. Appl.,* **25***, 1215-1221*, 2014.

[37] R. A. Thietart and B. Forgues, "Chaos theory and organization," *Organization Science,* **6** *(1), 19-31*, 1995.

Paper 3

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MODELING CRYPTOCURRENCIES TRANSACTION COUNTS USING VARIABLE-ORDER FRACTIONAL GREY LOTKA-VOLTERRA DYNAMICAL SYSTEM

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Abstract: Fractional Grey Lotka-Volterra Model with variable order is introduced and used for modeling the transaction counts of three cryptocurrencies namely Bitcoin, Litecoin and Ripple. Bitcoin and Litecoin then both three cryptocurrencies transaction counts are modeled in 2 and 3-dimensional framework respectively. Dataset include transaction counts of cryptocurrencies of interest. The 2-dimensional model uses Bitcoin and Litecoin transactions from April, 28, 2013 to February, 10, 2018. The 3-dimensional model uses transactions from August, 7, 2013 to February, 10, 2018. The actual values and the model values of n-dimensional model $n = \{2,3\}$ are displayed. The Mean Absolute Percentage Error (MAPE) suggests a high accuracy of the 3-dimensional Variable-order Fractional Lotka-Volterra model (VFGLVM) for the overall model values of Bitcoin and a reasonable accuracy for both model values of Litecoin and Ripple. The 2-dimensional VFGLVM has a good accuracy for the overall forecasting values of Bitcoin and a reasonable accuracy for the forecasting values of Litecoin. By analysing values of Lyapunov exponents and patterns of the corresponding Lotka-Volterra models, the 2 and 3-dimensional models show a chaotic behavior. Forecasting values indicate a future slight linear increase in transacting Bitcoin and a future decreasing transaction of Litecoin and Ripple. Bitcoin will keep relatively higher transaction counts and Litecoin transaction counts will be everywhere higher than that of Ripple.

Keywords: Fractional derivative, Lotka-Volterra, Grey Model, Mean Absolute Percentage Error, chaos, Lyapunov exponents.

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1. Introduction

The fractional calculus consists of defining real or complex powers of the integration operator $\mathscr{I}$ and differentiation operator $\mathscr{D}$. Several ways of defining fractional integral and differentiation include Riemann-Liouville, Hadamard, Caputo and Grünwald-Letnikov approaches [1].
In Caputo fractional derivative, we consider a continuously differentiable function $\theta(t)$ on [a,b] and a fractional order q, $n-1<q<n$, $n\in\mathbb{Z}$:

$$\mathscr{D}_t^q\theta(t)=\frac{1}{\Gamma(n-a)}\int_a^t\frac{\theta^{(n)}(s)ds}{(t-s)^{q-n+1}},$$

where $\Gamma(x)$ is given by

$$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt,\ x\in\mathbb{C}.$$

The discrete form of (1) for any sequence of complex numbers $f(n)$, can be written as the following difference equation [11]:

$$\mathscr{D}_t^q\theta(n)=\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(q+1)}{\Gamma(q-k+1)\Gamma(k+1)}\theta(n+q-k).$$

The recent manuscripts on the theory on fractional calculus include [4], [5], [17], [7], [6], [11], [15], and [22]. Fractional calculus has many applications in various studies such as the discretization in fractional differentiation [3], the iterative methods in fractional calculus found in [22], the fractional order determination [21], the study on numerical approach of fractional differentiation [27], the algorithm of the variable fractional order [29], the study on discrete time fractional calculus [11], study on numerical discrete time fractional calculus [20] and many others recent studies such as for example [8], [24], [23], [31], [28], [10], [1], [14], [16] and [34].

In the present study, the fractional Lotka-Volterra Model with variable order $q(t)$ at time t is applied to the cryptocurrency adoption. A pair of Bitcoin and Litecoin is considered by the 2-dimensional model while a triplet Bitcoin, Litecoin and Ripple is the interest of the 3-dimensional model. The details on cryptocurrencies is found for example in [2], [9], [18], [35] and [33]. Grey Lotka-Volterra model were applied to the concurrency adoption in [13] and presented better results than that of classical Grey Model. However, the results by applying fractional differentiation with constant order rendered the model much more accurate as shown in [12]. The natural high variability observed in transaction counts brings idea on a relatively better model based on fractional differentiation with specific order $q(t)$ at time t. In the present study, the order $q(t)$ as a rate of change at time t is estimated by the slops of the regression lines of transaction counts of Bitcoin and counterpart cryptocurrencies.

This study assesses a chaotic behavior of the model by checking the values of the Lyapunov exponents (described in [25]) and by observing the pattern of the corresponding estimated LVM. Lyapunov [25] shows that for a regular dynamical system of the first approximation is where maximal Lyapunov exponent is negative, the solution of the original system is asymptotically stable, while for a dynamical system with at least one positive Lyapunov exponent, a strange attractor is generated by a chaotic dynamical system.

The accuracy of VFGLVM will be measured by the Mean Absolute Percentage Error (MAPE) criterion used for example in [19], [26, 36], [38] or [39]. The MAPE is given by the formula $\text{MAPE} = \frac{100}{n}\sum_{i=1}^{n}\left|\frac{X_i-\widehat{X}_i}{X_i}\right|$, where X_i and $\widehat{X}_i$ are respectively the i^{th} observed and estimated quantities. The model is highly accurate for MAPE less than 10. The accuracy of the model is good when MAPE range from 10 to 20, reasonably good if MAPE is between 20 and 50. There is lack of accuracy if MAPE is 50 or above.

Including the introduction, the study comprises 4 sections: Section 2 presents the methodology of the study, that is a description of the VFGLVM and a description of the datasets. Section 3 presents the main results and their interpretation and Section 4 gives a conclusion.

2. Methodology

2.1. q(t)-Fractional accumulation

Let $X_i^{(0)}$ be the original data sequences, that is

$$X_i^{(0)} = \left(x_i^{(0)}(1), x_i^{(0)}(2), \ldots, x_i^{(0)}(n)\right)$$

with the corresponding first order accumulation generating operations (1-AGO) given by:

$$X_i^{(1)} = \left(x_i^{(1)}(1), x_i^{(1)}(2), \ldots, x_i^{(1)}(n)\right)$$

with

$$x_i^{(1)}(k) = \sum_{j=1}^{k} x_i^{(0)}(j), \; k = 1, 2, \ldots, n. \quad (1)$$

Equation (1) can be written in matrix form as

$$\mathbf{X}^{(1)} = \mathbf{X}^{(0)}\mathbf{U} \tag{2}$$

where $\mathbf{U}$ is given by

$$\mathbf{U} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 0 & 1 & \dots & 1 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}. \tag{3}$$

The second order accumulation generating operations (2-AGO) is then given by

$$\begin{aligned} \mathbf{X}^{(2)} &= \mathbf{X}^{(1)}\mathbf{U} \\ &= \mathbf{X}^{(0)}\mathbf{U}^2. \end{aligned}$$

The M^{th} order accumulated sequence, $M \in \mathbb{N}$ is given by

$$\mathbf{X}^{(M)} = \mathbf{X}^{(0)}\mathbf{U}^M, \tag{4}$$

where elements of $\mathbf{U}^M$ are

$$u_{ik}^M = \begin{cases} M(M+1)(M+2)\dots(M+k-i-1) & \text{if } i < k \\ 1 & \text{if } i = k\,. \\ 0 & \text{if } i > k \end{cases} \tag{5}$$

The k^{th} accumulation in $X^{(M)}$ is then given by

$$x^{(M)}(k) = \sum_{i=1}^{k} u_{ik}^M x^{(0)}(i). \tag{6}$$

The fractional accumulation generating operations of order q $\forall q \in \mathbb{R}^+$ $(q - AGO)$ for any sequence $X^{(q)} = \{x^{(q)}(1), x^{(q)}(2), \dots, x^{(q)}(n)\}$ is then given by

$$\mathbf{X}^{(q)} = \mathbf{X}^{(0)}\mathbf{U}^q \tag{7}$$

with elements of $\mathbf{U}^q$ given by

$$u_{ik}^q = \begin{cases} \frac{\Gamma(q+k-i)}{\Gamma(q)\Gamma(k-i+1)} & \text{if } i < k \\ 1 & \text{if } i = k\,; \\ 0 & \text{if } i > k \end{cases} \tag{8}$$

[37]. Equation (6) yields the k^{th} fractional accumulation as

$$\begin{aligned} x^{(q)}(k) &= \sum_{i=1}^{k} u_{ik}^{q} x^{(0)}(i) \\ &= \sum_{i=1}^{k} \frac{\Gamma(q+k-i)}{\Gamma(q)\Gamma(k-i+1)} x^{(0)}(i). \end{aligned} \tag{9}$$

Assuming that the order q is variable along $[1,n]$, $q = q(t)\ \forall t \in [1,n]$, Equation (9) can be written as

$$x^{[q(t)]}(k) = \sum_{i=1}^{k} \frac{\Gamma[q(t)+k-i]}{\Gamma[q(t)]\Gamma(k-i+1)} x^{(0)}(i). \tag{10}$$

Equation (10) is the expression of the k^{th} fractional accumulation with variable order $q(t)$, $\forall t \in [1,n]$.

2.2. Variable-order Fractional Grey Lotka Volterra Model (VFGLVM)

Consider the general Lotka-Volterra model of competing relationships between n species [30], that is

$$\begin{cases} \frac{dX_1}{dt} = X_1 \left(a_1 - \sum_{j=1}^{n} \alpha_j X_j \right) \\ \frac{dX_2}{dt} = X_2 \left(a_2 - \sum_{j=1}^{n} \alpha_j X_j \right) \\ \vdots \\ \frac{dX_n}{dt} = X_n \left(a_n - \sum_{j=1}^{n} \alpha_j X_j \right) \end{cases} \tag{11}$$

or equivalently

$$\frac{dX_i}{dt} = X_i \left(a_i - \sum_{j=1}^{n} \alpha_j X_j \right) \tag{12}$$

where parameters $a_{i,\ i\in[1,n]}$ represent the capacity of growing of populations $X_{i,\ i\in[1,n]}$, while parameters $\alpha_{j,\ j\in[1,n]}$ represent the effect species j has on species i, the expressions X_i^2 are interactions within species, $X_i\, X_j$, $i \neq j$ are interactions of different species.
Let $Z_i^{[q(t)]}$ be the mean sequence of $X_i^{[q(t)]}$, that is

$$Z_i^{[q(t)]} = \left(z_i^{[q(t)]}(2), z_i^{[q(t)]}(3), \ldots, z_i^{[q(t)]}(n) \right) \tag{13}$$

with

$$z_i^{(q(t))}(k) = \frac{x^{[q(t)]}(k) + x^{[q(t)]}(k-1)}{2}, \; k = 2,3,\ldots,n \tag{14}$$

Applying the variable-order fractional grey model to the system (12) yields the following approximations:

$$x_i^{(0)}(k+1) \approx a_i z_i^{[q(t)]}(k) - b_i \left(z_i^{[q(t)]}(k)\right)^2 - \sum_{j\neq i}^{n} c_j z_i^{[q(t)]}(k) z_j^{[q(t)]}(k); \tag{15}$$

with error sequences expressed by

$$\varepsilon_i = x_i^{(0)}(k+1) - \left(a_i z_i^{[q(t)]}(k) - b_i \left(z_i^{[q(t)]}(k)\right)^2 - \sum_{j\neq i}^{n} c_i z_i^{[q(t)]}(k) z_j^{[q(t)]}(k)\right); \tag{16}$$

The least square estimates of parameters in (15) are given by

$$\begin{pmatrix} \hat{a}_i \\ \hat{b}_i \\ \hat{c}_i \end{pmatrix} = (B_i' B_i)^{-1} B_i' M_i \tag{17}$$

where,

$$B_i = \begin{pmatrix} z_i^{[q(t)]}(2) & -\left(z_i^{[q(t)]}(2)\right)^2 & -z_i^{[q(t)]}(2) z_1^{[q(t)]}(2) & \ldots & -z_i^{[q(t)]}(2) z_j^{[q(t)]}(2) \\ z_i^{[q(t)]}(3) & -\left(z_i^{[q(t)]}(3)\right)^2 & -z_i^{[q(t)]}(3) z_1^{[q(t)]}(3) & \ldots & -z_i^{[q(t)]}(3) z_j^{[q(t)]}(2) \\ \vdots & \vdots & \vdots & & \\ z_i^{[q(t)]}(n) & -\left(z_i^{[q(t)]}(n)\right)^2 & -z_i^{[q(t)]}(n) z_1^{[q(t)]}(n) & \ldots & -z_i^{[q(t)]}(n) z_j^{[q(t)]}(2) \end{pmatrix}; \forall j \neq i;$$

$$M_i = \begin{pmatrix} x_i^{(0)}(2) \\ x_i^{(0)}(3) \\ \vdots \\ x_i^{(0)}(n) \end{pmatrix}$$

2.3. Datasets

Data on daily transaction counts of Bitcoin and Litecoin from 28-April-2013 up to 10-February-2018 is considered for the 2-dimensional analysis while 3-dimensional study takes daily transaction counts of Bitcoin, Litecoin and Ripple from 7-August-2013 up to 10-February-2018. The evolution in transaction counts of Bitcoin and Litecoin along the study time and that of both Bitcoin, Litecoin and Ripple recorded daily with the related graphs are presented in [13] and can also be found via the authors of this paper.

3. Results and interpretation

3.1. 2-dimensional Variable-order Fractional Grey Lotka-Volterra model for Bitcoin and Litecoin

Applying model (15) to the dataset, Equations (17) give the following least square estimates of 2-dimensional model $q(t)$-parameters with $q(t) = \begin{cases} 0.0196 \text{ if } t \leq 1665 \\ 0.2717 \text{ if } t > 1665 \end{cases}$.

$$\begin{pmatrix} \hat{a}_1 \\ \hat{b}_1 \\ \hat{c}_1 \end{pmatrix} = \begin{pmatrix} 1.045 \\ 6.553 \times 10^{-7} \\ 1.843 \times 10^{-7} \end{pmatrix}, \quad \begin{pmatrix} \hat{a}_2 \\ \hat{b}_2 \\ \hat{c}_2 \end{pmatrix} = \begin{pmatrix} 1.000 \\ -2.711 \times 10^{-20} \\ 2.711 \times 10^{-20} \end{pmatrix}.$$

The $q(t)$-Fractional Grey Lotka-Volterra model (15) can be written as follows :

$$\begin{cases} x^{(0)}(k+1) \approx 1.045 z_x^{[q(t)]}(k) - 6.553 \times 10^{-7} \left(z_x^{[q(t)]}(k) \right)^2 \\ \qquad -1.843 \times 10^{-7} z_x^{[q(t)]}(k) z_y^{[q(k)]}(k) \\ y^{(0)}(k+1) \approx 1.000 z_y^{[q(t)]}(k) + 2.711 \times 10^{-20} \left(z_y^{[q(t)]}(k) \right)^2 \\ \qquad -2.711 \times 10^{-20} z_y^{[q(t)]}(k) z_x^{[q(t)]}(k) \end{cases} \tag{18}$$

$k = 1, 2, \ldots, n$.

The Lyapunov exponents at the trivial point of equilibrium of the corresponding LVM are all positive ($\boldsymbol{\lambda}_1 = 1.045$, $\boldsymbol{\lambda}_2 = 1.000$) and therefore the model is a chaotic dynamical system. The chaotic behavior of the model in the sens of Thietart and Forgues [32] is confirmed by the pattern of the system (Figure 1) which connects incompatible figures.

Under the MAPE, Model (18) is good for the overall model values of Bitcoin (MAPE=10) and reasonably good for the overall model values of Litecoin (MAPE=27). The accuracy of the VFGLVM is relatively better than that of GLVM and FGLVM where MAPE=22 and MAPE=16 for the overall model values of Bitcoin while reasonable accuracy is found at the last 300 model values of Litecoin [12, 13].

The last 50 values of Bitcoin (BTC) and Litecoin (LTC) for GLVM, FGLVM and VFGLVM are recorded in Table 1 and the VFGLVM values are relatively good as compared to that of GLVM and FGLVM. Forecasting values show that in future, there will be a linear slight increase in adopting Bitcoin and a decrease in adopting Litecoin as confirm Figures 2 and 3.

Table 1: Last 50 forecasting values of GLVM , FGLVM and VFGLVM for Bitcoin and Litecoin daily transaction counts.

N0	Actual values		GLVM values		FGLVM values		VFGLVM values	
	BTC	LTC	BTC	LTC	BTC	LTC	BTC	LTC
1	308072	117738	281883	58816	117738	83781	298770	329598
2	279371	81111	281846	59721	81111	81279	324419	297909
3	228791	77925	281806	60450	77925	78595	345468	272642
4	247298	82613	281775	61191	82613	77655	355103	266293
5	307486	112765	281742	62098	112765	79292	345748	280244
6	304904	111207	281710	63145	111207	81881	333217	297317
7	353659	155481	281689	64404	155481	86079	317014	324493
8	344260	141900	281675	65822	141900	90707	299515	349882
9	290259	105948	281648	67016	105948	90967	309281	336587
10	241601	83076	281614	67934	83076	88634	332162	309167
11	340809	127924	281579	68965	127924	89616	325977	315242
12	395806	186764	281560	70518	186764	96864	283735	368258
13	424840	225860	281573	72580	225860	107323	237373	432628
14	342564	197217	281606	74724	197217	114660	231025	461614
15	358679	173712	281626	76629	173712	117045	243974	453823
16	368025	143412	281607	78278	143412	116989	239729	434392
17	345506	146511	281575	79800	146511	116824	241463	421387
18	360101	145848	281546	81350	145848	118028	240218	421827
19	347227	140304	281512	82881	140304	119080	237125	420327
20	337766	120843	281469	84292	120843	118748	242430	409282
21	299913	106887	281418	85531	106887	117063	257413	391282
22	265586	93443	281372	86629	93443	114847	280219	373291
23	234890	88779	281334	87633	88779	112714	301614	358561
24	273473	90381	281291	88627	90381	111522	306931	351332
25	303566	117447	281244	89786	117447	112833	293661	361655
26	315604	113111	281197	91081	113111	114988	279673	374176
27	309322	95276	281134	92259	95276	114860	275959	366696
28	243454	70009	281068	93199	70009	112304	294099	344551
29	240433	66798	281004	93981	66798	109364	314103	324536
30	215435	55466	280940	94683	55466	106862	326028	310023
31	245395	61730	280870	95358	61730	105015	330419	300748
32	271759	59717	280786	96060	59717	104087	323302	297390
33	250247	59072	280698	96749	59072	103090	322181	292475
34	236422	61836	280622	97453	61836	102372	328233	290187
35	220304	57452	280553	98150	57452	101598	335037	286928
36	193421	49382	280489	98777	49382	100148	345008	278358
37	213288	51278	280424	99369	51278	98836	349854	271605
38	232028	50067	280346	99967	50067	98020	347093	268334
39	236442	55270	280264	100590	55270	97598	343861	267599
40	204159	54531	280192	101242	54531	97346	347186	268072
41	257504	57962	280114	101912	57962	97228	344846	268381
42	235750	66669	280034	102656	66669	97823	338642	273718
43	194733	49384	279967	103352	49384	97381	347238	270270
44	173509	45225	279907	103921	45225	95762	358705	258648
45	216178	51043	279841	104502	51043	95045	359321	255887
46	243950	59946	279761	105173	59946	95480	350603	261038
47	213578	50320	279681	105842	50320	95451	349323	260901
48	173158	37148	279609	106375	37148	93887	359780	249179
49	177725	44811	279546	106875	44811	92676	366417	242930
50	181640	46594	279485	107434	46594	92432	367165	244531

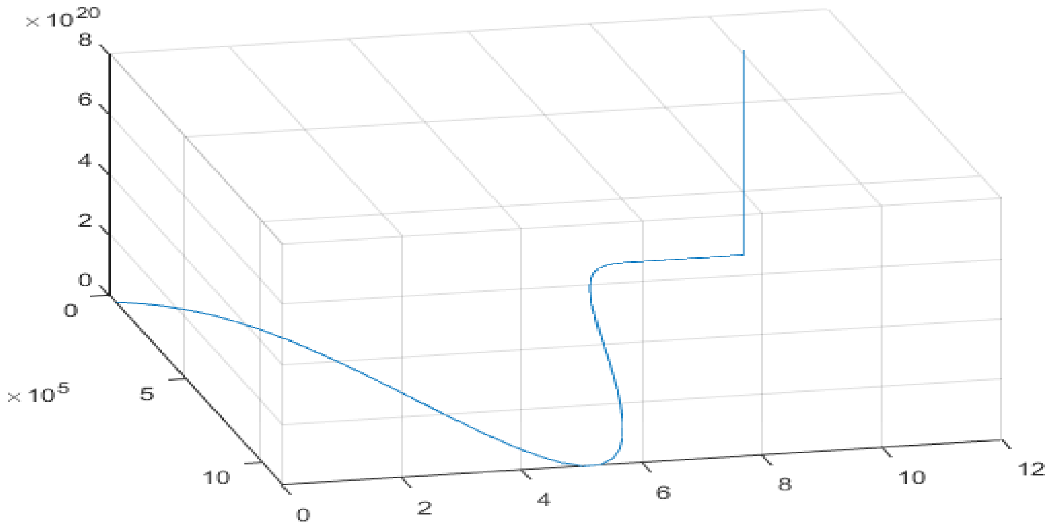

Figure 1: 2-dimensional LVM plot with initial conditions $X(0) = 40035$; $Y(0) = 9408$.

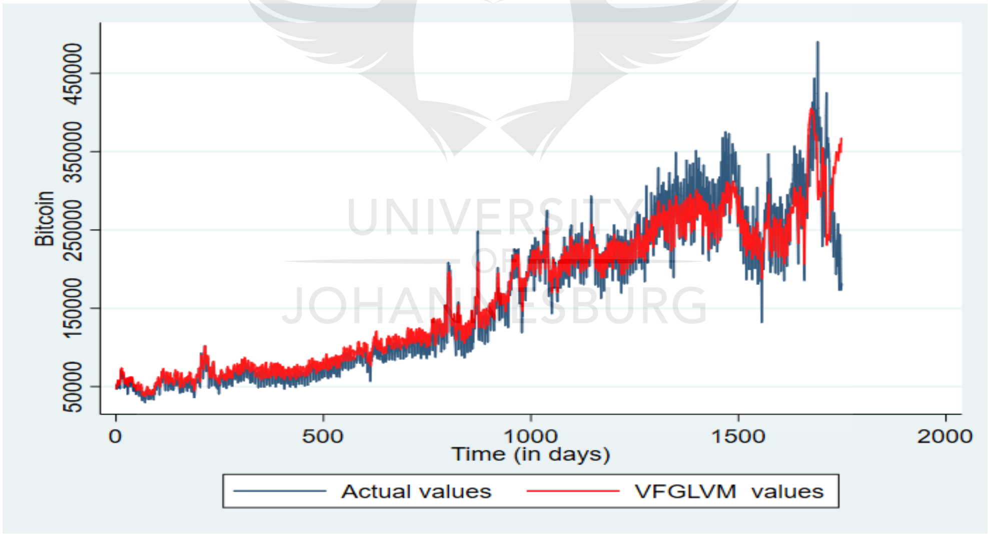

Figure 2: Transaction counts and forecasting values of Bitcoin

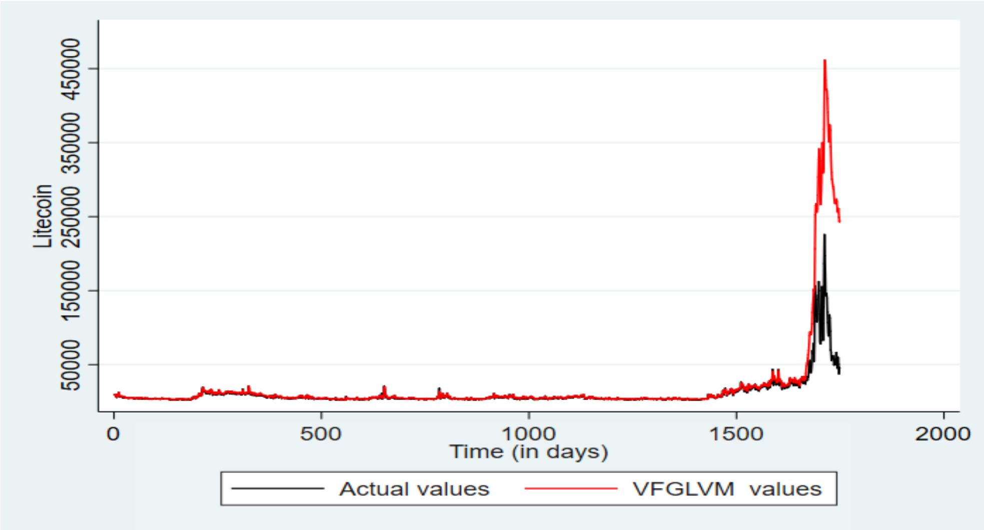

Figure 3: Transaction counts and forecasting values of Litecoin

3.2. 3-dimensional Variable-order Fractional Grey Lotka-Volterra model for Bitcoin, Litecoin and Ripple

We Apply Model (15) to the dataset. Equation (17) gives the following least square estimation of model $q(t)$-parameters with $q(t) = \begin{cases} 0.03385 \text{ if } t \leq 1551 \\ 0.21315 \text{ if } t > 1551 \end{cases}$.

$$\begin{pmatrix} \hat{a}_1 \\ \hat{b}_1 \\ \hat{c}_1 \\ \hat{d}_1 \end{pmatrix} = \begin{pmatrix} 9.377 \times 10^{-1} \\ 4.862 \times 10^{-7} \\ 4.804 \times 10^{-7} \\ 1.789 \times 10^{-7} \end{pmatrix}, \begin{pmatrix} \hat{a}_2 \\ \hat{b}_2 \\ \hat{c}_2 \\ \hat{d}_2 \end{pmatrix} = \begin{pmatrix} 7.359 \times 10^{-1} \\ -2.586 \times 10^{-6} \\ 4.047 \times 10^{-7} \\ 4.248 \times 10^{-6} \end{pmatrix}, \begin{pmatrix} \hat{a}_3 \\ \hat{b}_3 \\ \hat{c}_3 \\ \hat{d}_3 \end{pmatrix} = \begin{pmatrix} 9.145 \times 10^{-1} \\ -6.356 \times 10^{-7} \\ 6.163 \times 10^{-7} \\ 1.717 \times 10^{-7} \end{pmatrix}.$$

The expression of the q(t)-Fractional Grey Lotka-Volterra model (15) can be written as follows :

$$\begin{cases} x^{(0)}(k+1) \approx 9.377 \times 10^{-1} z_x^{[q(t)]}(k) - 4.862 \times 10^{-7} \left(z_x^{[q(t)]}(k)\right)^2 \\ \qquad - 4.804 \times 10^{-7} z_x^{[q(t)]}(k) z_y^{[q(t)]}(k) - 1.789 \times 10^{-7} z_x^{[q(t)]}(k) z_w^{[q(t)]}(k) \\ y^{(0)}(k+1) \approx 7.359 \times 10^{-1} z_y^{[q(t)]}(k) + 2.586 \times 10^{-6} \left(z_y^{[q(t)]}(k)\right)^2 \\ \qquad - 4.047 \times 10^{-7} z_y^{[q(t)]}(k) z_x^{[q(t)]}(k) - 4.248 \times 10^{-6} z_y^{[q(t)]}(k) z_w^{[q(t)]}(k) \\ w^{(0)}(k+1) \approx 9.145 \times 10^{-1} z_w^{[q(t)]}(k) + 6.356 \times 10^{-7} \left(z_w^{[q(t)]}(k)\right)^2 \\ \qquad - 6.163 \times 10^{-7} z_w^{[q(t)]}(k) z_x^{[q(t)]}(k) - 1.717 \times 10^{-7} z_w^{[q(t)]}(k) z_y^{[q(t)]}(k) \end{cases} \tag{19}$$

$k = 1, 2, \ldots, n$.

The Lyapunov exponents at the trivial point of equilibrium $(0,0,0)$ of the corresponding LVM are all positive ($\boldsymbol{\lambda}_1 = 9.377 \times 10^{-1}$, $\boldsymbol{\lambda}_2 = 7.359 \times 10^{-1}$, $\boldsymbol{\lambda}_3 = 9.145 \times 10^{-1}$) and therefore the model is a chaotic dynamical system. The pattern of the LLVM shows also a chaotic behavior confirmed by the connected incompatible figures (Figure 4).

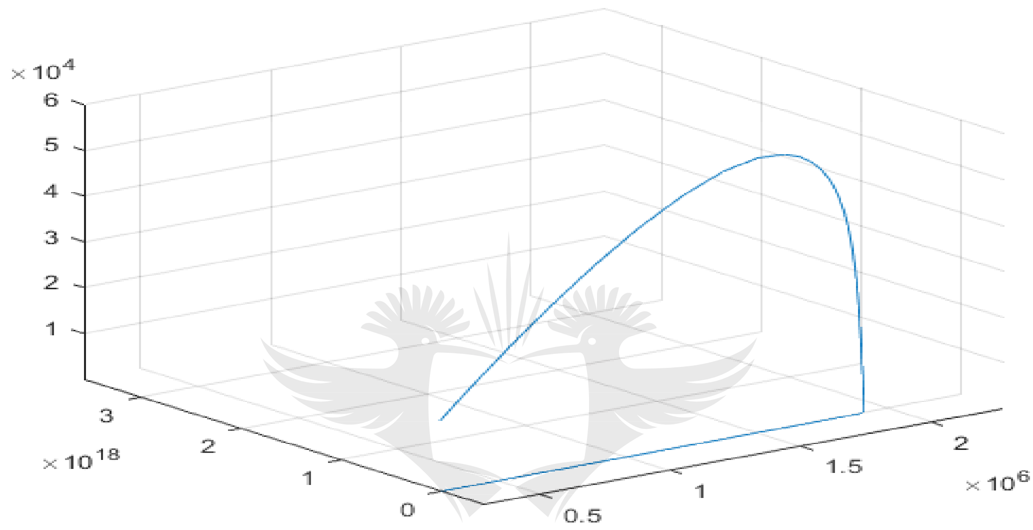

Figure 4: 3-dimensional LVM plot with initial conditions $X(0) = 56974$; $Y(0) = 4385$; $W(0) = 3335$.

Under the MAPE, Model (19) is highly accurate for the overall values of Bitcoin (MAPE=9). Model (19) is reasonably accurate also for the overall values of Litecoin and Ripple with MAPE=24 and MAPE=41 respectively. As for the 2-dimensional VFGLVM, the accuracy of the 3-dimensional VFGLVM is relatively better than that of 3-dimensional GLVM and FGLVM where MAPE=24 and MAPE=16 for the overall values of Bitcoin and reasonable accuracy found at the last 300 values of Litecoin as presented in [12] and in [13].

The last 50 values of Bitcoin (BTC), Litecoin (LTC) and Ripple (RPL) for GLVM, FGLVM and VFGLVM are displayed in Table 1 and relatively better accuracy is observed in VFGLVM values. Forecasting values show that in future, there will be a linear slight increase in adopting Bitcoin and a decrease in adopting both Litecoin and Ripple as confirm Figures 5, 6 and 7.

Table 2: Last 50 forecasting values of GLVM , FGLVM and VFGLVM for Bitcoin, Litecoin and Ripple daily transaction counts.

N0	Actual values			GLVM values			FGLVM values			VFGLVM values		
	BTC	LTC	RPL	BTC	LTC	RPL	BTC	LTC	RPL	BTC	LTC	RPL
1	308072	117738	58024	383561	97843	71338	299966	101178	45364	304031	134565	55359
2	279371	81111	42168	358516	74034	52397	300274	98468	44316	325275	119914	52310
3	228791	77925	37359	342633	60291	41521	300560	95568	43208	338141	109830	50165
4	247298	82613	37608	338600	57434	39126	300755	94913	42822	340047	111602	49595
5	307486	112765	63774	357028	76300	53030	299526	95480	43876	333662	114047	53277
6	304904	111207	67160	381001	97792	68656	296950	95306	45743	324479	112268	59199
7	353659	155481	108608	413486	131100	92518	292179	93931	49018	308371	108069	69945
8	344260	141900	114645	447660	165983	117974	285382	89660	53227	291069	92060	83186
9	290259	105948	64844	417567	136237	94520	283998	87962	53839	302225	91638	79740
10	241601	83076	50779	368411	90252	60567	286447	88978	52087	321094	100392	69439
11	340809	127924	71079	372240	95482	63862	287046	91082	52294	318213	106762	67284
12	395806	186764	98324	409663	131504	89140	285880	98041	55641	285628	130620	70158
13	424840	225860	121276	445436	171415	116034	282879	105998	60988	241948	158394	76636
14	342564	197217	125177	467630	194408	130526	278006	106726	65888	228941	156289	88176
15	358679	173712	92750	444875	174657	115150	276165	106431	67641	241458	151491	89157
16	368025	143412	78515	409256	139902	90154	278011	109535	66841	252400	150607	77698
17	345506	146511	84686	404440	135056	85850	278169	109573	66681	260205	140523	75497
18	360101	145848	95356	416086	150049	94846	276134	107440	67958	259134	129183	79600
19	347227	140304	86853	416445	153163	96006	274463	105721	69025	259079	121159	80869
20	337766	120843	78796	405457	140322	87165	274118	104716	69014	267226	116755	78949
21	299913	106887	55717	381082	116497	70611	275573	105307	67775	281969	119106	74099
22	265586	93443	52404	360107	95630	56629	277526	106166	66186	297193	123689	70527
23	234890	88779	41844	348145	84806	49302	278936	106205	64868	307837	125588	69149
24	273473	90381	53869	351867	86473	50089	279542	105894	64189	311192	122832	68381
25	303566	117447	70937	372524	110858	65465	278125	104991	65255	303150	117897	71962
26	315604	113111	66633	381985	122844	72246	277044	105640	66536	294612	119533	73317
27	309322	95276	58456	373812	113014	65623	277143	105606	66405	298054	114024	70439
28	243454	70009	39989	352884	91056	51534	278184	104407	64981	312477	108421	67216
29	240433	66798	40426	337520	76134	42026	279258	102852	63424	323603	104678	65056
30	215435	55466	34088	332361	71408	38923	279854	100980	62215	329685	99770	63938
31	245395	61730	40442	334979	71658	38945	280154	99332	61360	332989	95087	63104
32	271759	59717	39433	337661	76482	41752	280356	98478	60864	334049	90216	61633
33	250247	59072	42078	338134	78223	42610	280321	97194	60446	335696	85984	61294
34	236422	61836	37857	336354	77178	41783	280317	96412	60182	336338	87383	61912
35	220304	57452	36261	333296	72354	38722	280677	96198	59771	337381	89812	61420
36	193421	49382	26703	324308	62961	32870	281382	95790	58963	339905	90438	59855
37	213288	51278	28291	317595	56103	28692	282385	95967	58080	342023	90919	57391
38	232028	50067	30034	319532	59246	30438	282941	95861	57529	344051	88060	56115
39	236442	55270	33106	322652	63678	32966	283158	95664	57263	344684	86177	55905
40	204159	54531	30543	324494	64331	33233	283262	95630	57147	343536	88376	56598
41	257504	57962	34994	325361	66274	34226	283454	95795	57028	343827	88006	56201
42	235750	66669	44598	335915	78957	41622	282708	95265	57536	341732	86203	58529
43	194733	49384	27038	328841	72145	37431	282535	94628	57461	341775	87571	59122
44	173509	45225	22840	314057	53009	26016	283628	94684	56440	343893	88806	55542
45	216178	51043	31168	316086	56796	28181	283907	94353	56056	344822	87154	55219
46	243950	59946	37098	326527	69782	35664	283609	94282	56322	345180	85198	56297
47	213578	50320	27775	323283	66952	33883	283861	94634	56215	345642	85732	55457
48	173158	37148	16700	306803	48817	23190	284971	94700	55193	347821	85649	52107
49	177725	44811	30748	311191	51615	24747	284825	93239	54780	346552	82967	53254
50	181640	46594	36859	326307	70083	35322	283140	90443	55321	344137	78624	58647

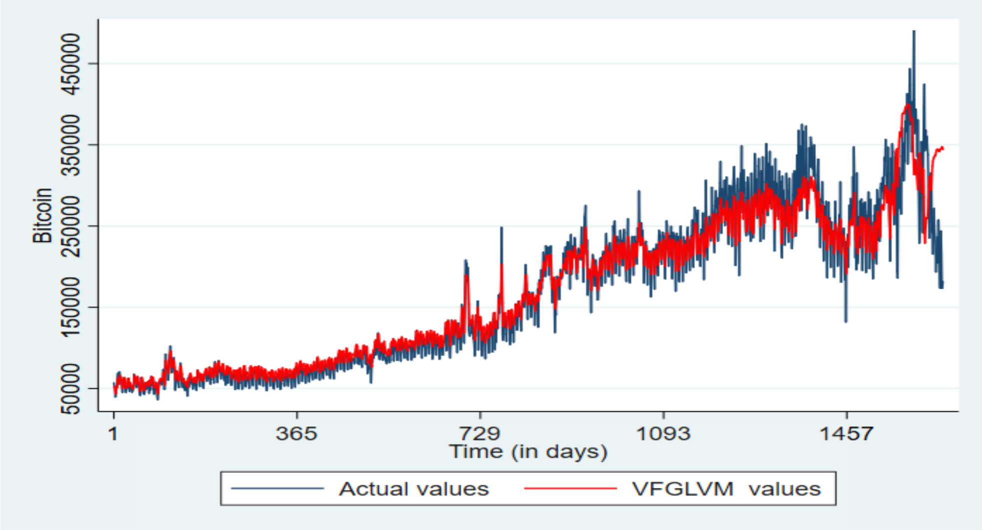

Figure 5: Transaction counts and VFGLVM values of Bitcoin

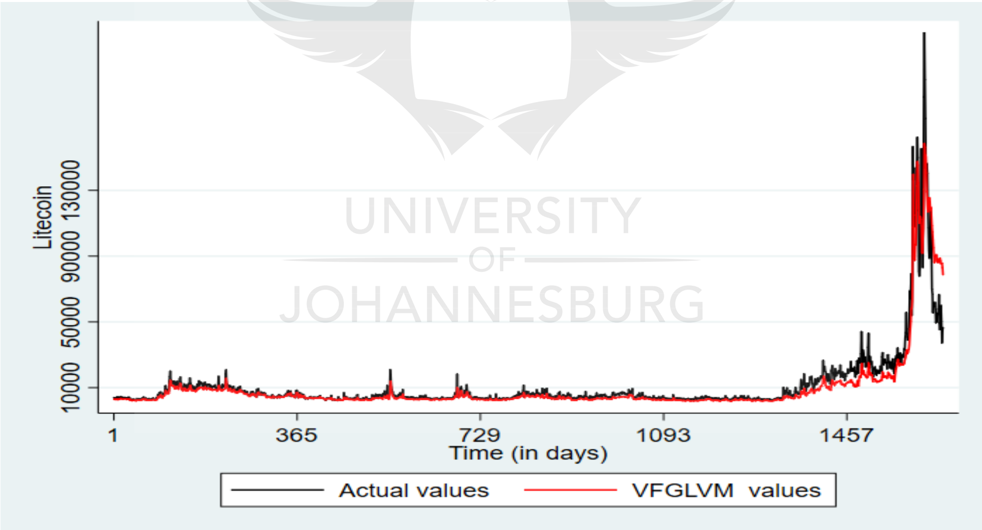

Figure 6: Transaction counts and VFGLVM values of Litecoin

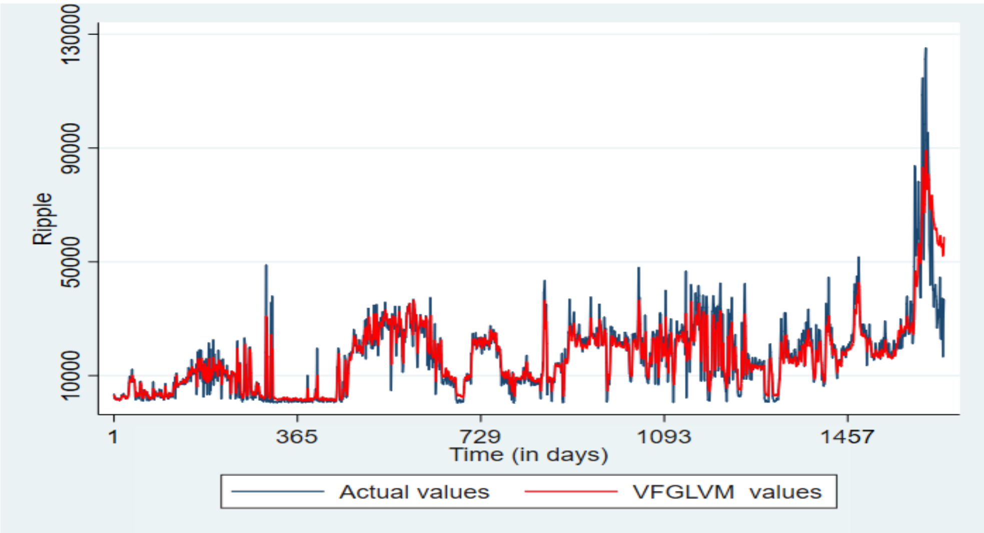

Figure 7: Transaction counts and VFGLVM values of Ripple

4. Conclusions

The variable-order is applied to the fractional discrete differentiation on the Grey Lotka-Volterra Model (GLVM). The Variable-order Fractional Grey Lotka-Volterra Model (VFGLVM) is then formulated and applied to the adoption of cryptocurrencies namely Bitcoin, Litecoin and Ripple. The chaotic behavior of the LVM through the VFGLVM is indicated by the Lyapunov exponents and graphs of the model using dataset on cryptocurrencies. Both patterns of the 2 and 3-dimensional models suggested a chaotic dynamical system. The chaotic behavior was also confirmed the positive Lyapunov exponents at the trivial equilibrium points $(0,0)$ and $(0,0,0)$. The accuracy of VFGLVM was checked by the Mean Absolute Percentage Error (MAPE) and was found relatively better than that of GLVM and FVLVM.

The MAPE for 2-dimensional study suggested that the model accuracy is good for overall forecasting values of Bitcoin (MAPE=10) and reasonable good Litecoin (MAPE=27). The 3-dimensional VFGLVM for Bitcoin, Litecoin and Ripple suggested high accuracy (MAPE=9) for all Bitcoin model and reasonable good accuracy for the model values of Litecoin and Ripple with MAPE=24 and MAPE=41 respectively. The VFGLVM revealed that in future, there will be a linear slight increase in adopting Bitcoin and a decrease in adopting both Litecoin and Ripple.
This study shows that transaction counts of Bitcoin will remain relatively higher than that of both Litecoin and Ripple with Ripple transaction counts less than that of Litecoin.

The future work will check the performance of fractional Grey modeling with variable order by different types of fractional differentiation.

References

[1] Almeida, R., Tavares, D., and Torres, D. F. M. (2018). *The variable-order fractional calculus of variations*. SpringerBriefs in Applied Sciences and Technology, Springer, Cham.

[2] Angela, S. B. (2016). Ten types of digital currencies and how they work. *Online trading: Free introductory eBook, September 24, 2016*.
URL http://www.techbullion.com/10-types-digital-currencies-work/

[3] Angstmann, C. N., Henry, B. I., Jacobs, B. A., and McGann, A. V. (2017). Discretization of fractional differential equations by a piecewise constant approximation. *Math. Model. Nat. Phenom.*, **0** *(0), 0-11*.

[4] Atangana, A., and Gómez-Aguilar, J. F. (2017). Hyperchaotic behaviour obtained via a nonlocal operator with exponential decay and Mittag-Leffler laws. *Chaos, Solitons & Fractals,* **102** *doi: 10.1016/j.chaos.2017.03.022*.

[5] Atangana, A., and Gómez-Aguilar, J. F. (2018). Decolonisation of fractional calculus rules: Breaking commutativity and associativity to capture more natural phenomena. *The European Physical Journal Plus,* **133***, 1-23*.

[6] Atangana, A., and Gómez-Aguilar, J. F. (2018). Fractional derivatives with no-index law property: application to chaos and statistics. *Chaos, Solitons & Fractals,* **114***, 516-535*.

[7] Atangana, A., and Gómez-Aguilar, J. F. (2018). A new derivative with normal distribution kernel: Theory, methods and applications. *Physica A: Statistical mechanics and its applications,* **476***, 1-14*.

[8] Baleanu, D., and Fernandez, A. (2017). On some new properties of fractional derivatives with Mittag-Leffler kernel. *Communications in Nonlinear Science and Numerical Simulation* **59***, 444-462*.

[9] Blundell-Wignall, A. (2014). The Bitcoin question: Currency versus trust-less transfer technology. *OECD Working Papers on Finance, Insurance and Private Pensions, No 37, OECD Publishing*.
URL http://dx.doi.org/10.1787/5jz2pwjd9t20-en

[10] Ciesielski, M., and Blaszczyk, T. (2017). The multiple composition of the left and right fractional Riemann-Liouville integrals-analytical and numerical calculations. *Filomat* **31** *(19), 6087-6099*.

[11] Farid, H. (2004). Discrete-time fractional differentiation from integer derivatives. *TR* **2004***, 528-536*.

[12] Gatabazi, P., Mba, J. C., and Pindza, E. (2019). Fractional Grey Lotka-Volterra model with application to cryptocurrencies adoption. *Chaos: An Interdisciplinary Journal of Nonlinear Science,* **29** *(7), 073116*.

[13] Gatabazi, P., Mba, J. C., Pindza, E., and Labuschagne, C. (2019). Grey Lotka-Volterra model with application to cryptocurrencies adoption. *Chaos, Solitons & Fractals,* **122***, 47-57*.

[14] Ghanbari, B., and Gómez-Aguilar, J. (2018). Modeling the dynamics of nutrient-phytoplankton-zooplankton system with variable-order fractional derivatives. *Chaos, Solitons & Fractals,* **116***, 114-120, doi: https://doi.org/10.1016/j.chaos.2018.09.026*.

[15] Ghanbari, B., and Kumar, D. (2019). Numerical solution of predator-prey model with Beddington-DeAngelis functional response and fractional derivatives with Mittag-Leffler kernel. *Chaos: An Interdisciplinary Journal of Nonlinear Science,* **29***, 063103, doi: 10.1063/1.5094546*.

[16] Ghanbari, B., Osman, M., and Baleanu, D. (2019). Generalized exponential rational function method for extended Zakharov-Kuzetsov equation with conformable derivative. *Modern Physics Letters A, doi: 10.1142/S0217732319501554*.

[17] Gómez-Aguilar, J. F., and Atangana, A. (2017). New insight in fractional differentiation: power, exponential decay and Mittag-Leffler laws and applications. *The European Physical Journal Plus,* **133***, 1-23*.

[18] Hendrickson, J. R., Hogan, T. L., and Luther, W. J. (2016). The political economy of Bitcoin. *Economics Inquiry,* **54** *(2), 925-939*.

[19] Hsi-Tse Wang, and Ta-Chu Wang (2016). Application of grey Lotka-Volterra model to forecast the diffusion and competition analysis of the TV and smart-phone industries. *Technological Forecasting Social Change* **106***, 37-44*.

[20] Ikeda, F. (2007). A numerical algorithm of discrete fractional calculus by using inhomogeneous sampling data. *Trans. of the Society of Instrument and Control Engineers* **6** *(1), 1-8*.

[21] Janno, J. (2016). Determination of the order of fractional derivative and kernel in an inverse problem for a generalized time fractional diffusion equation. *Electronic Journal of Differential Equations,* **2016** *(199), 1-28*.

[22] Koçak, H., and Yildirim, A. (2011). An efficient new iterative method for finding exact solutions of non-linear time-fractional partial differential equations. *Modelling and Control* **16** *(4), 403-414*.

[23] Kotha, P., and Krishna, B. T. (2014). Comparative study of fractional order derivative based on image enhancement techniques. *International Journal of Research in Computer and Communication Technology* **3** *(2), 231-235*.

[24] Liu, D. Y., Gibaru, O., Perruquetti, W., and Laleg-Kirati, T. M. (2015). Fractional order differentiation by integration and error analysis in noisy environment. *IEEE Transactions on Automatic Control* **60** *(11), 2945-2960*.

[25] Lyapunov, A. M. (1992). *The general problem of the stability of motion*. Taylor & Francis, London.

[26] Nai-Ming Xie, and Si-Feng Liu (2009). Discrete grey forecasting model and its optimization. *Applied Mathematical Modelling* **33***, 1173-1186*.

[27] Ostalczyk, P., and Brzezinński, D. (2011). Numerical evaluation of Variable-Fractional-Order derivatives. *Automatyka* **15** *(3), 431-441*.

[28] Shen, Y., He, B., and Qin, P. (2016). Fractional-order grey prediction method for non-equidistant sequences. *Entropy 2016,* **18** *(6), 227*.

[29] Sierociuk, D., and Wiraszka, M. S. (2018). A new variable fractional-order PI algorithm. *IFAC PapersOnLine,* **51** *(4), 745-750*.

[30] Strobeck, C. (1973). N species competition. *Ecology,* **54** *(3), 650-654*.

[31] Tarasov, V. E. (2009). Differential equations with fractional derivative and universal map with memory. *J. Phys. A: Math. Theor.* **42***, doi: 10.1088/1751-8113/42/46/465102*.

[32] Thietart, R. A., and Forgues, B. (1995). Chaos theory and organization. *Organization Science,* **6** *(1), 19-31*.

[33] Urquhart, A. (2016). The inefficiency of Bitcoin. *Economics Letters,* **148***, 80-82*.

[34] V. Zakharchenko and I. G. Kovalenko (2018). Best approximation of the fractional semi-derivative operator by exponential series. *Mathematics* **6** *(12), doi:10.3390/math6010012*.

[35] Wayner, P. (1997). *Digital Cash*, 2^{nd} edition. AP Professional, London.

[36] Wu, L., Liu, S., and Wang, Y. (2012). Grey Lotka-Volterra model and its applications. *Technological Forecasting & Social Change* **79***, 1720-1730*.

[37] Wu, L. F., Liu, S. F., Cui, W., Liu, D. L., and Yao, T. X. (2014). Non-homogeneous discrete grey model with fractional-order accumulation. *Neural Comput. Appl.,* **25***, 1215-1221*.

[38] Xie, N. M., Liu, S. F., Yang, Y. J., and Yuan, C. Q. (2013). On a novel grey forecasting model based on no-homogeneous index sequence. *Applied Mathematical Modelling* **37** *, 5059-5068*.

[39] Zhou, W., and Jian-Min He (2013). Generalised GM(1,1) model and its application in forecasting of fuel production. *Applied Mathematical Modelling* **37***, 6234-6243*.

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ERROR ASSESSMENT IN FORECASTING CRYPTOCURRENCIES TRANSACTION COUNTS USING VARIANTS OF THE GREY LOTKA-VOLTERRA DYNAMICAL SYSTEM

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Abstract: The error assessment is made on the classical Grey Model (GM(1,1)) and the variants of the Grey Lotka-Volterra dynamical system namely the Grey Lotka-Volterra Model (GLVM), the Fractional Grey Lotka-Volterra Model (FGLVM) and the Variable-order Fractional Grey Lotka-Volterra Model (VFGLVM) for modeling the transaction counts of three cryptocurrencies: Bitcoin, Litecoin and Ripple. The error in transacting Bitcoin and Litecoin is assessed for the 2-dimensional study, while error in transacting three cryptocurrencies is assessed for the 3-dimensional study. The 2-dimensional models use Bitcoin and Litecoin transactions from April, 28, 2013 to February, 10, 2018. The 3-dimensional model uses transactions from August, 7, 2013 to February, 10, 2018. The error sequence patterns and the the Mean Absolute Percentage Error (MAPE) suggest a relatively higher accuracy of the VFLVM in 2- and 3-dimensional framework. The results show that in most of the cases, the descending order in performance is VFGLVM, FGLVM, GLVM and then GLVM.

Keywords: Error, Fractional derivative, Lotka-Volterra, Grey Model, Mean Absolute Percentage Error.

The error in modeling as a tool of measuring accuracy is assessed for four models of the cryptocurrencies adoption namely the classical Grey modeling, the Grey Lotka-Voltera model, the Fractional Grey Lotka-Volterra model and the Variable-order Fractional Grey Lotka-Volterra model. The results suggest that the 2- and 3-dimensional Variable-order Fractional Grey Lotka-Volterra model is relatively better. In most of the cases, the variants of Lotka-Volterra model perform better than the classical Grey model.

1. Introduction

Digital currencies also known as cryptocurrencies consist of directly trading, third-party free and without intermediary with the banks [3]. Transaction data and all records on cryptocurrencies constitute a *blockchain*. Bitcoin is one of the existing cryptocurrencies initiated in 2008 [1], which performs the processing of a block every 10 minutes [4]. Litecoin and Ripple are other cryptocurrencies of interest in this study where the variants of Grey Lotka-Volterra dynamical system are used for analysing the competition in adopting these cryptocurrencies. Further description on Bitcoin, Litecoin, Ripple and other cryptocurrencies can be found in [18].

The variants of the Grey Lotka-Volterra model in discrete framework were applied in Gatabazi et al. [8, 9, 10]. In Gatabazi et al. [10], the Grey Lotka-Volterra Model (GLVM) was described and applied to the cryptocurrencies adoption in 2-and 3-dimensional systems. The study applied the dataset of Bitcoin and Litecoin as a 2-dimensional study; and Bitcoin, Litecoin and Ripple as a 3-dimensional study. The description on these cryptocurrencies is developed in Gatabazi et al. [8]. The same study using fractional calculus instead of total differentiation applied in GLVM was done in [8] and in [9] where the order of fractional differentiation is variable along the study time. In these 3 studies, the accuracy was measured by the Mean Absolute Percentage Error while the predictability was measured by using properties of the Lyapunov exponents as described in [17]. The need of comparing adequacy of the classical GM(1,1), GLVM, FGLVM and VFGLVM gives idea on the computational analysis of errors of the models.

The error is defined as the difference between an observed or calculated value and the true value [20]. The error analysis in scientific research for measuring adequacy of models of interest is found in several manuscripts such as for example [2]. The recent works on the error analysis include [16] where different structures of errors in science and engineering are presented, and [5] where the error is analysed in physical sciences. Many other studies are devoted on applying the standard error rather than the error such as statistical studies found in [7], [6], [11], [13], [12] and [14].

The present study assesses accuracy of the variants of Lotka-Volterra by observing the variation of the error along the study time, with adequacy attributed to the model whose error tends to zero. The error assessment of models uses both graphical error analysis and the Mean Absolute Percentage Error (MAPE) which is the measurement of the accuracy for the variants of Lotka-Volterra analysed in [10], [8] and [9].

Including the introduction, the study comprises 4 sections: Section 2 presents the methodology of the study. Section 3 presents the main results and their interpretation and Section 4 gives a conclusion.

2. Methodology

2.1. $q(t)$-Accumulation generating operations

Solving analytically nonlinear dynamical system is often problematic. The $q(t)$-Accumulation generating operations provide a formulation compiled in the next definition, and lead to the discrete form whose properties are relatively close to that of corresponding nonlinear systems.

Definition 1. Assume an original data sequence $X_i^{(0)} = \left(x_i^{(0)}(1), x_i^{(0)}(2), \ldots, x_i^{(0)}(n)\right)$. The k^{th} $q(t)$-Accumulation generating operation (q(t)-AGO), for the sequence $X_i^{(0)}$ is denoted by $x^{[q(t)]}(k)$ and defined by

$$x^{[q(t)]}(k) = \sum_{i=1}^{k} \frac{\Gamma[q(t)+k-i]}{\Gamma[q(t)]\Gamma(k-i+1)} x^{(0)}(i), \tag{1}$$

where $q(t)$ is the variable order of accumulation along the time interval of length T.

Expression (1) yields the following particular cases:
For $q(t) = 1$, Equation (1) becomes

$$\begin{aligned} x^{(1)}(k) &= \sum_{i=1}^{k} \frac{\Gamma(1+k-i)}{\Gamma(1)\Gamma(k-i+1)} x^{(0)}(i) \\ &= \sum_{i=1}^{k} x^{(0)}(i). \end{aligned} \tag{2}$$

which is known as the first order accumulation generating operation (1-AGO).

For $q(t) = q$ where q is a real number such that $n-1 < q < n$, $n \in \mathbb{Z}$, Equation (1) becomes

$$x^{(q)}(k) = \sum_{i=1}^{k} \frac{\Gamma(q+k-i)}{\Gamma(q)\Gamma(k-i+1)} x^{(0)}(i), \tag{3}$$

which is known as the accumulation generating operation of order q (q-AGO) or equivalently, the fractional accumulation generating operation of order q.

2.2. q(t)-Mean sequence and Grey Modeling

The accumulations in Equation (1) yields the mean sequences. Grey difference equation resulting from the classical grey differential equation applies the mean sequences for constructing the discrete model and related extensions upon the following definitions.

Definition 2. Consider the $q(t)$-AGO for an original data sequence

$$X_i^{(0)} = \left(x_i^{(0)}(1), x_i^{(0)}(2), \ldots, x_i^{(0)}(n)\right).$$

The k^{th} $q(t)$-mean sequence for $X_i^{(0)}$ is denoted by $z^{[q(t)]}(k)$ and defined by

$$Z_i^{[q(t)]} = \left(z_i^{[q(t)]}(2), z_i^{[q(t)]}(3), \ldots, z_i^{[q(t)]}(n)\right) \tag{4}$$

with

$$z_i^{(q(t))}(k) = \frac{x^{[q(t)]}(k) + x^{[q(t)]}(k-1)}{2}, \ k = 2, 3, \ldots, \ n \in \mathbb{Z}. \tag{5}$$

As for Definition 1, Expression (5) yields the following particular cases:
For $q(t) = 1$, Equation (5) becomes

$$z_i^{(1)}(k) = \frac{x^{(1)}(k) + x^{(1)}(k-1)}{2}, \ k = 2, 3, \ldots, n. \tag{6}$$

Equation (6) is known as the first order mean accumulation.

For $q(t) = q$ where q is a real number such that $n-1 < q < n$, $n \in \mathbb{Z}$, , Equation (5) becomes

$$z_i^{(q)}(k) = \frac{x^{(q)}(k) + x^{(q)}(k-1)}{2}, \ k = 2, 3, \ldots, n., \tag{7}$$

which is known as the mean fractional accumulation of order q.

Definition 3. The 1-order and 1-variable Grey Model (GM(1,1)) based on the series $X^{(1)} = \left(x^{(1)}(1), x^{(1)}(2), \ldots, x^{(1)}(n)\right)$ is defined by the following differential equation

$$\begin{cases} \frac{dx^{(1)}(t)}{dt} + ax^{(1)}(t) = b \\ x^{(1)}(1) = x^{(0)}(1) \end{cases}. \tag{8}$$

In Equation (8), the parameters a and b are calculated by the least square method proposed in [15] as

$$\begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix} = (B'B)^{-1}B'M$$

where,

$$B = \begin{pmatrix} -z^{(1)}(2) & 1 \\ -z^{(1)}(3) & 1 \\ \vdots & \vdots \\ -z^{(1)}(n) & 1 \end{pmatrix}; \quad M = \begin{pmatrix} x^{(0)}(2) \\ x^{(0)}(3) \\ \vdots \\ x^{(0)}(n) \end{pmatrix}.$$

Proposition 1. The difference equation corresponding to Equation (8) can be written as:

$$x^{(0)}(k+1) + az^{(1)}(k) = b. \tag{9}$$

The proof of Proposition 1 is found in [10]. The approximation

$$x^{(0)}(k+1) = b - az^{(1)}(k), \; k = 2, 3, \ldots, n \tag{10}$$

is known as the Grey Model (GM(1,1)) with error sequence expressed by

$$\varepsilon = x^{(0)}(k+1) - \left[b - az^{(1)}(k)\right]. \tag{11}$$

2.3. Mean sequences and Lotka-Volterra dynamical system

Grey modeling is applied to the Lotka-Volterra dynamical system for formulating various models of competition in discrete framework. Below the Lotka-Volterra dynamical system is presented and then the models of interest are derived.

Definition 4. The general Lotka-Volterra model of competing relationships between n species is given by

$$\frac{dX_i}{dt} = X_i \left(a_i - \sum_{j=1}^{n} \alpha_j X_j\right) \tag{12}$$

where parameters $a_{i,\ i\in[1,n]}$ represent the capacity of growing of populations $X_{i,\ i\in[1,n]}$, while parameters $\alpha_{j,\ j\in[1,n]}$ represent the effect species j has on species i, the expressions X_i^2 are interactions within species, $X_i\, X_j$, $i \neq j$ are interactions of different species [19].

Proposition 2. Consider the Lotka-Volterra system (12) and the $q(t)$- fractional accumulation of the original data sequence $X_i^{(0)}$. Applying Grey Model yields the following approximations:

$$x_i^{(0)}(k+1) \approx a_i z_i^{[q(t)]}(k) - b_i \left(z_i^{[q(t)]}(k)\right)^2 - \sum_{j \neq i}^{n} c_j z_i^{[q(t)]}(k) z_j^{[q(t)]}(k); \tag{13}$$

with error sequences expressed by

$$\varepsilon_i = x_i^{(0)}(k+1) - \left(a_i z_i^{[q(t)]}(k) - b_i \left(z_i^{[q(t)]}(k)\right)^2 - \sum_{j \neq i}^{n} c_i z_i^{[q(t)]}(k) z_j^{[q(t)]}(k)\right); \tag{14}$$

and the least square estimates of parameters in (13) are given by

$$\begin{pmatrix} \hat{a}_i \\ \hat{b}_i \\ \hat{c}_i \end{pmatrix} = (B_i' B_i)^{-1} B_i' M_i \tag{15}$$

where,

$$B_i = \begin{pmatrix} z_i^{[q(t)]}(2) & -\left(z_i^{[q(t)]}(2)\right)^2 & -z_i^{[q(t)]}(2) z_1^{[q(t)]}(2) & \dots & -z_i^{[q(t)]}(2) z_j^{[q(t)]}(2) \\ z_i^{[q(t)]}(3) & -\left(z_i^{[q(t)]}(3)\right)^2 & -z_i^{[q(t)]}(3) z_1^{[q(t)]}(3) & \dots & -z_i^{[q(t)]}(3) z_j^{[q(t)]}(2) \\ \vdots & \vdots & \vdots & & \\ z_i^{[q(t)]}(n) & -\left(z_i^{[q(t)]}(n)\right)^2 & -z_i^{[q(t)]}(n) z_1^{[q(t)]}(n) & \dots & -z_i^{[q(t)]}(n) z_j^{[q(t)]}(2) \end{pmatrix}; \forall j \neq i;$$

$$M_i = \begin{pmatrix} x_i^{(0)}(2) \\ x_i^{(0)}(3) \\ \vdots \\ x_i^{(0)}(n) \end{pmatrix}$$

The proof of Proposition (2) is detailed in [9]. The results of Proposition (2) constitute the Variable-order Fractional Lotka Volterra Model (VFLVM) and the two important particular cases:

The order $q(t) = 1$ which yields the Grey Lotka-Volterra Model (GLVM) analysed in [10]. The GLVM values and corresponding error sequences are obtained by replacing $z_i^{[q(t)]}(k)$ and $z_j^{[q(t)]}(k)$ of the VFGLVM by $z_i^{(1)}(k)$ and $z_j^{(1)}(k)$ as defined in Equation (6).

The order $q(t) = q$ where q is a real number such that $n-1 < q < n$, $n \in \mathbb{Z}$, yields the Fractional Grey Lotka Volterra Model with details found in [8]. The FGLVM values and corresponding error sequences are obtained by replacing $z_i^{[q(t)]}(k)$ and $z_j^{[q(t)]}(k)$ of the VFGLVM by $z_i^{(1)}(k)$ and $z_j^{(1)}(k)$ as defined in Equation (7).

2.4. Datasets

Data on daily transaction counts of Bitcoin and Litecoin from 28-April-2013 up to 10-February-2018 is considered for the 2-dimensional models. The 3-dimensional study takes daily transaction counts of both Bitcoin, Litecoin and Ripple from 7-August-2013 up to 10-February-2018. The evolution in transaction counts of Bitcoin and Litecoin along the study time and that of both Bitcoin, Litecoin and Ripple recorded daily with the related graphs are presented in [10] and in [8] and can also be found via the authors of this paper.

3. Results and interpretation

3.1. Error of 2-dimensional GM(1,1), GLVM, FGLVM and VFGLVM for Bitcoin and Litecoin

Table 1 presents the last 50 actual and forecasting values of both GM(1,1), GLVM, FGLVM and VFGLVM with corresponding errors for Bitcoin and Litecoin. The entire error sequences are presented in Figures 1 and 2 in linear scales and in Figures 3 and 4 in logarithmic scales. Figure 1 presents the 2-dimensional error sequences of GM(1,1), GLVM, FGLVM and VFGLVM for Bitcoin. The patterns suggest that VFGLVM is relatively better than both GM(1,1), GLVM and FGLVM. However, the performance of VFGLVM is close to that of GLVM and FGLVM while the least fitting performance is shown by the GM(1,1). Figure 3 suggests better performance of GLVM at a later stage as compared to the earlier one. The Mean Absolute Percentage Error (MAPE) emphasises the suggestion of the patterns by pointing VFGLVM as the most accurate model (MAPE=10) followed by FGLVM with MAPE=16; GLVM (MAPE=22) and GM(1,1) with MAPE=49.

Figure 2 presents the 2-dimensional error sequences of GM(1,1), GLVM, FGLVM and VFGLVM for Litecoin. The patterns of both models are close to the zero line before the last 60 days of the study time. However, only the VFGLVM (MAPE=19) is accurate for the overall values of Litecoin. The results along the 60 last days of the study time present an abrupt change but the MAPE suggests that only the GM(1,1) values are reasonably accurate (MAPE=43). Figure 4 shows a relatively critical high performance of VFGLVM for Litecoin in the 1^{st} and 4^{th} year of the study time.

Table 1: Last 50 forecasting values of GLVM , FGLVM and VFGLVM for Bitcoin, Litecoin and Ripple daily transaction counts.

	Actual Values		GM(1,1) Values		GM(1,1) Errors		GLVM Values		GLVM Errors		FGLVM Values		FGLVM Errors		VFGLVM Values		VFGLVM Errors	
No	BTC	LTC	BTC	LTC	BTC	LTC	BTC	LTC	BTC	LTC	BTC	LTC	BTC	LTC	BTC	LTC	BTC	LTC
1	308072	117738	60015	43585	248057	74153	281883	58816	26189	58922	291039	83781	17033	33957	298770	329598	9302	-211860
2	279371	81111	59984	43951	219387	37160	281846	59721	-2475	21390	291139	81279	-11768	-168	324419	297909	-45048	-216798
3	228791	77925	59931	44243	168860	33682	281806	60450	-53015	17475	291164	78595	-62373	-670	345468	272642	-116677	-194717
4	247298	82613	59951	44539	187347	38074	281775	61191	-34477	21422	290908	77655	-43610	4958	355103	266293	-107805	-183680
5	307486	112765	60014	44898	247472	67867	281742	62098	25744	50667	290619	79292	16867	33473	345748	280244	-38262	-167479
6	304904	111207	60011	45310	244893	65897	281710	63145	23194	48062	290281	81881	14623	29326	333217	297317	-28313	-186110
7	353659	155481	60063	45801	293596	109680	281689	64404	71970	91077	289685	86079	63974	69402	317014	324493	36645	-169012
8	344260	141900	60053	46348	284207	95552	281675	65822	62585	76078	289057	90707	55203	51193	299515	349882	44745	-207982
9	290259	105948	59996	46804	230263	59144	281648	67016	8611	38932	288925	90967	1334	14981	309281	336587	-19022	-230639
10	241601	83076	59945	47152	181656	35924	281614	67934	-40013	15142	288968	88634	-47367	-5558	332162	309167	-90561	-226091
11	340809	127924	60049	47540	280760	80384	281579	68965	59230	58959	288803	89616	52006	38308	325977	315242	14832	-187318
12	395806	186764	60107	48119	335699	138645	281560	70518	114246	116246	288006	96864	107800	89900	283735	368258	112071	-181494
13	424840	225860	60138	48878	364702	176982	281573	72580	143267	153280	286628	107323	138212	118537	237373	432628	187467	-206768
14	342564	197217	60051	49657	282513	147560	281606	74724	60958	122493	285428	114660	57136	82557	231025	461614	111539	-264397
15	358679	173712	60068	50339	298611	123373	281626	76629	77053	97083	284937	117045	73742	56667	243974	453823	114705	-280111
16	368025	143412	60078	50922	307947	92490	281607	78278	86418	65134	285106	116989	82919	26423	239729	434392	128296	-290980
17	345506	146511	60054	51456	285452	95055	281575	79800	63931	66711	285214	116824	60292	29687	241463	421387	104043	-274876
18	360101	145848	60070	51994	300031	93854	281546	81350	78555	64498	285059	118028	75042	27820	240218	421827	119883	-275979
19	347227	140304	60056	52520	287171	87784	281512	82881	65715	57423	284955	119080	62272	21224	237125	420327	110102	-280023
20	337766	120843	60046	53001	277720	67842	281469	84292	56297	36551	285021	118748	52745	2095	242430	409282	95336	-288439
21	299913	106887	60006	53420	239907	53467	281418	85531	18495	21356	285202	117063	14711	-10176	257413	391282	42500	-284395
22	265586	93443	59970	53788	205616	39655	281372	86629	-15786	6814	285304	114847	-19718	-21404	280219	373291	-14633	-279848
23	234890	88779	59938	54124	174952	34655	281334	87633	-46444	1146	285299	112714	-50409	-23935	301614	358561	-66724	-269782
24	273473	90381	59978	54453	213495	35928	281291	88627	-7818	1754	285323	111522	-11850	-21141	306931	351332	-33458	-260951
25	303566	117447	60010	54836	243556	62611	281244	89786	22322	27661	285203	112833	18363	4614	293661	361655	9905	-244208
26	315604	113111	60023	55260	255581	57851	281197	91081	34407	22030	284987	114988	30617	-1877	279673	374176	35931	-261065
27	309322	95276	60016	55643	249306	39633	281134	92259	28188	3017	285117	114860	24205	-19584	275959	366696	33363	-271420
28	243454	70009	59947	55947	183507	14062	281068	93199	-37614	-23190	285379	112304	-41925	-42295	294099	344551	-50645	-274542
29	240433	66798	59943	56199	180490	10599	281004	93981	-40571	-27183	285565	109364	-45132	-42566	314103	324536	-73670	-257738
30	215435	55466	59917	56424	155518	-958	280940	94683	-65505	-39217	285721	106862	-70286	-51396	326028	310023	-110593	-254557
31	245395	61730	59949	56640	185446	5090	280870	95358	-35475	-33628	285883	105015	-40488	-43285	330419	300748	-85024	-239018
32	271759	59717	59976	56863	211783	2854	280786	96060	-9027	-36343	286143	104087	-14384	-44370	323302	297390	-51543	-237673
33	250247	59072	59954	57082	190293	1990	280698	96749	-30451	-37677	286341	103090	-36094	-44018	322181	292475	-71934	-233403
34	236422	61836	59939	57304	176483	4532	280622	97453	-44200	-35617	286313	102372	-49891	-40536	328233	290187	-91811	-228351
35	220304	57452	59922	57524	160382	-72	280553	98150	-60249	-40698	286235	101598	-65931	-44146	335037	286928	-114733	-229476
36	193421	49382	59894	57720	133527	-8338	280489	98777	-87068	-49395	286178	100148	-92757	-50766	345008	278358	-151587	-228976
37	213288	51278	59915	57905	153373	-6627	280424	99369	-67136	-48091	286183	98836	-72895	-47558	349854	271605	-136566	-220327
38	232028	50067	59935	58092	172093	-8025	280346	99967	-48318	-49900	286323	98020	-54295	-47953	347093	268334	-115065	-218267
39	236442	55270	59939	58286	176503	-3016	280264	100590	-43822	-45320	286432	97598	-49990	-42328	343861	267599	-107419	-212329
40	204159	54531	59905	58488	144254	-3957	280192	101242	-76033	-46711	286327	97346	-82168	-42815	347186	268072	-143027	-213541
41	257504	57962	59961	58695	197543	-733	280114	101912	-22610	-43950	286344	97228	-28840	-39266	344846	268381	-87342	-210419
42	235750	66669	59938	58924	175812	7745	280034	102656	-44284	-35987	286333	97823	-50583	-31154	338642	273718	-102892	-207049
43	194733	49384	59895	59137	134838	-9753	279967	103352	-85234	-53968	286160	97381	-91427	-47997	347238	270270	-152505	-220886
44	173509	45225	59873	59311	113636	-14086	279907	103921	-106398	-58696	286048	95762	-112539	-50537	358705	258648	-185196	-213423
45	216178	51043	59918	59489	156260	-8446	279841	104502	-63663	-53459	286033	95045	-69855	-44002	359321	255887	-143143	-204844
46	243950	59946	59947	59693	184003	253	279761	105173	-35811	-45227	286129	95480	-42179	-35534	350603	261038	-106653	-201092
47	213578	50320	59915	59896	153663	-9576	279681	105842	-66103	-55522	286151	95451	-72573	-45131	349323	260901	-135745	-210581
48	173158	37148	59872	60057	113286	-22909	279609	106375	-106451	-69227	286112	93887	-112954	-56739	359780	249179	-186622	-212031
49	177725	44811	59877	60207	117848	-15396	279546	106875	-101821	-62064	285987	92676	-108262	-47865	366417	242930	-188692	-198119
50	181640	46594	59881	60376	121759	-13782	279485	107434	-97845	-60840	285838	92432	-104198	-45838	367165	244531	-185525	-197937

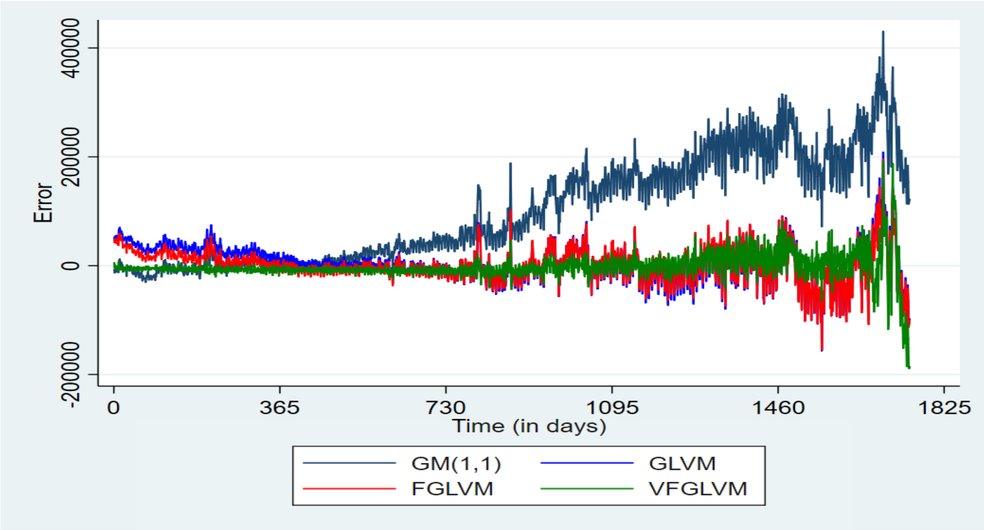

Figure 1: 2-dimensional error sequences of GM(1,1), GLVM, FGLVM and VFGLVM for Bitcoin

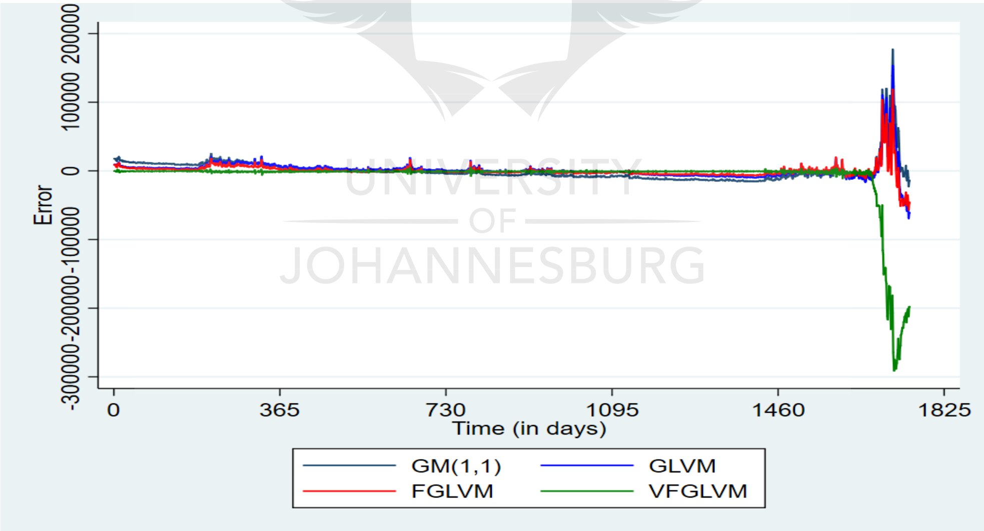

Figure 2: 2-dimensional error sequences of GM(1,1), GLVM, FGLVM and VFGLVM for Litecoin

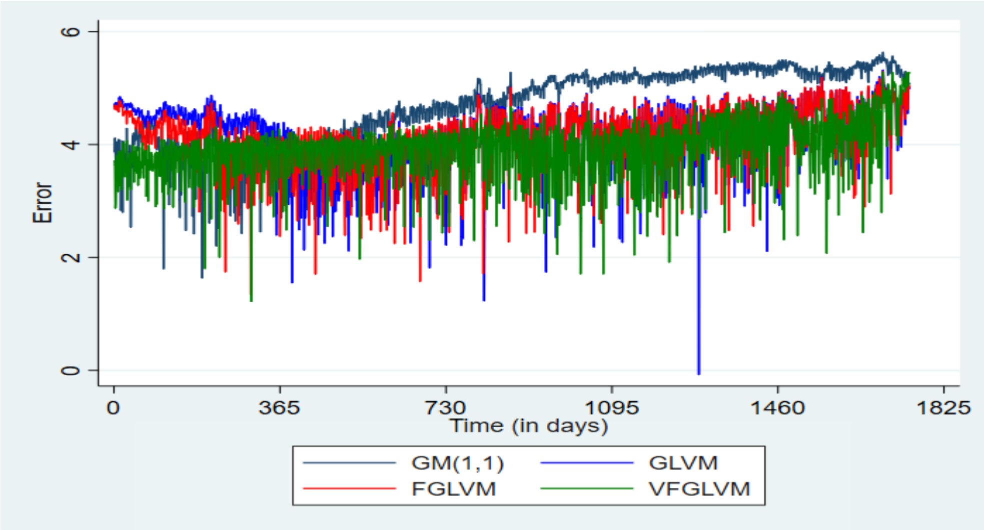

Figure 3: 2-dimensional error sequences of GM(1,1), GLVM, FGLVM and VFGLVM for Bitcoin in logarithmic scales

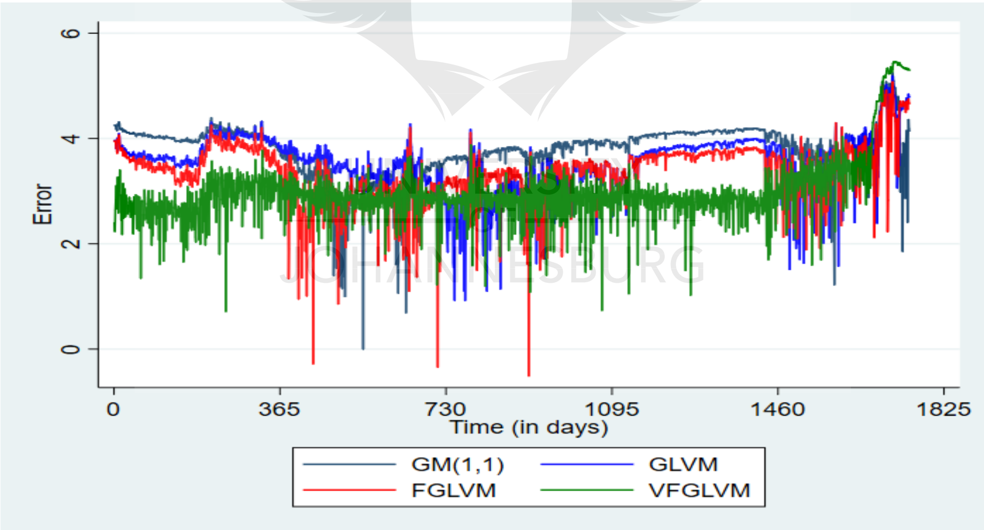

Figure 4: 2-dimensional error sequences of GM(1,1), GLVM, FGLVM and VFGLVM for Litecoin in logarithmic scales

3.2. Error of 3-dimensional GM(1,1), GLVM, FGLVM and VFGLVM for Bitcoin, Litecoin and Ripple

Table 2 presents the last 50 actual and forecasting values of both GM(1,1), GLVM, FGLVM and VFGLVM with corresponding errors for both Bitcoin, Litecoin and Ripple. Figures 5, 6 and 7 represent the entire error sequences for each model in linear scales and Figures 8, 9 and 10 in logarithmic scales.

Figure 5 and 8 present the 3-dimensional error sequences of GM(1,1), GLVM, FGLVM and VFGLVM for Bitcoin in linear and logarithmic scales respectively. The patterns in these different scales suggest that VFGLVM is relatively better than both GM(1,1), GLVM and FGLVM. As for the 2-dimensional study, the performance of VFGLVM is close to that of GLVM and FGLVM while the GM(1,1) shows the least fitting performance. The MAPE stresses the suggestion of the patterns and suggests no accuracy for GM(1,1). The VFGLVM remains the most accurate model (MAPE=9) as compared to FGLVM (MAPE=16) and GLVM (MAPE=24).

The Litecoin error sequences for GM(1,1), GLVM, FGLVM and VFGLVM in 3-dimensional framework are represented by Figure 6 and 9 in linear and logarithmic scales respectively. The major parts of the patterns of both models in linear scales fluctuate around the zero line but the VFGLVM is much more closer to the zero line. In logarithmic scales (Figure 9) the pattern of the FVGLVM is also relatively closer to the zero line especially at a later stage of the study time. The MAPE suggests that only the VFGLVM values are reasonably accurate with MAPE=24. The Ripple error sequences for GM(1,1), GLVM, FGLVM and VFGLVM in 3-dimensional framework (Figure 7) behaves similarly as that of Litecoin with relatively less accuracy for VFGLVM values where MAPE=41. Logarithmic scales show that the GLVM for Ripple performs well especially between the 2^{nd} and the 4^{th} year of the study time as shows Figure 10.

Table 2: Last 50 forecasting values of GLVM , FGLVM and VFGLVM for Bitcoin, Litecoin and Ripple daily transaction counts.

	Actual Values			GM(1,1) Values			GM(1,1) Errors			GLVM Values			GLVM Errors			FGLVM Values			FGLVM Errors			VFGLVM Values			VFGLVM Errors		
No	BTC	LTC	RPL	BTC	LTC	RPL	BTC	LTC	RPL	BTC	LTC	RPL	BTC	LTC	RPL	BTC	LTC	RPL	BTC	LTC	RPL	BTC	LTC	RPL	BTC	LTC	RPL
1	308072	117738	58024	1036330	45985	30110	-728258	71753	27914	383561	97843	71338	-75489	19895	-13314	299966	101178	45364	8106	16560	12660	304031	134565	55359	4041	-16827	2665
2	279371	81111	42168	1037410	46388	30157	-758039	34723	12011	358516	74034	52397	-79145	7077	-10229	300274	98468	44316	-20903	-17357	-2148	325275	119914	52310	-45904	-38803	-10142
3	228791	77925	37359	1038345	46710	30194	-809554	31215	7165	342633	60291	41521	-113842	17634	-4162	300560	95568	43208	-71769	-17643	-5849	338141	109830	50165	-109350	-31905	-12806
4	247298	82613	37608	1039221	47034	30229	-791923	35579	7379	338600	57434	39126	-91302	25179	-1518	300755	94913	42822	-53457	-12300	-5214	340047	111602	49595	-92749	-28989	-11987
5	307486	112765	63774	1040242	47430	30277	-732756	65335	33497	357028	76300	53030	-49542	36465	10744	299526	95480	43876	7960	17285	19898	333662	114047	53277	-26176	-1282	10497
6	304904	111207	67160	1041369	47883	30338	-736465	63324	36822	381001	97792	68656	-76097	13415	-1496	296950	95306	45743	7954	15901	21417	324479	112268	59199	-19575	-1061	7961
7	353659	155481	108608	1042581	48423	30420	-688922	107058	78188	413486	131100	92518	-59827	24381	16090	292179	93931	49018	61480	61550	59590	308371	108069	69945	45288	47412	38663
8	344260	141900	114645	1043865	49025	30524	-699605	92875	84121	447660	165983	117974	-103400	-24083	-3329	285382	89660	53227	58878	52240	61418	291069	92060	83186	53191	49840	31459
9	290259	105948	64844	1045032	49527	30608	-754773	56421	34236	417567	136237	94520	-127308	-30289	-29676	283998	87962	53839	6261	17986	11005	302225	91638	79740	-11966	14310	-14896
10	241601	83076	50779	1046011	49909	30662	-804410	33167	20117	368411	90252	60567	-126810	-7176	-9788	286447	88978	52087	-44846	-5902	-1308	321094	100392	69439	-79493	-17316	-18660
11	340809	127924	71079	1047082	50336	30719	-706273	77588	40360	372240	95482	63862	-31431	32442	7217	287046	91082	52294	53763	36842	18785	318213	106762	67284	22596	21162	3795
12	395806	186764	98324	1048437	50973	30798	-652631	135791	67526	409663	131504	89140	-13857	55260	9184	285880	98041	55641	109926	88723	42683	285628	130620	70158	110178	56144	28166
13	424840	225860	121276	1049947	51809	30900	-625107	174051	90376	445436	171415	116034	-20596	54445	5242	282879	105998	60988	141961	119862	60288	241948	158394	76636	182892	67466	44640
14	342564	197217	125177	1051359	52665	31016	-708795	144552	94161	467630	194408	130526	-125066	2809	-5349	278006	106726	65888	64558	90491	59289	228941	156289	88176	113623	40928	37001
15	358679	173712	92750	1052649	53416	31117	-693970	120296	61633	444875	174657	115150	-86196	-945	-22400	276165	106431	67641	82514	67281	25109	241458	151491	89157	117221	22221	3593
16	368025	143412	78515	1053987	54058	31197	-685962	89354	47318	409256	139902	90154	-41231	3510	-11639	278011	109535	66841	90014	33877	11674	252400	150607	77698	115625	-7195	817
17	345506	146511	84686	1055299	54645	31274	-709793	91866	53412	404440	135056	85850	-58934	11455	-1164	278169	109573	66681	67337	36938	18005	260205	140523	75497	85301	5988	9189
18	360101	145848	95356	1056598	55236	31358	-696497	90612	63998	416086	150049	94846	-55985	-4201	510	276134	107440	67958	83967	38408	27398	259134	129183	79600	100967	16665	15756
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22	265586	93443	52404	1061373	57211	31633	-795787	36232	20771	360107	95630	56629	-94521	-2187	-4225	277526	106166	66186	-11940	-12723	-13782	297193	123689	70527	-31607	-30246	-18123
23	234890	88779	41844	1062294	57579	31677	-827404	31200	10167	348145	84806	49302	-113255	3973	-7458	278936	106205	64868	-44046	-17426	-23024	307837	125588	69149	-72947	-36809	-27305
24	273473	90381	53869	1063229	57942	31722	-789756	32439	22147	351867	86473	50089	-78394	3908	3780	279542	105894	64189	-6069	-15513	-10320	311192	122832	68381	-37719	-32451	-14512
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49	177725	44811	30748	1084812	64272	32600	-907087	-19461	-1852	311191	51615	24747	-133466	-6804	6001	284825	93239	54780	-107100	-48428	-24032	346552	82967	53254	-168827	-38156	-22506
50	181640	46594	36859	1085473	64457	32631	-903833	-17863	4228	326307	70083	35322	-144667	-23489	1537	283140	90443	55321	-101500	-43849	-18462	344137	78624	58647	-162497	-32030	-21788

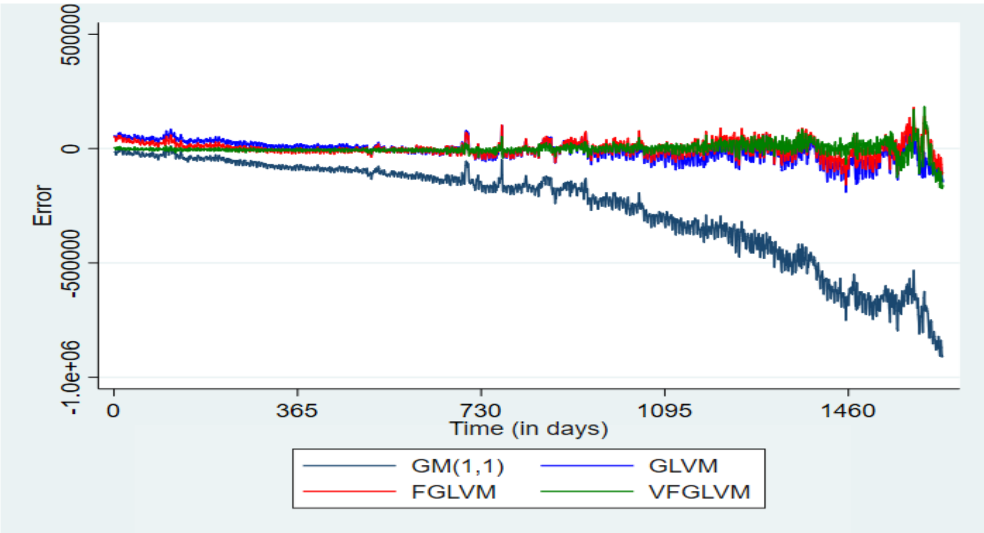

Figure 5: 3-dimensional error sequences of GM(1,1), GLVM, FGLVM and VFGLVM for Bitcoin

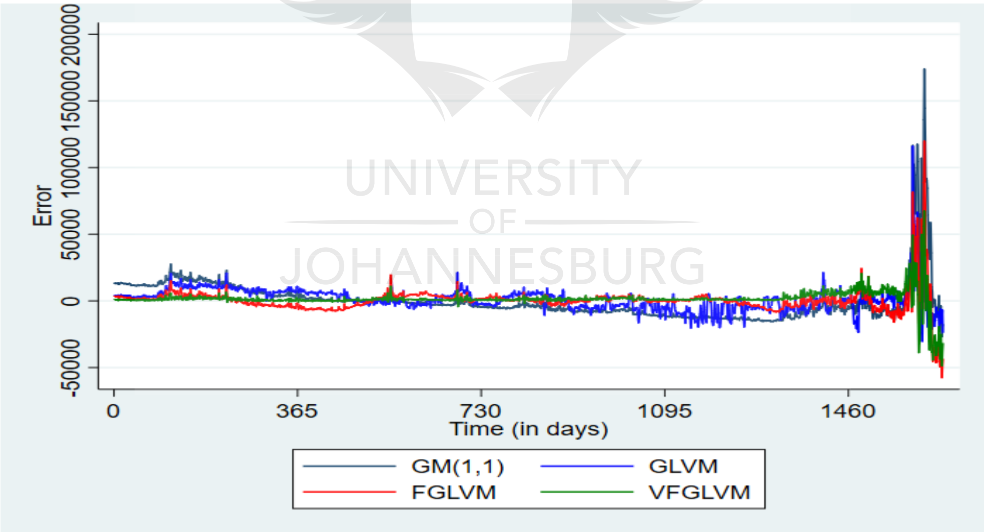

Figure 6: 3-dimensional error sequences of GM(1,1), GLVM, FGLVM and VFGLVM for Litecoin

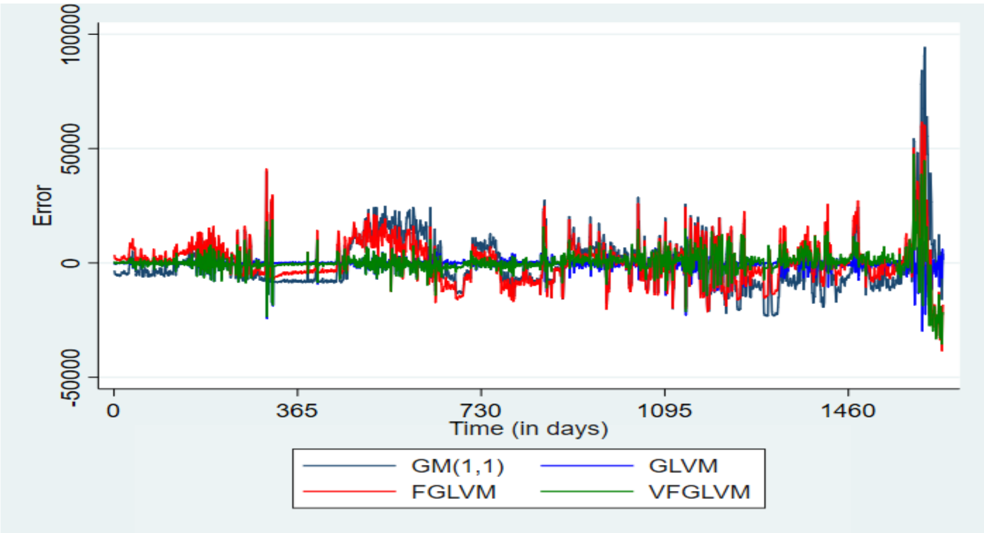

Figure 7: 3-dimensional error sequences of GM(1,1), GLVM, FGLVM and VFGLVM for Ripple

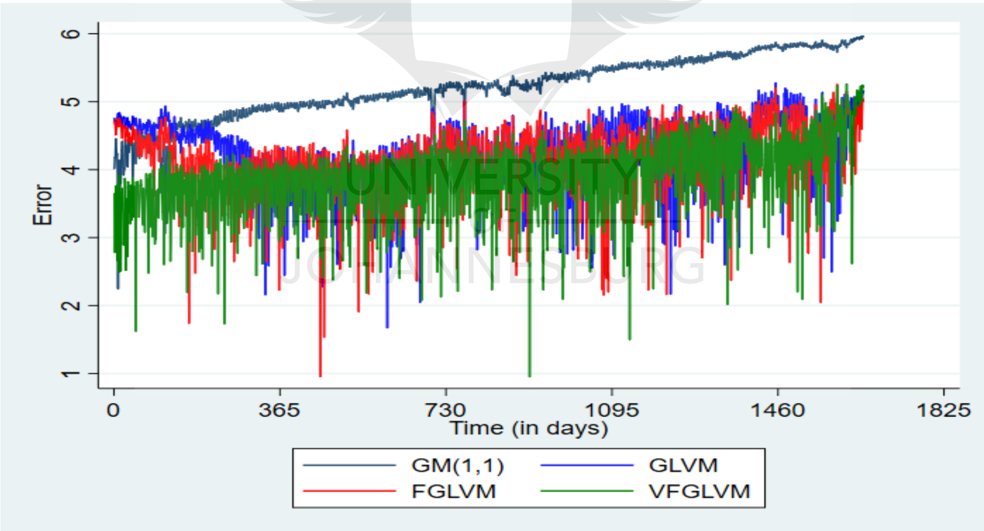

Figure 8: 3-dimensional error sequences of GM(1,1), GLVM, FGLVM and VFGLVM for Bitcoin in logarithmic scales

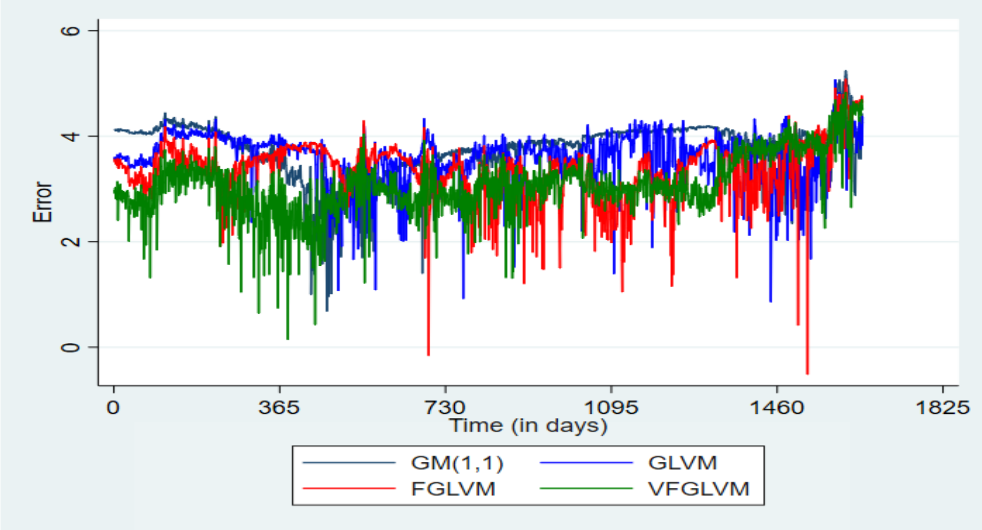

Figure 9: 3-dimensional error sequences of GM(1,1), GLVM, FGLVM and VFGLVM for Litecoin in logarithmic scales

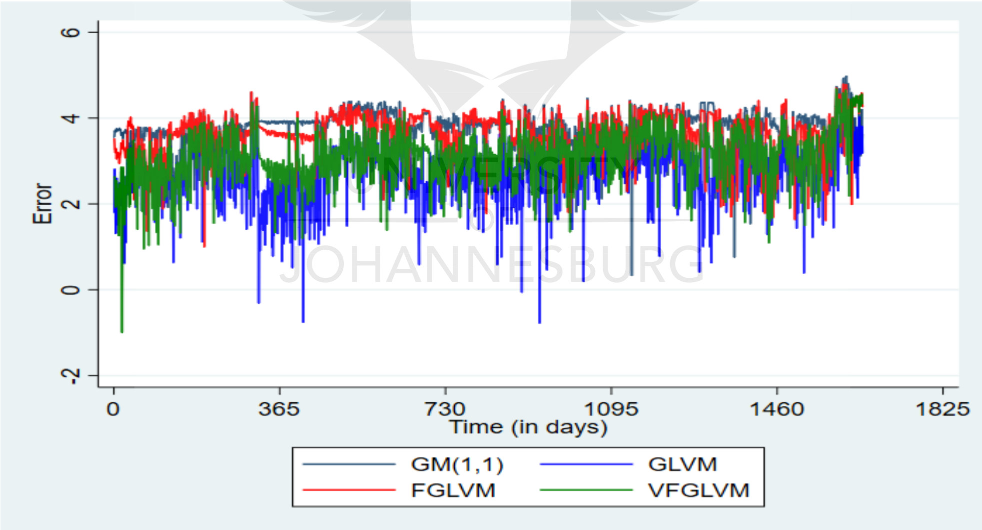

Figure 10: 3-dimensional error sequences of GM(1,1), GLVM, FGLVM and VFGLVM for Ripple in logarithmic scales

4. Conclusions

The study assessed the error in fitting the classical Grey Model (GM(1,1)) and the variants of the Grey Lotka-Volterra model namely the Grey Lotka-Volterra Model (GLVM), the Fractional Grey Lotka-Volterra Model (FGLVM) and the Variable-order Fractional Grey Lotka-Volterra Model (VFGLVM) and all these models were applied to the adoption of cryptocurrencies namely Bitcoin, Litecoin and Ripple. Models with higher performance are selected by considering the model with relatively least error terms taking account of the Mean Absolute Percentage Error of the fitted model.

The patterns of the 2-dimensional analysis which represent the errors along the study time suggested that VFGLVM is relatively the best model for the overall Bitcoin forecasting values (MAPE=10). VFGLVM is also the best for the Litecoin forecasting before 60 last days of study time (MAPE=19) while GM(1,1) takes over (MAPE=43) for the Litecoin forecasting values along the 60 last days of the study time.

The error sequence patterns of the 3-dimensional analysis suggested that VFGLVM is relatively the best model for the overall Bitcoin forecasting values (MAPE=9) followed by FGLVM with MAPE=16 and GLVM with MAPE=24. The MAPE suggested no accuracy of GM(1,1) for the overall Bitcoin forecasting values. The VFGLVM is also the best for the Litecoin and Ripple forecasting values. The VFGLVM is reasonably accurate with MAPE=24 for Litecoin and MAPE=41 for Ripple while the MAPE suggests no accuracy for the rest of models.

This study justified the relatively higher performance of the VFGLVM as compared to the classical GM(1,1) and the other presented variants of the Lotka-Volterra models. The future work will analyse the behavior of the variants of the Lotka-Volterra models in continuous time scale with corresponding error assessment.

References

[1] Angela, S. B. (2016). Ten types of digital currencies and how they work. *Online trading: Free introductory eBook, September 24, 2016*.
URL http://www.techbullion.com/10-types-digital-currencies-work/

[2] Bevington, P. R., and Robinson, D. K. (2002). *Data Reduction and Error Analysis for the Physical Sciences*, 3^{rd} edition. McGraw-Hill, New York.

[3] Blundell-Wignall, A. (2014). The Bitcoin question: Currency versus trust-less transfer

technology. *OECD Working Papers on Finance, Insurance and Private Pensions, No 37, OECD Publishing*.
URL http://dx.doi.org/10.1787/5jz2pwjd9t20-en

[4] Chan, S., Chu, J., Nadarajah, S., and Osterrieder, J. (2017). A statistical analysis of cryptocurrencies. *Journal of Risk and Financial Management* **2017**.

[5] Chhetri, K. B. (2013). Computation of errors and their analysis on physics experiments. *Himalayan Physics,* **3***, 78, doi: 10.3126/hj.v3i0.7312*.

[6] Efron, B., and Tibshirani, R. J. (1994). *An introduction to the bootstrap*. Chapman & Hall/CRC.

[7] Gatabazi, P., and Kabera, G. (2015). Survival analysis and its stochastic process approach with application to diabetes data.

[8] Gatabazi, P., Mba, J. C., and Pindza, E. (2019). Fractional Grey Lotka-Volterra model with application to cryptocurrencies adoption. *Chaos: An Interdisciplinary Journal of Nonlinear Science,* **29** *(7), 073116*.

[9] Gatabazi, P., Mba, J. C., and Pindza, E. (2019). Modeling cryptocurrencies transaction counts using variable-order fractional grey Lotka-Volterra dynamical system. *Chaos Solitons & Fractals,* **127***, 283-290*.

[10] Gatabazi, P., Mba, J. C., Pindza, E., and Labuschagne, C. (2019). Grey Lotka-Volterra model with application to cryptocurrencies adoption. *Chaos, Solitons & Fractals,* **122***, 47-57*.

[11] Gatabazi, P., Melesse, S. F., and Ramroop, S. (2018). Multiple events model for the infant mortality at Kigali University Teaching Hospital. *The Open Public Health Journal* **11***, 464-473*.

[12] Gatabazi, P., Melesse, S. F., and Ramroop, S. (2019). Infant mortality at the Kigali University Teaching Hospital: Application of Aalen additive hazards model and comparison with other classical survival models. *African Population Studies* **33** *(2), 4834-4851*.

[13] Gatabazi, P., Melesse, S. F., and Ramroop, S. (2019). Resampled Cox proportional hazards model for the infant mortality at the Kigali University Teaching Hospital. *The Open Public Health Journal* **12***, 136-144*.

[14] Gatabazi, P., Melesse, S. F., and Ramroop, S. (2019). Risk prediction of infant mortality using re-sampled marginal risk set model for newborn babies at the Kigali University Teaching Hospital, Rwanda, 2016. *BMC Pediatrics No: BPED-D-19-00022R1*.

[15] Hsi-Tse Wang, and Ta-Chu Wang (2016). Application of grey Lotka-Volterra model to forecast the diffusion and competition analysis of the TV and smart-phone industries. *Technological Forecasting Social Change* **106***, 37-44*.

[16] Lakshmikantham, V., and Sen, S. K. (1997). *Computational Error and Complexity in Science and Engineering: Computational Error and Complexity*. Elsevier Science, United States.

[17] Lyapunov, A. M. (1992). *The general problem of the stability of motion*. Taylor & Francis, London.

[18] Mba, J. C., and Wang, Q. G. (2019). Multi-period portfolio optimization: A differential evolution copula-based approach.

[19] Strobeck, C. (1973). N species competition. *Ecology,* **54** *(3), 650-654*.

[20] Taylor, J. (1997). *Introduction to Error Analysis*, 2^{nd} edition. University Science Books, Sausalito.

General conclusion

This study discussed threes versions of 2- and 3- dimensional Lotka-Volterra dynamical system namely Grey Lotka-Volterra Model (GLVM), Fractional Grey Lotka-Volterra Model (FGLVM) and Variable-order Fractional Grey Lotka-Volterra Model (VFGLVM) with an application to cryptocurrencies adoption. The 3 cryptocurrencies of interest are Bitcoin, Litecoin and Ripple. The study comprises two main parts: The first part of the study reviewed the elements of fractional calculus, the Lotka-Volterra dynamical system and the elements of Grey modeling. The second part consisted of applying successively Grey modeling and fractional derivative to the Lotka-Volterra dynamical system with an application on forecasting adoption of cryptocurrencies.

The 2-dimensional study considered Bitcoin and Litecoin while the 3-dimensional study used Bitcoin, Litecoin and Ripple. The dataset used include records from 28-April-2013 to 10-February-2018 which provided forecasting values for Bitcoin and Litecoin through 2-dimensional study, while records from 7-August-2013 to 10- February-2018 provided forecasting values of Bitcoin, Litecoin and Ripple through 3-dimensional study. Predictability of the models is checked by the estimated Lyapunov exponents while the accuracy is checked by the Mean Absolute Percentage Error (MAPE). The thesis has produced the following four papers.

In Paper 1, models for competing species, namely the Grey Model (GM(1,1)), the Lotka-Volterra Model (LVM) and Grey Lotka-Volterra Model (GLVM) are reviewed. The LVM shows a chaotic behavior for the dataset at hand. The results for GLVM show accurately that transaction counts of Bitcoin are relatively higher than that of Ripple and Litecoin along the study time and suggests a long term strength in transacting Bitcoin relatively to Litecoin and Ripple.

In Paper 2, Fractional Grey Lotka-Volterra Model (FGLVM) is introduced. Forecasting values of cryptocurrencies for n-dimensional FGLVM study, $n = \{2,3\}$ along 100 days of study time are displayed. The results of the FGLVM reveals that the 2- and 3-dimensional Lotka-Volterra system are chaotic dynamical systems. The MAPE indicates that FGLVM is better than GM(1,1) and GLVM. The 2- and 3-dimensional FGLVMs analysis suggest a future constant trend in transacting Bitcoin and a future decreasing trend in transacting Litecoin and Ripple with Bitcoin at a relatively higher transaction while Litecoin transaction counts are everywhere superior to that of Ripple.

In Paper 3, Fractional Grey Lotka-Volterra Model with variable order is introduced. The MAPE suggests a high accuracy of the 3-dimensional Variable-order Fractional Lotka-Volterra model (VFGLVM) for the overall model values of Bitcoin and a reasonable accuracy for both model values of Litecoin and Ripple. The 2-dimensional VFGLVM has a good accuracy for the overall forecasting values of Bitcoin and a reasonable accuracy for the forecasting values of Litecoin.

As for the GLVM and the FGLVM, the 2- and 3-dimensional VFGLVM show a chaotic behavior. Forecasting values indicate a future slight linear increase in transacting Bitcoin and a future decreasing transaction of Litecoin and Ripple. The VFGLVM suggests that Bitcoin will keep relatively higher transaction counts while Litecoin transaction counts will be everywhere higher than that of Ripple.

In Paper 4, the error assessment is made on GM(1,1), GLVM, FGLVM and VFGLVM. The error sequence patterns and the MAPE suggest a relatively higher accuracy of the VFLVM in 2- and 3-dimensions. Mostly the VFGLVM is relatively the best model followed by the FGLVM, the GLVM and then the GM(1,1).

The important limitation of this work is that the estimates are based on difference equations, rather than differential equations. This is due to the difficulty of generalising solutions for a wide class of differential equations in continuous framework.